

**Fractional Analysis**  
**(Lessons and Exercises with Solutions)**  
Master 2, PDEs & Applications

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# Chapter 1

## Special functions

### 1.1 Gamma function

The Gamma function is a special function defined by Leonhard Euler (1707-1783), its objective is to extend the factorial function to the set of complex numbers. Several mathematicians have given particular importance to this function, namely, Adrien-Marie Legendre (1752-1833), Carl Friedrich Gauss...

**Definition 1.1.** *Euler's infinite limit* (Euler, 1729 and Gauss, 1811)

Letting

$$\Gamma_n(s) := \frac{1.2.3 \dots n}{s(s+1)(s+2) \dots (s+n)} n^s, \quad s \neq 0, -1, -2, \dots$$

The Gamma function is defined by limit as follows

$$\Gamma(s) = \lim_{n \rightarrow \infty} \Gamma_n(s).$$

**Examples 1.1.** 1. We have  $\Gamma(1) = 1$ . In fact,

$$\begin{aligned} \Gamma_n(1) &= \frac{1.2.3 \dots n}{1.2.3 \dots n(n+1)} n \\ &= \frac{n}{n+1}. \end{aligned}$$

This implies that

$$\Gamma(1) = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1.$$

2. Note that since

$$\Gamma_n\left(\frac{16}{3}\right) = \frac{1.2.3 \dots n}{\left(\frac{16}{3}\right)\left(\frac{16}{3}+1\right) \dots \left(\frac{16}{3}+n\right)} n^{\frac{16}{3}},$$

$$\Gamma_n\left(\frac{10}{3}\right) = \frac{1.2.3 \dots n}{\left(\frac{10}{3}\right)\left(\frac{10}{3}+1\right) \dots \left(\frac{10}{3}+n\right)} n^{\frac{10}{3}},$$

we have

$$\begin{aligned} \frac{\Gamma_n\left(\frac{16}{3}\right)}{\Gamma_n\left(\frac{10}{3}\right)} &= \lim_{n \rightarrow \infty} \frac{\left(\frac{10}{3}\right)\left(\frac{10}{3}+1\right)}{\left(\frac{10}{3}+1+n\right)\left(\frac{10}{3}+2+n\right)} \\ &= \lim_{n \rightarrow \infty} \frac{\frac{130}{9}n^2}{n^2} = \frac{130}{9}. \end{aligned}$$

3.  $0! = 1$ .

Since  $\Gamma(1) = 0!$  and since  $\Gamma(1) = 1$  we obtain  $0! = 1$ .

**Definition 1.2.** Weierstrass's infinite product

$$\frac{1}{\Gamma(x)} = x e^{\gamma x} \prod_{n=1}^{+\infty} \left(1 + \frac{x}{n}\right) e^{-\frac{x}{n}}.$$

**Lemma 1.1** (Stirling's Formula). For  $z \in \mathbb{C}$ ,  $|\arg(z)| \leq \delta < \pi$  and  $|z| \rightarrow \infty$ , we have

$$\frac{1}{\Gamma(z)} = \frac{1}{\sqrt{2\pi}} e^{z} z^{-z+1/2} \left(1 + O\left(z^{-1}\right)\right),$$

that is

$$\Gamma(z) \sim \sqrt{2\pi} e^{-z} z^{z-\frac{1}{2}}.$$

**Lemma 1.2.** For  $z \in \mathbb{C}$ ,  $|z| \rightarrow \infty$ , we have

$$\frac{1}{\Gamma(z)} = \begin{cases} \frac{e^z z^{-z+1/2}}{\sqrt{2\pi}} \left(1 + O\left(z^{-1}\right)\right), & \text{if } \operatorname{Re}(z) > 1, \\ \sqrt{\frac{2}{\pi}} e^{z-1} (1-z)^{-z+1/2} \sin(\pi z) \left(1 + O\left(z^{-1}\right)\right), & \text{if } 0 < \operatorname{Re}(z) < 1. \end{cases}$$

**Definition 1.3. (Euler's integral)(Euler, 1730)**

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt, \quad \operatorname{Re}(x) > 0. \quad (1.1)$$

**Remark 1.1.** For the divergence's reasons of the integral, the definition of the Gamma function by the above integral formula is not applicable for negative values.

**Example 1.1.**

$$\Gamma(1) = \int_0^{\infty} e^{-t} dt = [-e^{-t}]_0^{\infty} = 1.$$

**Theorem 1.1.** 1. The Gamma function(1.1) is defined, continuous on  $]0, +\infty[$ .

2. Moreover,  $\Gamma \in C^{\infty}$  and

$$\forall k \in \mathbb{N}, \quad \forall x > 0, \quad \Gamma^{(k)}(x) = \int_0^{+\infty} (\ln t)^k e^{-t} t^{x-1} dt.$$

**Proof.**

1. (a) Let us consider the function  $f$

$$f : \mathbb{R} \times \mathbb{R}_+^* \mapsto \mathbb{R}, \quad (x, t) \mapsto e^{-t} t^{(x-1)}, \quad \text{for all } x \in \mathbb{R}.$$

Define the function  $f_x$  as follows

$$f_x : \mathbb{R}_+^* \mapsto \mathbb{R}, \quad t \mapsto f(t, x).$$

We note the  $f_x$  est continuous, positive on  $]0, +\infty[$  therefore, it is locally integrable.

