Fractional integrals and derivatives
(Exercises with Solutions)

Master 2, PDEs & Applications

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Chapter 1

Fractional integrals and derivatives

1.1 Riemann-Liouville fractional integral

Definition 1.1. *Let $x : \mathbb{R}^+ \to \mathbb{R}$, be a piecewise continuous function on $(0,\infty)$ and integrable over any finite interval of $\mathbb{R}^+$, the left fractional Riemann-Liouville integral of order $0 < \alpha < 1$ is defined by*

$$J^{\alpha}_{a_+} x(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{x(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \quad t > 0, \tag{1.1}$$

Γ is the Euler gamma function.

Equation (1.1) reads as

$$J^{\alpha}_{a_+} x(t) = (\phi_\alpha \star x)(t), \tag{1.2}$$

with

$$\phi_\alpha := \begin{cases} t^{\alpha-1}/\Gamma(\alpha), & t > a, \\ 0 & t \leq a. \end{cases} \tag{1.3}$$

We remark that

$$J^{\alpha}_{a_+} x(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \tau^{\alpha-1} x(t-\tau) d\tau, \quad t > 0.$$

Definition 1.2. *Let $x : \mathbb{R}^+ \to \mathbb{R}$, be a piecewise continuous function on $(0,\infty)$ and integrable over any finite interval of $\mathbb{R}^+$, the right* *fractional Riemann-Liouville integral of order* $0<\alpha<1$ *is defined as*

$$J_{b_-}^{\alpha}x(t) := \frac{1}{\Gamma(\alpha)}\int_t^b \frac{x(\tau)}{(t-\tau)^{1-\alpha}}d\tau, \quad t>0.$$

For $a=0$ we write J^{α} instead of J_{0+}^{α}.

Definition 1.3. *Let x be a function with two variables s and t, then the double fractional integral of the function x of order* α *and* β *is defined by*

$$J_a^{\alpha}\left[J_c^{\beta}x(s,t)\right] = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^s\int_c^t (s-\tau)^{(\alpha-1)}(t-\varsigma)^{(\beta-1)}x(\tau,\varsigma)d\tau d\varsigma,$$

where $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$ *and* Γ *is the Euler gamma function.*

Examples 1.1. *1. Letting*

$$x(s)=e^{as} \quad \text{for some constant} \quad a.$$

Hence

$$J^{\alpha}e^{at} = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{(\alpha-1)}e^{a\xi}d\xi.$$

Let $s=t-\xi$, *we get*

$$J^{\alpha}e^{at} = \frac{e^{at}}{\Gamma(\alpha)}\int_0^t s^{(\alpha-1)}e^{-as}ds.$$

Let us consider the following function, with $\mathrm{Re}(\alpha)>0$,

$$\gamma^*(\alpha,t) = \frac{1}{\Gamma(\alpha)t^{\alpha}}\int_0^t \xi^{(\alpha-1)}e^{-\xi}d\xi.$$

Thus,

$$J^{\alpha}e^{at} = t^{\alpha}e^{at}\gamma^*(\alpha,at) = E_t(\alpha,a).$$

2. *Let us consider the Fresnel integrals*

$$C_t(\alpha,a) = \frac{1}{\Gamma(\alpha)}\int_0^t y^{(\alpha-1)}\cos[a(t-y)]dy, \qquad Re(\alpha)>0,$$

$$S_t(\alpha,a) = \frac{1}{\Gamma(\alpha)}\int_0^t y^{(\alpha-1)}\sin[a(t-y)]dy, \qquad Re(\alpha)>0,$$

$$C(s) = \int_0^s \cos(t^2)dt, \qquad S(x) = \int_0^s \sin(t^2)dt,$$

so that

$$J^{\alpha}\cos(at) = \frac{1}{\Gamma(\alpha)}\int_0^t \xi^{(\alpha-1)}\cos[a(t-\xi)]d\xi, \qquad Re(\alpha) > 0,$$
$$J^{\alpha}\sin(at) = \frac{1}{\Gamma(\alpha)}\int_0^t \xi^{(\alpha-1)}\sin[a(t-\xi)]d\xi, \qquad Re(\alpha) > 0,$$

and therefore

$$J^{\alpha}e^{at} = E_t(\alpha,a),$$
$$J^{\alpha}\cos(at) = C_t(\alpha,a),$$
$$J^{\alpha}\sin(at) = S_t(\alpha,a).$$

