Laplace Transform II
Exercises with Solutions

by

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0.1 Convolution Integral

Definition 0.1. *The convolution integral of two piecewise continuous functions* ξ *and* ζ *on* $\mathbb{R}$ *is defined by*

$$(\xi * \zeta)(s) = \int_{-\infty}^{+\infty} \xi(s-\tau)\,\zeta(\tau)\,d\tau$$

or equivalently

$$(\xi * \zeta)(s) = \int_{0}^{s} \xi(s-\tau)\,\zeta(\tau)\,d\tau.$$

if ξ *and* ζ *are two piecewise continuous function on* $[0,\infty)$ *that is* $\xi(s) = 0, s < 0$ *and* $\zeta(t) = 0, t < 0$.

Remark 0.1.

1. $\xi * \zeta$ *is also called the generalized product of* ξ *and* ζ.
2. $$\int_0^s \xi(s-\tau)\,\zeta(\tau)\,d\tau = \int_0^s \xi(s-\tau)\,\zeta(\tau)\,d\tau,$$
   that is
   $$\xi * \zeta = \zeta * \xi.$$

Theorem 0.1. *If* ξ, ζ *have well-defined Laplace Transforms* Ξ, Z, *then*

$$\mathscr{L}\{\xi * \zeta\}(s) = \Xi(s)Z(s) \qquad \mathscr{L}^{-1}\{\Xi(.)Z(.)\}(.) = (\xi * \zeta)(t).$$

Proof. We have

$$\begin{aligned}\mathscr{L}(\xi * \zeta)(s) &= \int_0^\infty e^{-st}(\xi * \zeta)(t)\,dt \\ &= \int_0^\infty e^{-st} \int_0^t \xi(\tau)\zeta(t-\tau)\,d\tau\,dt.\end{aligned}$$

Hence

$$\mathscr{L}(\xi * \zeta)(s) = \int_0^\infty \xi(\tau) \int_\tau^\infty e^{-st}\zeta(t-\tau)\,dt\,d\tau.$$

By interchanging two integrals, we obtain

$$\begin{aligned}\int_\tau^\infty e^{-st}\zeta(t-\tau)\,dt &= \int_0^\infty e^{-s(x+\tau)}\zeta(x)\,dx \\ &= e^{-s\tau}\int_0^\infty e^{-sx}\zeta(x)\,dx \\ &= e^{-s\tau}Z(s).\end{aligned}$$

Hence

$$\begin{aligned}\mathscr{L}(\xi * \zeta)(s) &= \int_0^\infty e^{-s\tau}\xi(\tau)Z(s)\,d\tau \\ &= Z(s)\int_0^\infty e^{-st}\xi(\tau)\,d\tau \\ &= \Xi(s)Z(s).\end{aligned}$$

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Example 0.1. *We want to evaluate the Laplace transform of the convolution integral*

$$\mathscr{L}\left[\int_0^t e^{-2\tau}\sinh(t-\tau)\,d\tau\right].$$

Letting

$$u(t) := e^{-2t} \quad v(t) := \sinh(t).$$

we get

$$\int_0^t e^{-2\tau}\sinh(t-\tau)\,d\tau = (u*v)(t),$$

so that

$$\mathscr{L}\left[\int_0^t e^{-2\tau}\sin(t-\tau)\,d\tau\right] = \mathscr{L}\left[u*v\right](s).$$

$$\begin{aligned}\mathscr{L}\left[u\right](s) &= \mathscr{L}\left[e^{-2t}\right](s) = \frac{1}{s+2} \\ \mathscr{L}\left[v\right](s) &= \mathscr{L}\left[\cos t\right] = \frac{1}{s^2-1}.\end{aligned}$$

$$\mathscr{L}\left[\int_0^t e^{-2\tau}\sinh(t-\tau)\,d\tau\right] = \frac{1}{(s+2)(s^2-1)}.$$