In the neighborhood of zero:

$$e^{-t}t^{(x-1)} \sim t^{(x-1)}.$$

The function  $f_x$  is integrable for all  $t \in ]0, 1[$ , if and only if,  $x \in ]0, +\infty[$ .

In the neighborhood of infinity:

$$e^{-t}t^{(x-1)} \sim o\left(\frac{1}{t^2}\right),$$

hence  $f_x$  is integrable on  $]1, +\infty[$ .

(b) Continuity of function  $\Gamma$ :

Following the theorem of continuity under integral sign, we have

i.  $x \mapsto e^{-t}t^{x-1}$  is continuous on  $]0, +\infty[$ .

ii. Let  $a, b \in \mathbb{R}^*$ ,  $0 < a < b < +\infty$ :

For all  $0 < t < 1$ , the function  $x \mapsto t^{(x-1)}$  is decreasing on  $[a, b]$ , so that

$$\forall x \in [a, b] : 0 < f(t, x) < t^{(a-1)}.$$

For all  $t \geq 1$  we have  $x \mapsto t^{(x-1)}$  is increasing on  $[a, b]$ , hence

$$0 < f(t, x) < e^{-t}t^{(b-1)},$$

and hence

$$\varphi : ]0, +\infty[ \mapsto \mathbb{R}, \quad t \mapsto \varphi(t) = \begin{cases} t^{(a-1)}, & t \in ]0, 1], \\ e^{-t}t^{(b-1)} & t \in [1, +\infty[ \end{cases}$$

is positive, continuous, integrable on  $]0, +\infty[$  and

$$\forall (x, t) \in [a, b] \times ]0, +\infty[, \quad 0 < f(t, x) < \varphi(t),$$

Thus,  $\Gamma$  is continuous on  $]0, +\infty[$ .

2 For  $k = 1$

By the theorem of differentiation under the integral sign.

(a)

$$\begin{aligned}\frac{\partial}{\partial x}(e^{-t}t^{(x-1)}) &= e^{-t}\frac{\partial}{\partial x}t^{x-1} \\ &= e^{-t}\ln(t)t^{(x-1)}.\end{aligned}$$

(b)

$$|e^{-t}\ln(t)t^{(x-1)}| \leq \begin{cases} |\ln(t)|t^{(a-1)}, & 0 < t < 1, \\ |t^{(b-1)}e^{-t}\ln(t)|, & 1 < t < +\infty. \end{cases}$$

$$\begin{aligned}|e^{-t}\ln(t)t^{(x-1)}| &= [-\ln(t)e^{-t}t^{(x-1)}]_{t \in ]0,1[} + [\ln(t)e^{-t}t^{(x-1)}]_{t \in [1,+\infty[} \\ &\leq [-\ln(t)t^{(a-1)}]_{t \in ]0,1[} + [\ln(t)e^{-t}t^{(b-1)}]_{t \in [1,+\infty[},\end{aligned}$$

so that  $\varphi$  integrable on  $]0, +\infty[$ , moreover,

$$\Gamma'(t) = \int_0^{\infty} \ln t e^{-t} t^{x-1} dt.$$

By induction using the same process we get

$$\Gamma^{(k)}(x) = \int_0^{\infty} \ln(t)^k e^{-t} t^{(x-1)} dt.$$

■

**Remark 1.2.** We can use the following formula to define  $\Gamma(x)$  for non-integer negative values

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}, \quad x \neq 0, -1, -2, \dots$$

**Example 1.2.**

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi} \\ &\approx 1.77245\end{aligned}$$

$$\begin{aligned}\Gamma\left(-\frac{3}{2}\right) &= \frac{4}{3}\sqrt{\pi} \\ &\approx 2.36327\end{aligned}$$

$$\begin{aligned}\Gamma\left(-\frac{1}{2}\right) &= -2\sqrt{\pi} \\ &\approx -3.54490\end{aligned}$$

**Remark 1.3.** 1. *Stirling's formula*

$$\Gamma(x) \sim \sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x}, \quad \operatorname{Re}(x) \rightarrow +\infty.$$

**1.2 Volume of n-dimensional sphere via Gamma function**

This section is based on the lectures of Professor Chaur-Chin Chen. We show the importance use of the Gamma function to calculate the volume  $V_n$ , of an  $n$ -dimensional sphere. For this purpose, let us introduce the Cartesian coordinates  $(x_1, x_2, x_3, \dots, x_n)$ , we recall that the sphere is defined by the equation

$$x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2 = \rho^2$$

**Proposition 1.1.** *The volume  $V_2$  of the 2-dimensional sphere with radius  $\rho$  is given by*

$$V_2 = \pi\rho^2.$$

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**Proof.** Letting

$$\begin{aligned}x_1 &= r \cos \theta, \quad 0 \leq \theta \leq 2\pi, \\x_2 &= r \sin \theta, \quad 0 \leq r \leq \rho.\end{aligned}$$