In particular $\alpha = \frac{1}{2}$,

$$\begin{aligned}
J^{\frac{1}{2}}e^{at} &= E_t(\frac{1}{2},a) \\
&= a^{-\frac{1}{2}}e^{at}Erf(at)^{\frac{1}{2}}. \\
J^{\frac{1}{2}}\cos(at) &= C_t(\frac{1}{2},a) \\
&= \sqrt{\frac{2}{a}}\,[(\cos at)C(s) + (\sin at)S(s)]\,. \\
J^{\frac{1}{2}}\sin(at) &= S_t(\frac{1}{2},a) \\
&= \sqrt{\frac{2}{a}}\,[(\sin at)C(s) + (\cos at)S(s)]\,, \quad \textit{with} \qquad s = \sqrt{\frac{2at}{\pi}}
\end{aligned}$$

3. *Following the formula*

$$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta,$$

we obtain

$$J_x^{\alpha}\cos^2 at = \frac{t^{\alpha}}{2\Gamma(\alpha+1)} + \frac{1}{2}C_t(\alpha,2a),$$
$$J_x^{\alpha}\sin^2 at = \frac{t^{\alpha}}{2\Gamma(\alpha+1)} - \frac{1}{2}S_t(\alpha,2a).$$

4. *Let* $x(t) = \ln t$,

$$J^{\alpha} \ln t = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{(\alpha-1)} \ln s \, ds, \qquad \alpha > 0,$$

Letting $\xi = ts$, *we get*

$$J^{\alpha} \ln t = \frac{t^{\alpha}}{\Gamma(\alpha+1)} \ln t + \frac{t^{\alpha}}{\Gamma(\alpha)} \int_0^1 (1-s)^{(\alpha-1)} \ln s \, ds.$$

Exercise 1.1. 1. *Calculate the fractional integral* J^{α} *of the function* $x(s) = s^{\rho}$, *for* $\rho > -1$ *and* $\mathrm{Re}\,\rho > 0$.

2. *Deduce* $J^{\frac{1}{2}}(1+s+s^2)$.

Exercise 1.2. 1. *Calculate the fractional integral* J_a^{α} *of the function* $x(s) = (s-a)^{\rho-1}$, *for* $\mathrm{Re}\,\rho > 0$.

2. *Calculate the fractional integral* J_b^{α} *of the function* $y(s) = (b-s)^{\rho-1}$, *for* $\mathrm{Re}\,\rho > 0$.

1.2 Dirichlet's formula

Let F be a continous function and $\lambda,\ \mu,\ \nu \in \mathbb{R}^+$. Hence

$$\int_a^t (t-s)^{(\mu-1)} ds \int_a^s (y-a)^{(\lambda-1)} (s-y)^{(\nu-1)} F(s,y) dy$$
$$= \int_a^t (y-a)^{(\lambda-1)} dy \int_y^t (t-s)^{(\mu-1)} (s-y)^{(\nu-1)} F(s,y) ds.$$

Exercise 1.3. 1. *Prove that*

$$\int_0^t (t-s)^{(\mu-1)} g(s) ds \int_0^s (s-y)^{(\nu-1)} f(y) dy$$

$$= \int_0^t f(y) dy \int_y^t (t-s)^{(\mu-1)} (s-y)^{(\nu-1)} g(s) ds.$$

2. *Prove that*

$$\int_0^t (t-s)^{(\mu-1)} ds \int_0^s (s-y)^{(\nu-1)} f(y) dy$$
$$= B(\mu,\nu) \int_0^t (t-y)^{(\mu+\nu-1)} f(y) dy,$$