Example 0.2. *Let* $\vartheta(s) := \frac{1}{3}\sin(3t)$. *We have*

$$
\begin{aligned}
(\vartheta * \vartheta)(s) &= \int_0^s \vartheta(s-\tau)\,\vartheta(\tau)\,d\tau \\
&= \frac{1}{9}\int_0^s \sin(3s-3\tau)\sin(3\tau)\,d\tau \\
&= \frac{1}{9}\int_0^s [\sin(3s)\cos(3\tau)-\cos(3s)\sin(3\tau)]\sin(3\tau)\,d\tau \\
&= \frac{1}{9}\int_0^s \sin(3s)\cos(3\tau)\sin(3\tau)-\cos(3s)\sin^2(3\tau)\,d\tau \\
&= \frac{1}{18}\sin(3s)\int_0^s \sin(6\tau)\,d\tau-\frac{1}{18}\cos(3s)\int_0^s 1-\cos(6\tau)\,d\tau \\
&= \frac{1}{18}\sin(3s)\left(-\frac{1}{6}\cos(6\tau)\right)\Big|_0^s-\frac{1}{18}\cos(3s)\left(\tau-\frac{1}{6}\sin(6\tau)\right)\Big|_0^s \\
&= \frac{1}{108}(\sin(3s)-\sin(3s)\cos(6s))-\frac{1}{18}\cos(3s)\left(s-\frac{1}{6}\sin(6s)\right) \\
&= \frac{1}{104}\sin(3s)-\frac{1}{18}s\cos(3s)-\frac{1}{104}\sin(3s)\cos(6s)+\frac{1}{104}\cos(3s)\sin(6s) \\
&= \frac{1}{104}\sin(3s)-\frac{1}{18}s\cos(3s)-\frac{1}{216}[\sin(9s)+\sin(-3s)]+\frac{1}{216}[\sin(9s)-\sin(-3s)] \\
&= \frac{1}{216}\sin(3s)-\frac{1}{18}s\cos(3s)+\frac{1}{216}\sin(3s)+\frac{1}{216}\sin(3s) \\
&= \frac{1}{54}\sin(3s)-\frac{1}{18}s\cos(3s) \\
&= \frac{1}{54}[\sin(3s)-3s\cos(3s)]
\end{aligned}
$$

Example 0.3. *Let* $\xi(s) = e^{-2s}, \quad \zeta(s) = 2, \quad 0 < s < 1$. *We have*

$$\begin{aligned}
(\xi * \zeta)(s) &= \int\limits_0^s \xi(\tau) \cdot \zeta(s-\tau) d\tau \\
&= \int\limits_0^s 2 \cdot e^{-2(s-\tau)} d\tau \\
&= 2e^{-2s} \int\limits_0^s e^{2\tau} d\tau \\
&= 2e^{-2s} \left(\frac{1}{2} \, e^{2s} \big|_0^s \right) \\
&= 2e^{-2s} \left(\frac{1}{2} \left(e^{2s} - 1 \right) \right) \\
&= 1 - e^{-2s}.
\end{aligned}$$

0.2 Solving Volterra integral equations via Laplace transform

Integral equations are one of the most useful mathematical tools in both pure and applied analysis. Integral equations are used as mathematical models for many and varied physical situations.

Integral equations of the form

$$\int_a^s k(s,\tau)\varphi(\tau)d\tau = g(s), \quad a \leq s \leq b,$$

and

$$\varphi(s) - \int_a^s k(s,\tau)\varphi(\tau)d\tau = g(s), \quad a \leq s \leq b,$$

with variable limit of integration are called linear Volterra integral equations of the first and second kind, respectively. Here k and g are given functions and φ is unknown.

This section discusses the linear Volterra integral equations of the second kind with regular kernel.

Theorem 0.2. *For each right-hand side $g \in C^0[a,b]$, the Volterra integral equations of the second kind*

$$\varphi(s) - \int_a^s k(s,\tau)\varphi(\tau)d\tau = g(s), \quad a \le s \le b,$$

with continuous kernel k has a unique solution $\varphi \in C^0[a,b]$.

Let us consider the following Volterra integral equation of the form

$$\varphi(s) - \int_0^s k(s-\tau)\varphi(\tau)\,d\tau = g(s),$$

The above integral equation reads as

$$\varphi(s) - (k * \varphi)(s) = g(s),$$

Taking Laplace transforms in this equation, we obtain

$$\phi(s) - (K \cdot \phi)(s) = G(s),$$

where ϕ, K and G are the Laplace transforms of φ, k and g, respectively.