We have

$$\begin{aligned}J_2 &= \frac{\partial(x_1, x_2)}{\partial(r, \theta)} \\&= \begin{vmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_2}{\partial r} \\ \frac{\partial x_1}{\partial \theta} & \frac{\partial x_2}{\partial \theta} \end{vmatrix} \\&= r.\end{aligned}$$

Thus, we obtain

$$V_2 = \int_0^\rho \int_0^{2\pi} J_2 dr d\theta = \int_0^\rho \int_0^{2\pi} r dr d\theta = \pi \rho^2.$$

■

**Proposition 1.2.** *The volume  $V_3$  of the 3-dimensional sphere of radius  $\rho$  is given by*

$$V_3 = \frac{4\pi\rho^3}{3}.$$

**Proof.** For

$$\begin{aligned}x_1 &= r \cos \theta_1 \cos \theta_2, \quad 0 \leq \theta_2 \leq 2\pi \\x_2 &= r \cos \theta_1 \sin \theta_2, \quad -\frac{\pi}{2} \leq \theta_1 \leq \frac{\pi}{2} \\x_3 &= r \sin \theta_1, \quad 0 \leq r \leq \rho.\end{aligned}$$

We get

$$\begin{aligned}
 J_3 &= \frac{\partial(x_1, x_2, x_3)}{\partial(r, \theta_1, \theta_2)} \\
 &= \begin{vmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_2}{\partial r} & \frac{\partial x_3}{\partial r} \\ \frac{\partial x_1}{\partial \theta_1} & \frac{\partial x_2}{\partial \theta_1} & \frac{\partial x_3}{\partial \theta_1} \\ \frac{\partial x_1}{\partial \theta_2} & \frac{\partial x_2}{\partial \theta_2} & \frac{\partial x_3}{\partial \theta_2} \end{vmatrix} \\
 &= r^2 \cos \theta_1,
 \end{aligned}$$

and therefore

$$\begin{aligned}
 V_3 &= \int_0^\rho \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} J_3 dr d\theta_1 d\theta_2 \\
 &= \int_0^\rho \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} r^2 \cos \theta_1 dr d\theta_1 d\theta_2 \\
 &= \frac{4\pi\rho^3}{3}.
 \end{aligned}$$

■

**Theorem 1.2.** *The volume  $V_n$  of the  $n$ -dimensional sphere of radius  $\rho$  is given by*

$$V_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)} \cdot \rho^n.$$

**Proof.** Let us consider the following polar coordinates

$$\begin{aligned}
 x_1 &= r \cos \theta_1 \cos \theta_2 \cdots \cos \theta_{n-2} \cos \theta_{n-1}, \quad 0 \leq \theta_{n-1} \leq 2\pi \\
 x_{n-1} &= r \cos \theta_1 \sin \theta_2, \quad -\frac{\pi}{2} \leq \theta_1 \leq \frac{\pi}{2} \\
 x_n &= r \sin \theta_1, \quad 0 \leq r \leq \rho,
 \end{aligned}$$

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and for  $2 \leq k \leq n-2$ ,

$$x_k = r \cos \theta_1 \cos \theta_2 \cdots \cos \theta_{n-k} \sin \theta_{n-k+1}, \quad -\frac{\pi}{2} \leq \theta_{n-k} \leq \frac{\pi}{2}.$$

This implies that

$$J_n = r^{n-1} c_1^{n-1} c_2^{n-2} \cdots c_{n-1}^1 s_1 s_2 \cdots s_{n-1} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ -t_1 & -t_1 & -t_1 & \cdots & \frac{1}{t_1} \\ -t_2 & -t_2 & -t_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{n-2} & -t_{n-2} & \frac{1}{t_{n-2}} & \cdots & 0 \\ -t_{n-1} & \frac{1}{t_{n-1}} & 0 & \cdots & 0 \end{vmatrix},$$

with  $t_i = \frac{\sin \theta_i}{\cos \theta_i}$ .

Subtracting each column from the preceding one, we get

$$\begin{aligned} J_n &= r^{n-1} c_1^{n-1} c_2^{n-2} \cdots c_{n-1}^1 s_1 s_2 \cdots s_{n-1} \prod_{j=1}^{n-1} \left( t_j + \frac{1}{t_j} \right) \\ &= r^{n-1} c_1^{n-1} c_2^{n-2} \cdots c_{n-1}^1 s_1 s_2 \cdots s_{n-1} \prod_{j=1}^{n-1} \left( \frac{1}{s_j \cdot c_j} \right) \\ &= r^{n-1} c_1^{n-2} c_2^{n-3} \cdots c_{n-3}^2 c_{n-2}^1 \\ &= r^{n-1} \cos^{n-2} \theta_1 \cos^{n-3} \theta_2 \cdots \cos^2 \theta_{n-3} \cos^1 \theta_{n-2} \end{aligned}$$