Exercise 1.4. *Prove that*

$$\int_a^s ds_1 \int_a^{s_1} ds_2 \int_a^{s_2} f(t) dt = J^3 f(s).$$

Exercise 1.5. *Let us consider the following problem*

$$\begin{cases} y^{(n)}(s) = f(s), & a \le s \le b, \\ y(a) = y'(a) = \ldots = y^{(n-1)}(a) = 0. \end{cases}$$

Prove that the solution of this problem is given by

$$F(s) = \frac{1}{(n-1)!} \int_a^s (s-t)^{n-1} f(t) dt, \quad a \le s \le b.$$

Exercise 1.6. *Prove that*

$$\int_0^t (t-\tau)^{\nu\alpha-\nu} e^{-(\nu t-\nu\tau)} d\tau \le \frac{1}{\nu^{\nu\alpha-\nu+1}} \Gamma(\nu a-\nu+1).$$

Exercise 1.7. *Prove that*

$$J_0^{\alpha-j} e^{\delta t} \le \delta^{-\alpha+j} e^{\delta t} \quad \delta > 0.$$

Exercise 1.8. *Let f be a continuous function on $J = [0,\infty[$ and $\mu,\nu >$ 0. Prove that $\forall t > 0$*

$$J^{\mu}\left[J^{\nu} f(t)\right] = J^{\mu+\nu} f(t) = J^{\nu}\left[J^{\mu} f(t)\right].$$

1.3 Leibniz integral rule

We present the following Leibniz integral rule

$$\frac{d}{dt}\left[\int_a^{b(t)} f(t,s) ds\right] = f(t,b(t))b'(t) - f(t,a(t))a'(t) + \int_0^{b(t)} \frac{\partial}{\partial t} f(t,s) ds,$$

We use this rule to prove the derivative theorem of the Riemann-Liouville integral.

Theorem 1.1. *Let f be a continuous function on $J=[0,\infty[$ and Df is continuous, with $\nu>0$, so that*

$$\forall t>0,\quad D[J^{\nu}f(t)]=J^{\nu}[Df(t)]+\frac{f(0)}{\Gamma(\nu)}t^{(\nu-1)}.$$

Proof. We have

$$J^{\nu}f(t)=\frac{1}{\Gamma(\nu)}\int_0^t(t-z)^{(\nu-1)}f(z)dz,$$

Letting $z=t-s^{\lambda}$, $\lambda=\dfrac{1}{\nu}$, we get

$$\begin{aligned} J^{\nu}f(t) &= \frac{1}{\Gamma(\nu)}\int_{t^{\nu}}^{0}(s^{\lambda})^{(\nu-1)}f(t-s^{\lambda})(-\lambda s^{(\lambda-1)})ds \\ &= \frac{1}{\Gamma(\nu+1)}\int_0^{t^{\nu}}f(t-s^{\lambda})ds, \end{aligned}$$

by the Leibniz integral rule, we get

$$D[J^{\nu}f(t)]=\frac{1}{\Gamma(\nu+1)}\left[f(0)\nu t^{(\nu-1)}+\int_0^{t^{\nu}}\frac{\partial}{\partial t}f(t-s^{\lambda})ds\right],$$

again letting $t-s^{\lambda}=z$, we obtain

$$D[J^{\nu}f(t)]=\frac{f(0)}{\nu\Gamma(\nu)}\nu t^{(\nu-1)}+\frac{1}{\nu\Gamma(\nu)}\int_t^0\frac{\partial}{\partial t}f(z)(\frac{-1}{\lambda}s^{(1-\lambda)})dz.$$

Finally, for $\lambda=\dfrac{1}{\nu}$ and $s=(t-z)^{\frac{1}{\nu}}$ we get

$$\begin{aligned} D\left[J^{\nu}f(t)\right] &= \frac{f(0)}{\Gamma(\nu)}t^{(\nu-1)}+\frac{1}{\Gamma(\nu)}\int_0^t(t-z)^{(\nu-1)}\frac{\partial}{\partial t}f(z)dz \\ &= J^{\nu}\left[Df(t)\right]+\frac{f(0)}{\Gamma(\nu)}t^{(\nu-1)}. \end{aligned}$$