Hence

$$\phi = \frac{G}{1-K}.$$

Consequently,

$$\varphi = \mathscr{L}^{-1}\left(\frac{G}{1-K}\right).$$

Example 0.4. *Let us consider the following Volterra integral equation of the second kind*

$$\varphi(s) - \int_0^s e^{2s-2\tau}\,\varphi(\tau)\,d\tau = 3s^2 - e^{-2s}.$$

We want to find the Laplace transform of the solution of this integral equation.

For this purpose, letting

$$\xi(s) := e^{2s}.$$

So that the integral equation reads as

$$\varphi(s) - (\xi * \varphi)(s) = 3s^2 - e^{-2s}.$$

By taking the Laplace transforms we get

$$\mathscr{L}[\varphi] - \mathscr{L}(\xi * \varphi)(s) = \mathscr{L}\left[3t^2\right] - \mathscr{L}\left[e^{-2t}\right]$$

$$\mathscr{L}\left[3t^2\right](s) = \frac{6}{s^3},$$

$$\mathscr{L}\left[e^{-2t}\right](s) = \frac{1}{s+2},$$

$$\mathscr{L}\left[\int_0^t \varphi(\tau)\, e^{2t-2\tau}\, d\tau\right] = \mathscr{L}(\varphi)(s)\mathscr{L}\left[e^{2t}\right](s) = \mathscr{L}(\varphi)(s)\,\frac{1}{s-2}.$$

$$\mathscr{L}(\varphi)(s) = \frac{6}{s^3} - \frac{1}{s+2} + \mathscr{L}(\varphi)(s)\,\frac{1}{s-2}.$$

$$\mathscr{L}(\varphi)(s) = \frac{s-2}{s-3}\left(\frac{6}{s^3} - \frac{1}{s+2}\right).$$

Example 0.5. *Let us consider the following Volterra integral equation of the second kind*

$$\varphi(s) = 1 - 3\int_0^s e^{3(s-\tau)}\varphi(\tau)\, d\tau.$$

Letting

$$\xi(s) := e^{3s}.$$

So that, the integral equation reads as

$$\varphi(s) + 3(\xi * \varphi)(s) = 1.$$

Taking the Laplace transforms we obtain

$$\mathscr{L}(\varphi)(s) + \frac{3}{s-3}\mathscr{L}(\varphi)(s) = \frac{1}{s},$$

that it is

$$\mathscr{L}(\varphi)(s) = \frac{1}{s} - \frac{3}{s^2}.$$

Thus,

$$\varphi(s) = 1 - 3s.$$

0.3 Solving some initial value problems

Let us consider the following initial value problem :

$$\varphi'' - 2\varphi' + \varphi = g(t), \quad \varphi(0) = \alpha, \quad \varphi'(0) = \beta.$$

Taking the Laplace transforms we obtain

$$(s^2-2s+1)\mathscr{L}(\varphi)(s) = G(s) + (\beta+\alpha s) - 2\alpha.$$

Hence

$$\begin{aligned}\mathscr{L}(\varphi)(s) &= \frac{1}{(s-1)^2}G(s) + \frac{\beta+\alpha s-2\alpha}{(s-1)^2}\\ &= \frac{1}{(s-1)^2}G(s) + \frac{\alpha}{s-1} + \frac{\beta-\alpha}{(s-1)^2}.\end{aligned}$$

But

$$\mathscr{L}^{-1}\left(\frac{\alpha}{s-1} + \frac{\beta-\alpha}{(s-1)^2}\right) = e^t\left(\alpha + (\beta-\alpha)t\right),$$

and

$$\mathscr{L}^{-1}\left(\frac{1}{(s-1)^2}G(s)\right) = \int_0^t \tau e^\tau g(t-\tau)\,d\tau.$$

Thus

$$\varphi(t) = [\alpha + (\beta-\alpha)t]e^t + \int_0^t \tau e^\tau g(t-\tau)\,d\tau.$$