Thus,

$$\begin{aligned}
 V_n &= \int_0^\rho \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \cdots \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} (J_n) d\theta_1 d\theta_2 \cdots d\theta_{n-2} d\theta_{n-1} dr \\
 &= \int_0^\rho \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \cdots \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} [r^{n-1} \cos^{n-2} \theta_1 \cos^{n-3} \theta_2 \cdots \cos \theta_{n-2}] d\theta_1 d\theta_2 \cdots d\theta_{n-1} dr \\
 &= \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} \cdot \rho^n.
 \end{aligned}$$

■

**Remark 1.4.** 1. The surface  $S_n$  for a sphere with radius  $r$  in  $\mathbb{R}^n$  is given by

$$S_n = \frac{2\Gamma(\frac{1}{2})^n r^{n-1}}{\Gamma(\frac{n}{2})}.$$

### 1.3 Incomplete Gamma functions

**Definition 1.4.** The complete Gamma function  $\Gamma$  can be generalized to the upper incomplete Gamma function and the lower incomplete one respectively as follows:

$$\begin{aligned}
 \gamma(a, z) &= \int_0^z e^{-t} t^{a-1} dt, & \operatorname{Re}(z) > 0, \\
 \Gamma(a, z) &= \int_z^\infty e^{-t} t^{a-1} dt, & \operatorname{Re}(z) > 0.
 \end{aligned}$$

**Remark 1.5.** We note that

1.  $\Gamma(a) = \lim_{x \rightarrow \infty} \gamma(a, x)$ ;
2.  $\Gamma(a, x) = \Gamma(a) - \gamma(a, x)$ ;
3.  $\Gamma(a, 0) = \Gamma(a)$ ;

$$4. \Gamma(a+1, z) = a\Gamma(a, z) + z^a e^{-z};$$

$$5. \Gamma(a+1, z) = a\Gamma(a, z) + z^a e^{-z};$$

$$6. \gamma(a+1, z) = a\gamma(a, z) - z^a e^{-z}.$$

**Proposition 1.3.** *The recursive formula for the serial*

$$\gamma(a, z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{a+n}}{n! (a+n)}$$

is given by

$$\gamma(a, z) = z^a \sum_{n=0}^{\infty} \frac{\alpha_n}{a+n}, \quad \text{with } \alpha_0 = 1 \quad \text{and} \quad \alpha_n = \alpha_{n-1} \frac{(-z)}{n}.$$

**Proof.** We have

$$\gamma(s, z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{s+n}}{n! (s+n)} = z^s \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n! (s+n)} = z^s \sum_{n=0}^{\infty} \frac{\alpha_n}{s+n},$$

for some constant  $g$  and  $\operatorname{Re}(z) + g + \frac{1}{2} > 0$

$$b_i(g) = \frac{\sqrt{2}e^{g+\frac{1}{2}}}{\pi} \sum_{n=0}^N B_{i,n} \sum_{l=0}^N C_{n,l} \Gamma\left(l + \frac{1}{2}\right) \left(l + g + \frac{1}{2}\right)^{-(l+\frac{1}{2})} e^l.$$

$$B_{i,j} := \begin{cases} 0, & i > j, \\ \frac{1}{2}, & i = j = 0, \\ 1, & i = 0, j > 0, \\ \frac{(-1)^{j-i+1} (i+j-1)!}{(j-i)! ((n-1)!)^2}, & \text{otherwise,} \end{cases}$$

and

$$C_{i,j} := \begin{cases} 0, & i < j, \\ 1, & i = 0, j = 0, \\ \frac{(-1)^{i-j} i(i+j)! 4^j}{(i+1)(2j)! (i-j)!}, & \text{otherwise.} \end{cases}$$

■

For more details see [2].

## 1.4 Beta function

**Definition 1.5.** (Euler 1730 & Jacques Binet 1839, (1786-1856))  
The Beta function  $B(.,.)$  or the Beta integral is the name used for the Eulerian integral of the first kind. The Beta function is usually defined by

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad \operatorname{Re}(x) > 0, \operatorname{Re}(y) > 0.$$

In the following proposition we introduce the relationship between the Gamma function and the Beta function.