□

1.4 Fractional Legendre polynomials

The fractional Legendre polynomials (FLP) $FL_n^{\vartheta}(.)$, are particular solutions of the following singular Strum-Liouville problem:

$$\left(\left(x-x^{1+\vartheta}\right)FL_n'^{\vartheta}(x)\right)' + \vartheta^2 n(n+1)x^{\vartheta-1}FL_n^{\vartheta}(x) = 0, \quad \vartheta > 0.$$

The Fractional Legendre polynomials (FLP) $\{FL_n^{\vartheta}(x)\}_{n=0}^{\infty}$ are orthogonal with respect to the weight function $w^{\vartheta}(x) = x^{\vartheta-1}$ over the interval $[0,1]$.

Let δ_{mn} the Kronecker's symbol. Indeed,

$$\int_0^1 FL_m^{\vartheta}(x)FL_n^{\vartheta}(x)w^{\vartheta}(x)dx = \frac{1}{(2n+1)\vartheta}\delta_{mn}.$$

For all $n > 1$, we can establish the relationship

$$(n+1)FL_{n+1}^{\vartheta}(x) = (2n+1)(2x^{\vartheta}-1)FL_n^{\vartheta}(x) - nFL_{n-1}^{\vartheta}(x), \quad n \in \mathbb{N},$$

with

$$FL_0^{\vartheta}(x) = 1,$$

and

$$FL_1^{\vartheta}(x) = 2x^{\vartheta} - 1.$$

The analytical form of $FL_n^{\vartheta}(x)$ of degree $n\vartheta$ is given by:

$$FL_n^{\vartheta}(x) = \sum_{i=0}^{n} b_{ni}x^{i\vartheta},, \quad n = 0, 1, 2, \ldots,$$

with

$$b_{ni} = \frac{(-1)^{n+i}(n+i)!}{(n-i)!(i!)^2}. \tag{1.4}$$

We can develop any function $u(.)$ defined on the interval $[0,1]$ as follows

$$u(x) = \sum_{n=0}^{\infty} c_n FL_n^{\vartheta}(x),$$

where the coefficients c_n are as follows:

$$c_n = (2n+1)\vartheta \int_0^1 u(x) FL_n^{\vartheta}(x) w^{\vartheta}(x) dx.$$

1.5 Fractional derivation

Definition 1.4. *The fractional Riemann-Liouville derivation of order* $0 < \; alpha < 1$ *for a continuous function* f *is defined by*

$$D_{0_+}^{\alpha} f(t) := \frac{d}{dt} J_{0_+}^{1-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha}} d\tau.$$

Note that Riemann-Liouville fractional derivative of a constant is non-zero.

For $0 < \alpha < 1$, we have

$$J_{0_+}^{\alpha} D_{0_+}^{\alpha} \Big(f(t) - f(0) \Big) = f(t) - f(0).$$

Definition 1.5. *Let* $\nu \in \mathbb{R}_+$, $n = [\nu] + 1$. *The Caputo fractional derivative of order* ν *for a continous function* f *is defined by*

$$\begin{aligned} D_{*a}^{\nu} f(t) &= J_{t,a}^{n-\nu} D^n f(t) \\ &= \frac{1}{\Gamma(n-\nu)} \int_a^t (t-\tau)^{(n-\nu-1)} \left(\frac{\partial}{\partial \tau} \right)^n f(\tau)\, d\tau, \quad a \leq t \leq b. \end{aligned}$$

1.6 Solution of exercises

Solution 1.1. *1. We have*

$$\begin{aligned}
J^{\alpha}s^{\rho} &= \frac{1}{\Gamma(\alpha)}\int_0^s (s-t)^{(\alpha-1)}t^{\rho}dt \\
&= \frac{1}{\Gamma(\alpha)}\int_0^s (1-\frac{t}{s})^{(\alpha-1)}t^{\rho}s^{(\alpha-1)}dt \\
&= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-u)^{(\alpha-1)}x^{(\alpha-1)}(su)^{\rho}du, \ (u=\frac{t}{s}) \\
&= \frac{1}{\Gamma(\alpha)}s^{(\rho+\alpha)}\int_0^1 u^{\rho}(1-u)^{(\alpha-1)}du \\
&= \frac{1}{\Gamma(\alpha)}s^{(\rho+\alpha)}B(\rho+1,\alpha) \\
&= \frac{\Gamma(\rho+1)}{\Gamma(\rho+\alpha+1)}s^{(\rho+\alpha)}.
\end{aligned}$$