0.4 Exercises with solutions

Exercise 0.1. *Find the Laplace transform of the following function*

$$\psi_1(s) := \frac{s^2}{4} + 1.$$

Solution 0.1. *Since the Laplace transform is a linear operator, we get*

$$\begin{aligned}\mathscr{L}(\psi_1(.))(s) &= \frac{1}{4}\mathscr{L}(t^2)(s) + \mathscr{L}(1)(s)\\ &= \frac{1}{4} \times \frac{2!}{s^3} + \frac{1}{s}\\ &= \frac{1}{2s^3} + \frac{1}{s}.\end{aligned}$$

Exercise 0.2. *Find the Laplace transform of the following function*

$$\psi_2(s) := e^{-2s} + 5\cos\left(\frac{5}{2}s\right)e^{3s}.$$

Solution 0.2. *We recall that*

$$\mathscr{L}(e^{\alpha t}\varphi(t))(s) = \mathscr{L}(\varphi(.))(s-\alpha).$$

Since

$$\mathscr{L}\left(\cos\left(\frac{5}{2}t\right)\right)(s) = \frac{s}{s^2 + \frac{25}{4}},$$

and since

$$\mathscr{L}(e^{-2t})(s) = \frac{1}{s+2},$$

we get

$$\mathscr{L}\left(e^{3t}\cos\left(\frac{5}{2}t\right)\right)(s) = \frac{s-3}{(s-3)^2 + \frac{25}{4}}.$$

Thus,

$$\mathscr{L}(\psi_2(.))(s) = \frac{1}{s+2} + \frac{5(s-3)}{(s-3)^2 + \frac{25}{4}}.$$

Exercise 0.3. *Find the Laplace transform of the following function*

$$\psi_3(s) := -3se^{-\frac{1}{2}s}.$$

Solution 0.3. *Since*

$$\mathscr{L}(t)(s) = \frac{1}{s^2},$$

we obtain

$$\mathscr{L}(-3te^{-\frac{1}{2}t})(s) = \frac{-3}{(s+\frac{1}{2})^2}.$$

Exercise 0.4. **1**. *Prove that*

$$\cos^3(s) = \frac{1}{4}\cos(3s) + \frac{3}{4}\cos(s).$$

2. *Find the Laplace transform of the function* $s \longmapsto \cos^3(s)$.
3. *Deduce the Laplace transform of the following function :*

$$\psi_4(s) := \cos^3(s)e^{-2s}.$$

Solution 0.4. **1**. *We have*

$$\begin{aligned}\cos^3(s) &= \left(\frac{e^{is}+e^{-is}}{2}\right)^3 \\ &= \frac{1}{8}\left(e^{3is}+3e^{is}+3e^{-is}+e^{-i3s}\right) \\ &= \frac{1}{4}\cos(3s) + \frac{3}{4}\cos(s).\end{aligned}$$

2. *We recall that*

$$\mathscr{L}(\cos(at))(s) = \frac{s}{s^2+a^2},$$

hence

$$\mathscr{L}(\cos^3(t))(s) = \frac{s}{4\,[s^2+9]} + \frac{3s}{4\,[s^2+1]}.$$

3. *It follows from above that*

$$\mathscr{L}(\cos^3(t)e^{-2t})(s) = \frac{s+2}{4\,[(s+2)^2+9]} + \frac{3\,(s+2)}{4\,[(s+2)^2+1]}.$$

Exercise 0.5. *Letting*

$$\xi(s) := \frac{5s^2-15s-11}{(s+1)(s-2)^3}.$$

Find the inverse Laplace transform of the function ξ.