**Proposition 1.4.**

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

**Proof.** We have

$$\begin{aligned} \Gamma(x)\Gamma(y) &= \int_0^\infty e^{-s} s^{(x-1)} ds \int_0^\infty e^{-t} t^{(y-1)} dt \\ &= \int_0^\infty \int_0^\infty e^{-(s+t)} s^{(x-1)} t^{(y-1)} ds dt. \end{aligned}$$

We use the change of variables

$$\begin{cases} u = s + t, \\ v = \frac{s}{s+t}, \end{cases} \implies \begin{cases} s = uv, \\ t = u(1-v), \end{cases}$$

with the domain of integration  $D = \{u, v \in \mathbb{R}, u > 0 \text{ et } 0 < v < 1\}$ ,

we get

$$\begin{aligned}
 \Gamma(x)\Gamma(y) &= \int \int_D (uv)^{(x-1)} [u(1-v)]^{(y-1)} e^{-u} | -u | dudv \\
 &= \int \int_D u^{(x-1+y)} v^{(x-1)} (1-v)^{(y-1)} e^{-u} dudv \\
 &= \int_0^\infty \left[ \int_0^1 u^{(x-1+y)} v^{(x-1)} (1-v)^{(y-1)} e^{-u} du \right] dv \\
 &= \int_0^\infty u^{(x-1+y)} e^{-u} du \int_0^1 v^{(x-1)} (1-v)^{(y-1)} dv \\
 &= \Gamma(x+y) \int_0^1 v^{(x-1)} (1-v)^{(y-1)} dv \\
 &= \Gamma(x+y) B(x, y).
 \end{aligned}$$

■

**Examples 1.2.** *It is easy to see that*

1.

$$B(1, 1) = 1.$$

2.

$$B\left(\frac{1}{3}, \frac{2}{3}\right) = \frac{2\sqrt{3}}{3}\pi.$$

3.

$$B\left(\frac{1}{4}, \frac{3}{4}\right) = \pi\sqrt{2}.$$

## 1.5 Mittag-Leffler function

The Mittag-Leffler function is an important special function with generalize of the exponential, it has many applications in mathematics sciences namely in fractional calculus. The Mittag-Leffler function is defined by

**Definition 1.6.** (Mittag-Leffler, 1903)

$$E_{\alpha}(x) = \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C} \text{ and } \operatorname{Re}(\alpha) > 0.$$

**Definition 1.7.** (Wiman, 1905)

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \quad \operatorname{Re}(\alpha) > 0 \text{ and } \operatorname{Re}(\beta) > 0.$$

The following results are inspired from [1].

**Theorem 1.3.** *The following integral formula holds:*

$$x^{\beta-1} E_{\alpha,\beta}(x^{\alpha}) = x^{\beta-1} E_{2\alpha,\beta}(x^{2\alpha}) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} E_{2\alpha,\beta}(\tau^{2\alpha}) \tau^{\beta-1} d\tau \quad (\beta > 0)$$

**Proof.** Letting

$$\gamma_{\alpha}(x, \tau) := \left[ 1 + \frac{(x-\tau)^{\alpha}}{\Gamma(\alpha+1)} \right].$$

On the one hand

$$\begin{aligned} \int_0^x E_{2\alpha,\beta}(\tau^{2\alpha}) \tau^{\beta-1} \gamma_{\alpha}(x, \tau) d\tau &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(2k\alpha + \beta)} \int_0^x \tau^{2k\alpha + \beta - 1} \gamma_{\alpha}(x, \tau) d\tau \\ &= x^{\beta} \sum_{k=0}^{\infty} \left[ \frac{x^{2k\alpha}}{\Gamma(2k\alpha + \beta + 1)} + \frac{x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha + \beta + 1)} \right] \\ &= x^{\beta} \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha + \beta + 1)} \\ &= x^{\beta} E_{\alpha,\beta+1}(x^{\alpha}). \end{aligned}$$

On the other hand, using the definition and by term-by-term integration, we get

$$\int_0^x E_{\alpha,\beta}(\lambda \tau^{\alpha}) \tau^{\beta-1} d\tau = x^{\beta} E_{\alpha,\beta+1}(\lambda x^{\alpha}), \quad \beta > 0.$$

Thus,

$$\int_0^x E_{2\alpha,\beta}(\tau^{2\alpha})\tau^{\beta-1} \left\{ 1 + \frac{(x-\tau)^\alpha}{\Gamma(\alpha+1)} \right\} d\tau = \int_0^x E_{\alpha,\beta}(\tau^\alpha)\tau^{\beta-1} d\tau, \quad \beta > 0.$$

By differentiation with respect to  $x$ , we get the desired result. ■

Let  $\varsigma_1$  and  $\varsigma_2$  two complex parameters. Define the following function

$$J_{\nu,\alpha,\beta,\varsigma_1,\varsigma_2}(\delta) := \int_0^\delta \tau^{\beta-1} E_{\alpha,\beta}(\varsigma_1 \tau^\alpha) (\delta - \tau)^{\nu-1} E_{\alpha,\nu}(\varsigma_2 (\delta - \tau)^\alpha) d\tau,$$

**Theorem 1.4.** *The following integral formula holds for some  $\beta > 0$  and  $\nu > 0$*

$$J_{\nu,\alpha,\beta,\varsigma_1,\varsigma_2}(\delta) = \frac{\varsigma_1 E_{\alpha,\beta+\nu}(\delta^\alpha \varsigma_1) - \varsigma_2 E_{\alpha,\beta+\nu}(\delta^\alpha \varsigma_2)}{\varsigma_1 - \varsigma_2} \delta^{\beta+\nu-1}$$