2. For $\alpha = \frac{1}{2}$, *we get*

$$J^{\frac{1}{2}}s^0 = \frac{\Gamma(1)}{\Gamma(\frac{3}{2})}s^{\frac{1}{2}} = 2\sqrt{\frac{s}{\pi}},$$

$$J^{\frac{1}{2}}s^1 = \frac{\Gamma(2)}{\Gamma(\frac{5}{2})}s^{\frac{3}{2}} = \frac{4}{3}\sqrt{\frac{s^3}{\pi}},$$

$$J^{\frac{1}{2}}s^2 = \frac{\Gamma(3)}{\Gamma(\frac{7}{2})}s^{\frac{5}{2}} = \frac{16}{15}\sqrt{\frac{s^5}{\pi}}.$$

$$J^{\frac{1}{2}}(1+s+s^2) = 2\sqrt{\frac{s}{\pi}} + \frac{4}{3}\sqrt{\frac{s^3}{\pi}} + \frac{16}{15}\sqrt{\frac{s^5}{\pi}}.$$

Solution 1.2. *1. We have*

$$\begin{aligned} J_a^{\alpha}x(s) &= \frac{1}{\Gamma(\alpha)}\int_a^s (s-t)^{(\alpha-1)}(t-a)^{\rho-1}dt \\ &= \frac{\Gamma(\rho)}{\Gamma(\alpha+\rho)}(s-a)^{\alpha+\rho-1}, \quad posons \quad u=\frac{t-a}{s-a}. \end{aligned}$$

2. We have

$$\begin{aligned} J_b^{\alpha}y(s) &= \frac{1}{\Gamma(\alpha)}\int_s^b (s-t)^{(\alpha-1)}(b-t)^{\rho-1}dt \\ &= \frac{\Gamma(\rho)}{\Gamma(\alpha+\rho)}(b-s)^{\alpha+\rho-1}. \end{aligned}$$

Solution 1.3. *1. For $a=0$, $\lambda=1$, et $F(s,y)=g(s)f(y)$ we obtain*

$$\int_0^t (t-s)^{(\mu-1)}g(s)ds\int_0^s (s-y)^{(\nu-1)}f(y)dy$$

$$=\int_0^t f(y)dy\int_y^t (t-s)^{(\mu-1)}(s-y)^{(\nu-1)}g(s)ds.$$

2. We prove that

$$\int_0^t (t-s)^{(\mu-1)}ds\int_0^s (s-y)^{(\nu-1)}f(y)dy = B(\mu,\nu)\int_0^t (t-y)^{(\mu+\nu-1)}f(y)dy.$$

Under the change of variables, $z=\frac{t-x}{t-y}$,

$$\int_0^t (t-s)^{(\mu-1)}ds\int_0^s (s-y)^{(\nu-1)}f(y)dy$$

$$=\int_0^t f(y)dy\int_0^1 z^{\mu-1}(1-z)^{\mu-1}(t-y)^{\mu+\nu-1}f(y)dy$$

$$=B(\mu,\nu)\int_0^t (t-y)^{\mu+\nu-1}f(y)dy.$$

Solution 1.4. *We have*

$$\begin{aligned} J^3 f(s) &= \int_a^s ds_1 \int_a^{s_1} ds_2 \int_a^{s_2} f(t)dt \\ &= \int_a^s ds_1 \left[\int_a^{s_1} ds_2 \int_a^{s_2} f(t)dt \right] \\ &= \int_a^s ds_1 \left[\int_a^{s_1} (s_1 - t) f(t)dt \right] \\ &= \int_a^s f(t)dt \int_t^s (s_1 - t) ds_1 \\ &= \int_a^s \frac{(s-t)^2}{2!} f(t)dt. \end{aligned}$$