Solution 0.5. *We assume that*

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3}=\frac{\alpha}{s+1}+\frac{\beta}{(s-2)^3}+\frac{\gamma}{(s-2)^2}+\frac{\delta}{s-2},$$

we get

$$\alpha=-\frac{1}{3},\ \beta=-7,\ \gamma=4\ \text{ and }\quad \delta=\frac{1}{3}.$$

Thus

$$\mathscr{L}^{-1}(\xi(.))(t)=-\frac{1}{3}e^{-t}-\frac{7}{2}t^2e^{2t}+4te^{2t}+\frac{1}{3}e^{2t}.$$

Exercise 0.6. *Let us consider the following two functions :*

$$\varphi(t)=1-\cos t,\ \phi(t)=e^{-t}\varphi(t).$$

1. *Find the Laplace transform of the function φ.*
2. *Find the Laplace transform of the function ϕ.*
2. *Deduce the Laplace transform of the function $e^{-2t}\phi''(t)$.*

$$\mathscr{L}\left(e^{-2t}\phi''\right)(s)=\frac{(s+2)^2}{(s+3)((s+3)^2+1)}.$$

Solution 0.6. 1. *We have*

$$\begin{aligned}\mathscr{L}(\varphi)(s) &= \mathscr{L}(1)(s)-\mathscr{L}(\cos t)(s)\\ &= \frac{1}{s}-\frac{s}{s^2+1}=\frac{1}{s(s^2+1)}.\end{aligned}$$

2. *It follows that*

$$\mathscr{L}(\phi)(s)=\frac{1}{(s+1)((s+1)^2+1)}.$$

3. *Since*

$$\begin{aligned}\mathscr{L}(\phi'')(s) &= s\mathscr{L}(\phi')(s)-\lim_{0}\phi'\\ &= s\big(s\mathscr{L}(\phi)(s)-\lim_{s\to 0}\phi\big)-\lim_{s\to 0}\phi'(s)\end{aligned}$$

$$\lim_{s\to 0}\phi(s)=\lim_{s\to 0}\phi'(s)=0,$$

we get

$$\mathscr{L}(\phi'')(s)=s^2\mathscr{L}(\phi)(s)=\frac{s^2}{(s+1)((s+1)^2+1)}.$$

4. *Using the above result we get*

$$\mathscr{L}\left(e^{-2t}\phi''\right)(s) = \frac{(s+2)^2}{(s+3)((s+3)^2+1)}.$$

Exercise 0.7. *Find the inverse Laplace transform of the function ζ given by*

$$\zeta(s) := \frac{2}{s^2-4} + \frac{s}{s^2+9}.$$

Solution 0.7.

$$\mathscr{L}^{-1}\left(\frac{2}{s^2-4} + \frac{s}{s^2+9}\right)(t) = \sinh 2t + \cos 3t.$$

0.5 Annex A

Define the following functions

$$\mathscr{U}_{\alpha}(s) = \begin{cases} 0 & \text{if } s < \alpha \\ 1, & \text{if } s \geq \alpha, \end{cases}$$

and

$$\mathscr{U}(s-\alpha) := u_{\alpha}(s).$$

We recall that

$$\sinh(s) = \frac{\mathbf{e}^{s} - \mathbf{e}^{-s}}{2}, \qquad \cosh(s) = \frac{\mathbf{e}^{s} + \mathbf{e}^{-s}}{2}.$$

$\varphi(.)$	$\psi(s) := \mathscr{L}(\varphi(.))(s)$
α	$\frac{\alpha}{s}$
$\delta(t-t_0)$	e^{-st_0}
$\delta(t)$	1
$\delta^{(n)}(t), \quad n=0,1,2,3,\ldots$	s^n
$t^n, \quad n=0,1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$
$t^p \ (p \in \mathbb{R} \quad \text{with} \quad p \geq -1)$	$\frac{\Gamma(p+1)}{s^{x+1}}$
$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$
$t^{n-\frac{1}{2}}, \quad n=1,2,3,\ldots$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
$e^{\alpha t}$	$\frac{1}{s-\alpha}$
$\frac{1}{\beta}e^{-\frac{t}{\beta}}$	$\frac{1}{1+\beta s}$
$1-e^{-\frac{t}{\beta}}$	$\frac{1}{s(1+\beta s)}$
$\frac{1}{\alpha}(e^{\alpha t}-1)$	$\frac{1}{s(s-\alpha)}$
$\frac{1}{\beta^2}te^{-\frac{t}{\beta}}$	$\frac{1}{(1+\beta s)^2}$
$\sin kt$	$\frac{k}{s^2+k^2}$
$\cos kt$	$\frac{s}{s^2+k^2}$
$\sinh kt$	$\frac{k}{s^2-k^2}$
$\cosh kt$	$\frac{s}{s^2-k^2}$
$\sin(\alpha t+\beta)$	$\frac{s\sin(\beta)+\alpha\cos(\beta)}{s^2+\alpha^2}$
$\cos(\alpha t+\beta)$	$\frac{s\cos(\beta)-\alpha\sin(\beta)}{s^2+\alpha^2}$
$\sin(\alpha t)-\alpha t\cos(\alpha t)$	$\frac{2\alpha^3}{(s^2+\alpha^2)^2}$
$\sin(\alpha t)+\alpha t\cos(\alpha t)$	$\frac{2\alpha s^2}{(s^2+\alpha^2)^2}$
$\cos(\alpha t)-\alpha t\sin(\alpha t)$	$\frac{s(s^2-\alpha^2)}{(s^2+\alpha^2)^2}$
$\cos(\alpha t)+\alpha t\sin(\alpha t)$	$\frac{s(s^2+3\alpha^2)}{(s^2+\alpha^2)^2}$