**Proof.** We have

$$\begin{aligned} J_{\nu,\alpha,\beta,\varsigma_1,\varsigma_2}(\delta) &= \sum_{n,m=0}^{\infty} \frac{\varsigma_1^n \varsigma_2^m}{\Gamma(n\alpha + \beta) \Gamma(m\alpha + \nu)} \int_0^\delta \tau^{n\alpha + \beta - 1} (\delta - \tau)^{m\alpha + \nu - 1} d\tau \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\varsigma_1^n \varsigma_2^m \delta^{(n+m)\alpha + \beta + \nu - 1}}{\Gamma((m+n)\alpha + \beta + \nu)} \\ &= \delta^{\beta + \nu - 1} \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{\varsigma_1^n (\varsigma_2)^{k-n} \delta^{k\alpha}}{\Gamma(k\alpha + \beta + \nu)} \\ &= \delta^{\beta + \nu - 1} \sum_{k=0}^{\infty} \frac{\varsigma_2^k \delta^{k\alpha}}{\Gamma(k\alpha + \beta + \nu)} \sum_{n=0}^k \left( \frac{\varsigma_1}{\varsigma_2} \right)^n \\ &= \frac{\delta^{\beta + \nu - 1}}{\varsigma_1 - \varsigma_2} \sum_{k=0}^{\infty} \frac{\delta^{k\alpha} (\varsigma_1^{k+1} - \varsigma_2^{k+1})}{\Gamma(k\alpha + \beta + \nu)}. \end{aligned}$$

By the definition of the generalized Mittag-Leffler function we obtain the desired result. ■

Denoting by  $(\delta)_n$  the following Pochhammer symbol

$$(\delta)_n = \delta(\delta + 1) \dots (\delta + n - 1), \quad \delta \neq 0.$$

Now, we give the second generalized Mittag-Leffler function

**Definition 1.8.**

$$E_{\beta, \gamma}^{\delta}(x) = \sum_{n=0}^{+\infty} \frac{(\delta)_n x^n}{\Gamma(\beta n + \gamma) n!}, \quad \text{such that } \alpha, \beta, \gamma \in \mathbb{C} \text{ and } \operatorname{Re}(\beta) > 0,$$

Let us consider the  $B_p(x, y)$  the generalized Beta function defined by

$$B_p(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} e^{\frac{-p}{t(1-t)}} dt.$$

We give the third generalized Mittag-Leffler function

**Definition 1.9.** (Ozarslan, Yasar 2013)

$$E_{\alpha, \beta}^{\gamma, c}(x, p) = \sum_{n=0}^{+\infty} \frac{B_p(\gamma + n, c - n)}{B_p(\gamma, c - n)} \frac{(c)_n x^n}{\Gamma(\beta n + \gamma) n!}, \quad p > 0, \quad \operatorname{Re}(c) > 0, \quad \operatorname{Re}(\gamma) > 0$$

**Examples 1.3.**

$$\begin{aligned} E_1(x) &= \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(k+1)} \\ &= e^x. \end{aligned}$$

$$\begin{aligned} E_2(x) &= \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(2k+1)} \\ &= \cos(x^2). \end{aligned}$$

$$\begin{aligned} E_{1,1}(x) &= \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(k+1)} \\ &= \sum_{k=0}^{+\infty} \frac{x^k}{k!} \\ &= e^x. \end{aligned}$$

**Remark 1.6.**

$$\begin{aligned}
 E_\beta(x) &= E_{\beta,1}(x). \\
 E_{\beta,1}(x) &= E_{\beta,1}^1(x). \\
 E_{\beta,\gamma}(x) &= E_{\beta,\gamma}^1(x).
 \end{aligned}$$

**1.6 Mellin-Ross function**

The Mellin-Ross function  $E_t(\nu, a)$  appears in the fractional integral of the exponential function of type  $e^{at}$ , it is related to the incomplete Gamma function and the Mittag-Leffler function. Letting

$$\begin{aligned}
 \gamma^*(a, x) &:= \frac{x^{-a}}{\Gamma(a)} \gamma(a, x) \\
 &= \frac{1}{\Gamma(a)x^a} \int_0^x e^{-t} t^{a-1} dt.
 \end{aligned}$$

**Definition 1.10.** *The Mellin-Ross function is defined by*

$$E_t(\nu, a) = t^\nu e^{at} \gamma^*(\nu, t),$$

*we can also write*

$$E_t(\nu, a) = t^\nu \sum_{k=0}^{+\infty} \frac{(at)^k}{\Gamma(\nu + k + 1)} = t^\nu E_{1,1+\nu}(at).$$