Solution 1.5. *By the Dirichlet's formula n times, we obtain the following Cauchy formula of the integral of the function f*

$$F(s) = \frac{1}{(n-1)!} \int_a^s (s-t)^{n-1} f(t)dt, \quad a \leq s \leq b.$$

In fact, by indication.
For $n = 2$, following Dirichlet's formula:

$$\int_a^b ds \int_a^s f(s,y)dy = \int_a^b dy \int_y^b f(s,y)ds,$$

we obtain

$$\begin{aligned} \int_a^s ds_1 \int_a^{s_1} f(t)dt &= \int_a^s f(t)dt \int_t^s ds_1 \\ &= \int_a^s (s-t) f(t)dt. \end{aligned}$$

We assume that

$$J^n f(s) = \int_a^s \frac{(s-t)^{(n-1)}}{(n-1)!} f(t)dt.$$

We have

$$\begin{aligned} J^{n+1}f(s) = & \int_a^y ds \int_a^s \frac{(s-t)^{(n-1)}}{(n-1)!} f(t)dt \\ = & \int_a^y f(t)dt \int_t^y \frac{(s-t)^{(n-1)}}{(n-1)!} ds \\ = & \int_a^y f(t)dt \left[\frac{(s-t)^{(n)}}{n(n-1)!} \right]_{s=t}^{s=y} \\ = & \int_a^y \frac{(y-t)^n}{n!} f(t)dt. \end{aligned}$$

Solution 1.6. *We have*

$$\begin{aligned} \int_0^t (t-\tau)^{v\alpha-v} e^{-(vt-v\tau)} d\tau &= \int_0^t s^{v\alpha-v} e^{-sv} ds \\ &= \frac{1}{v} \int_0^{tv} u^{v\alpha-v} e^{-u} du \\ &\leq \frac{1}{v^{v\alpha-v+1}} \int_0^{+\infty} u^{v\alpha-v} e^{-u} du \\ &= \frac{1}{v^{v\alpha-v+1}} \Gamma(va-v+1), \end{aligned}$$

with $s = t - \tau$, $u = sv$.

Solution 1.7.

$$\begin{aligned} J_0^{\alpha-j} e^{\delta t} &= \frac{1}{\Gamma(\alpha-j)} \int_0^t (t-\tau)^{\alpha-j-1} e^{\delta\tau} d\tau \\ &= \frac{1}{\Gamma(\alpha-j)} e^{\delta t} \int_0^t (t-\tau)^{\alpha-j-1} e^{\delta(\tau-t)} d\tau \\ &= \frac{1}{\Gamma(\alpha-j)} e^{\delta t} \int_0^t s^{\alpha-j-1} e^{-s\delta} ds \\ &= \frac{1}{\Gamma(\alpha-j)} e^{\delta t} \frac{1}{\delta^\alpha} \int_0^{\delta t} u^{\alpha-j-1} e^{-u} du \\ &\leq \frac{1}{\Gamma(\alpha-j)} \delta^{-\alpha+j} e^{\delta t} \int_0^{+\infty} u^{\alpha-j-1} e^{-u} du \\ &\leq \delta^{-\alpha+j} e^{\delta t}, \end{aligned}$$

with $s = t - \tau$, $u = \delta s$.

Solution 1.8. *We have*

$$\begin{aligned}
J^{\mu}\left[J^{\nu} f(t)\right] &= \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{(\mu-1)} \left[J^{\nu} f(t)\right] ds \\
&= \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{(\mu-1)} \left[\frac{1}{\Gamma(\nu)} \int_0^s (s-y)^{(\nu-1)} f(y) dy\right] ds \\
&= \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_0^t (t-s)^{(\mu-1)} ds \int_0^s (s-y)^{(\nu-1)} f(y) dy \\
&= \frac{1}{\Gamma(\mu+\nu)} \int_0^t (t-s)^{(\mu+\nu-1)} f(y) dy \qquad (\textit{Dirichlet's formula}) \\
&= J^{\mu+\nu} f(t).
\end{aligned}$$