Tab. 0.1: Laplace transform table

$\varphi(.)$	$\psi(s) := \mathscr{L}(\varphi(.))(s)$
$e^{-\alpha t}\varphi(t)$	$\psi(s+\alpha)$
$t^n e^{\alpha t}$	$\frac{n!}{(s-\alpha)^{n+1}}$
$e^{\alpha t}\sin kt$	$\frac{k}{(s-\alpha)^2+k^2}$
$e^{\alpha t}\cos kt$	$\frac{s-\alpha}{(s-\alpha)^2+k^2}$
$e^{\alpha t}\sinh kt$	$\frac{k}{(s-\alpha)^2-k^2}$
$e^{\alpha t}\cosh kt$	$\frac{s-\alpha}{(s-\alpha)^2-k^2}$
$t\sin kt$	$\frac{2ks}{(s^2+k^2)^2}$
$t\cos kt$	$\frac{s^2-k^2}{(s^2+k^2)^2}$
$t\sinh kt$	$\frac{2ks}{(s^2-k^2)^2}$
$t\cosh kt$	$\frac{s^2+k^2}{(s^2-k^2)^2}$
$te^{\alpha t}$	$\frac{1}{(s-\alpha)^2}$
$\frac{\sin \alpha t}{t}$	$\arctan\frac{\alpha}{s}$
$\frac{1}{\sqrt{\pi t}}e^{-\alpha^2/4t}$	$\frac{e^{-\alpha\sqrt{s}}}{\sqrt{s}}$
$\frac{a}{2\sqrt{\pi t^3}}e^{-\alpha^2/4t}$	$e^{-\alpha\sqrt{s}}$
$\mathrm{erfc}\left(\frac{\alpha}{2\sqrt{t}}\right)$	$\frac{e^{-\alpha\sqrt{s}}}{s}$

Tab. 0.2: Laplace transform table

$\varphi(.)$	$\psi(s) := \mathscr{L}(\varphi(.))(s)$
$\dfrac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta}$	$\dfrac{1}{(s-\alpha)(s-\beta)}$
$\dfrac{\alpha e^{\alpha t}-\beta e^{\beta t}}{\alpha-\beta}$	$\dfrac{s}{(s-\alpha)(s-\beta)}$
$\mathscr{U}(t-\alpha)$	$\dfrac{e^{-\alpha s}}{s}$
$\varphi(t-\alpha)\mathscr{U}(t-\alpha)$	$e^{-\alpha s}\psi(s)$
$t^n\varphi(t)$	$(-1)^n\dfrac{d^n\psi(s)}{ds^n}$
$\varphi'(t)$	$s\psi(s)-\varphi(0)$
$\varphi^{(n)}(t)$	$s^n\psi(s)-\sum_{k=0}^{n-1}\varphi^{(k)}(0)s^{n-k-1}$
$\int_0^t \varphi(\tau)g(t-\tau)d\tau$	$\psi(s)G(s)$
$\int_0^t \varphi(\tau)\,d\tau$	$\dfrac{\psi(s)}{s}$

Tab. 0.3: Laplace transform table