## 1.7 Exercises

**Exercise 1.1.** 1. Prove that

$$\Gamma(x+1) = x\Gamma(x), \quad x > 0.$$

2. Deduce that

$$\Gamma(n+1) = n! \quad \text{for all } n \in \mathbb{N}.$$

3. Calculate  $\Gamma(5)$  and  $\Gamma(7)$ .

4. Calculate  $\frac{\Gamma(\frac{22}{5})}{\Gamma(\frac{17}{5})}$ .

5. Calculate  $\frac{\Gamma(\frac{11}{4})}{\Gamma(\frac{23}{4})}$ .

**Exercise 1.2.** 1. Prove that

$$\Gamma_n(x) = \frac{e^{x(\log n - 1 - 1/2 - \dots - 1/n)} e^{x+x/2+\dots+x/n}}{x(1+x)(1+x/2)\dots(1+x/n)}.$$

2. Deduce that

$$\Gamma_n(x) = e^{x(\log n - 1 - 1/2 - \dots - 1/n)} \frac{1}{x} \frac{e^x}{1+x} \frac{e^{x/2}}{1+x/2} \times \dots \times \frac{e^{x/n}}{1+x/n}.$$

3. Prove that the sequence  $\gamma_n := 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n$  is decreasing.

4. Let us consider the Euler-Mascheron's constant  $\gamma$  given by

$$\gamma := \lim_{n \rightarrow \infty} \left[ 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n \right] = 0.57721\ 56649\ 01532\ 86060\ \dots$$

Deduce the value of  $\Gamma(x)$ .

**Exercise 1.3.** Let us consider the function  $\psi$  given by

$$\psi(x) := \frac{\Gamma(x)}{\Gamma'(x)}.$$

Prove that

1.

$$\frac{1}{\psi(x)} = -\gamma - \frac{1}{x} + \sum_{n \geq 1} \left( \frac{1}{n} - \frac{1}{x+n} \right).$$

2.

$$\left( \frac{1}{\psi(x)} \right)' = \sum_{n \geq 0} \frac{1}{(x+n)^2}.$$

**Exercise 1.4.** Prove that there are two constants  $c_1$  and  $c_2$  such that, for any positive real  $R$  large enough, we have

$$\left| \frac{1}{\Gamma(z)} \right| \leq \begin{cases} c_1 R^R & \text{if } \operatorname{Re}(z) > 1 \\ c_2 (R+1)^R & \text{if } 0 < \operatorname{Re}(z) < 1. \end{cases}$$

**Exercise 1.5.** Let us consider the function  $\xi(z) = 1/\Gamma(z)$ ,  $z \in [1-i, 1+i]$ . Prove that

$$|\xi(z)| \geq |x| e^{\gamma x} e^{-3} > \frac{1}{2}.$$

**Exercise 1.6.** Using the integral formula to prove that

$$\Gamma(z+1) = z\Gamma(z), \quad \text{for all } z \in \mathbb{C} \text{ with } \operatorname{Re}(z) > 0.$$

**Exercise 1.7.** Prove that for all  $x > 0$

1.

$$\Gamma(x) = \int_0^1 (-\log(u))^{x-1} du.$$

2.

$$\Gamma(x) = 2 \int_0^{+\infty} e^{-u^2} u^{2x-1} du.$$

**Exercise 1.8.** 1. Calculate the integral

$$I = 2 \int_0^{+\infty} e^{-u^2} du.$$

2. Deduce the value of  $\Gamma\left(\frac{1}{2}\right)$ .

3. Deduce  $\Gamma\left(\frac{3}{2}\right)$ ,  $\Gamma\left(-\frac{1}{2}\right)$  and  $\Gamma\left(-\frac{3}{2}\right)$ .

**Exercise 1.9.** Prove that

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma(2n+1)}{2^{2n}\Gamma(n+1)}, \quad \text{for all } n \in \mathbb{N}.$$

**Exercise 1.10.** Let us consider the following function

$$\varphi(s) := \int_0^1 \frac{u^{s-1}}{1+u} du.$$

1. Prove that

$$\Gamma(s)\Gamma(1-s) = \varphi(s) + \varphi(1-s).$$

2. Prove that

$$\varphi(s) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n+s}, \quad \text{pour tout } s \in \mathbb{C}, \quad \text{with } 0 < \operatorname{Re}(s) < 1.$$

3. Give the power series expansion for the function  $\Gamma(s)\Gamma(1-s)$ .

4. Letting

$$a_n := \frac{2 \sin \pi s}{\pi} \cdot \frac{(-1)^n s}{s^2 - n^2}, \quad n \in \mathbb{N}.$$

Use the formula

$$\begin{aligned} \forall t \in ]0, 1[ \quad \cos(st) &= \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos nt \\ &= \frac{\sin \pi s}{\pi s} + \sum_{n=1}^{+\infty} \frac{2 \sin \pi s}{\pi} \frac{(-1)^n s}{s^2 - n^2} \cos nt \end{aligned}$$

to prove that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \quad s \in \mathbb{C} \quad \text{such that} \quad 0 < \operatorname{Re}(s) < 1.$$

**Exercise 1.11.** Using the formula

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \quad x \geq 1,$$

calculate  $\Gamma\left(\frac{1}{2}\right)$ ,  $\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)$  and  $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$ .

**Exercise 1.12.** Let us consider the following integral

$$J_n(z) = \int_0^1 s^{z-1} (1-s)^n ds, \quad \operatorname{Re}(z) > 0.$$

1.

Prove that for all  $n \in \mathbb{N}$ , we have

$$J_n(z) = \frac{n!}{z(z+1)\dots(z+n)}.$$

Prove that

$$\Gamma(z) = \lim_{n \rightarrow +\infty} n^z J_n(z).$$

**Exercise 1.13.** Let  $x, y \in \mathbb{C}$  with  $\operatorname{Re}(x) > 0$ ,  $\operatorname{Re}(y) > 0$ . Prove that

1.

$$B(x, y) = B(y, x).$$

2.

$$B(x, y) = 2 \int_0^{\frac{\pi}{2}} \sin^{(2x-1)\theta} \cos^{(2y-1)\theta} d\theta.$$

**Exercise 1.14.** Let  $x, y \in \mathbb{C}$  with  $\operatorname{Re}(x) > 0$ ,  $\operatorname{Re}(y) > 0$ .

1. Prove that

$$B(x+1, y) = \frac{x}{x+y} B(x, y).$$

2. Deduce  $B(x, y+1)$ .

3. Deduce a relationship between the  $B(x, y)$  and  $B(x-1, y)$ .

**Exercise 1.15.** Prove that

$$\int_0^1 \frac{t dt}{\sqrt{1-t^5}} = \frac{1}{5} B\left(\frac{2}{5}, \frac{1}{2}\right).$$

**Exercise 1.16.** Prove that

$$\int_0^1 \frac{t^2 dt}{\sqrt{1-t^4}} = \frac{1}{4} B\left(\frac{3}{4}, \frac{1}{2}\right).$$

**Exercise 1.17.** 1. Calculate  $B\left(\frac{1}{2}, \frac{1}{2}\right)$ .

2. Deduce

$$\int_{-\infty}^{+\infty} e^{-u^2} du.$$

**Exercise 1.18.** Calculate  $E_{1,2}(z)$ ,  $E_{1,3}(z)$ ,  $E_{2,1}(z)$ ,  $z \in \mathbb{C}$ ,  $\operatorname{Re}(z) > 0$ .

**Exercise 1.19.** Prove that

$$\beta E_{\alpha, \beta}(x) = x E_{\alpha, \alpha+\beta}(x) + \frac{1}{\Gamma(\beta)}.$$

**Exercise 1.20.** Prove that

$$E_{\alpha, \beta+1}(x) + \alpha x \frac{d}{dx} [E_{\alpha, \beta+1}(x)] = E_{\alpha, \beta}(x)$$

**Exercise 1.21.** Let us consider the function  $\vartheta_\alpha$  given by

$$\vartheta_\alpha(x) := 1 - E(-x^\alpha), \quad 0 < \alpha \leq 1, \quad x > 0.$$

Prove that

$$\frac{d}{dx} \vartheta_\alpha(x) = x^{\alpha-1} E_{\alpha, \alpha}(-x^\alpha).$$

**Exercise 1.22.** Consider the generalized Pochhammer symbol defined by

$$(\lambda, p)_n = \frac{1}{\Gamma(\lambda)} \int_0^{+\infty} t^{\lambda+k-1} e^{-t-\frac{p}{t}} dt,$$

and consider the generalization

$$E_{\beta, \gamma}^{\lambda, p}(x) = \sum_{n=0}^{+\infty} \frac{(\lambda; p)_n x^n}{\Gamma(\lambda) \Gamma(\beta n + \gamma) n!},$$

such that  $\operatorname{Re}(p) > 0$ ,  $\operatorname{Re}(\lambda + n) > 0$  for  $p = 0$ .

Prove that

$$E_{\beta, \gamma}^{\lambda, p}(z) = \frac{1}{[\Gamma(\lambda)]^2} \int_0^{+\infty} t^{\lambda-1} e^{-t-\frac{p}{t}} E_{\beta, \gamma}(tz) dt.$$

**Exercise 1.23.** Calculate  $E_{1,2}(z)$ ,  $E_{1,3}(z)$ ,  $E_{2,1}(z)$ ,  $z \in \mathbb{C}$ ,  $\operatorname{Re}(z) > 0$ .

**Exercise 1.24.** Prove that

$$\beta E_{\alpha, \beta}(x) = x E_{\alpha, \alpha + \beta}(x) + \frac{1}{\Gamma(\beta)}.$$

**Exercise 1.25.** Prove that

$$E_{\alpha, \beta+1}(x) + \alpha x \frac{d}{dx} [E_{\alpha, \beta+1}(x)] = E_{\alpha, \beta}(x)$$

**Exercise 1.26.** Let us consider the function  $\vartheta_\alpha$  given by

$$\vartheta_\alpha(x) := 1 - E(-x^\alpha), \quad 0 < \alpha \leq 1, \quad x > 0.$$

Prove that

$$\frac{d}{dx} \vartheta_\alpha(x) = x^{\alpha-1} E_{\alpha, \alpha}(-x^\alpha).$$

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