Special functions
(Exercises with Solutions)
Master 2, PDEs & Applications

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Chapter 1

Special functions

1.1 Solutions

Solution 1.1. *1. We have*

$$\begin{aligned}
\Gamma(x+1) &= \lim_{n\to\infty} \frac{1.2.3\ldots n}{(x+1)(x+2)(x+3)\ldots(x+n+1)} n^{x+1} \\
&= \lim_{n\to\infty} \left[\frac{xn}{x+n+1} \times \frac{1.2.3\ldots n}{x(x+1)(x+2)\ldots(x+n)} n^{x} \right] \\
&= x\Gamma(x).
\end{aligned}$$

2. On the other hand,

$$\Gamma(1) = 1!.$$

On the one hand,

$$\begin{aligned}
\Gamma(n+1) &= n\Gamma(n) \\
&= n(n-1)\Gamma(n-1) \\
&. \\
&. \\
&. \\
&= n(n-1)(n-2)(n-3)\ldots+.2.1 \\
&= n!.
\end{aligned}$$

3. It is evident that

$$\Gamma(5) = 4! = 24.$$

$$\Gamma(7) = 6! = 720.$$

4. Note that

$$\Gamma\left(\frac{22}{5}\right) = \frac{17}{5}\Gamma\left(\frac{17}{5}\right).$$

This shows that

$$\Gamma\left(\frac{22}{5}\right) / \Gamma\left(\frac{17}{5}\right) = \frac{17}{5}\Gamma\left(\frac{17}{5}\right)$$

$$\frac{\Gamma\left(\frac{22}{5}\right)}{\Gamma\left(\frac{17}{5}\right)} = \frac{17}{5}.$$

5. *Also, we have*

$$\begin{aligned}
\Gamma\left(\frac{23}{4}\right) &= \frac{19}{4}\Gamma\left(\frac{19}{4}\right)\\
&= \frac{19}{4}\frac{15}{4}\Gamma\left(\frac{15}{4}\right)\\
&= \frac{19}{4}\frac{15}{4}\frac{11}{4}\Gamma\left(\frac{11}{4}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\Gamma\left(\frac{11}{4}\right)}{\Gamma\left(\frac{23}{4}\right)} &= \frac{1}{\frac{19}{4}\frac{15}{4}\frac{11}{4}}\\
&= \frac{64}{3135}.
\end{aligned}$$

Solution 1.2. *1. Since*

$$\begin{aligned}
n^x &= e^{x\log n}\\
&= e^{x(\log n-1-1/2-\ldots-1/n)}e^{x+x/2+\ldots+x/n},
\end{aligned}$$

we get

$$\Gamma_n(x) = \frac{1}{x}\frac{1}{x+1}\frac{2}{x+2}\cdots\frac{n}{x+n}n^x = \frac{e^{x(\log n-1-1/2-\ldots-1/n}e^{x+x/2+\ldots+x/n)}}{x(1+x)(1+x/2)\ldots(1+x/n)}.$$

2. *We have*

$$\Gamma_n(x) = e^{x(\log n)-1-1/2-\ldots-1/n}\frac{1}{x}\frac{e^x}{1+x}\frac{e^{x/2}}{1+x/2}+\cdots+\frac{e^{x/n}}{1+x/n}.$$

3. Note that

$$\begin{aligned}
\gamma_n - \gamma_{n+1} &= \log(n+1) - \log(n) - \frac{1}{n+1} \\
&= \log\left(1+\frac{1}{n}\right) - \frac{1}{n+1} \\
&= 1 - \frac{1}{n+1} - \frac{1}{2n^2} + 0\left(\frac{1}{n^3}\right) \\
&= \frac{1}{n(n+1)} - \frac{1}{2n^2} + 0\left(\frac{1}{n^3}\right) \\
&= \frac{n-1}{2n^2(n+1)} + 0\left(\frac{1}{n^3}\right).
\end{aligned}$$

This shows that, $\gamma_n - \gamma_{n+1} \geq 0$ *for all* $n \geq 1$.

4. Since

$$\Gamma(x) = \lim_{n\to\infty}\left[e^{x(\log n)-1-1/2-\ldots-1/n}\frac{1}{x}\frac{e^x}{1+x}\frac{e^{x/2}}{1+x/2} + \ldots + \frac{e^{x/n}}{1+x/n}\right],$$

we obtain

$$\Gamma(x) = \frac{e^{-\gamma x}}{x}\prod_{n=1}^{+\infty}\frac{e^{\frac{x}{n}}}{(1+\frac{x}{n})}.$$

Solution 1.3. *We have*

$$\psi(x) = \frac{-1}{x} - \gamma + \sum_{n\geq 1}\left(\frac{1}{n} - \frac{1}{x+n}\right).$$

By deriving the formula, we get

$$\psi'(x) = \sum_{n\geq 0}\frac{1}{(x+n)^2}.$$

Solution 1.4. *It follows from Stirling's Lemma that, for* $\mathrm{Re}(z) > 1$ *and* $|z| = R$ *large enough, we have*

$$\begin{aligned}\left|\frac{1}{\Gamma(z)}\right| &\leq \left|e^{z}z^{-z+1/2}\right| \\ &\leq e^{R}R^{R+1/2} \\ &\leq R^{R}e^{R}R^{\frac{1}{2}}. \\ &\leq c_1R^{R} \quad c_1 := e^{R}R^{\frac{1}{2}}.\end{aligned}$$

For $\mathrm{Re}(z) < 1$ *and* $|z| = R$*, large enough:*

$$\begin{aligned}\left|\frac{1}{\Gamma(z)}\right| &\leq \left|e^{z-1}(1-z)^{-z+1/2}\sin(\pi z)\right| \\ &\leq e^{|z|+1}(1+R)^{R+1/2} \\ &\leq c_2R^{R}.\end{aligned}$$

Solution 1.5. *We have*

$$\xi(z) = ze^{\gamma z}\prod_{n\geq 1}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}},$$

so that

$$|\xi(z)| = |z|e^{\gamma x}\prod_{n\geq 1}\left|1+\frac{z}{n}\right|e^{-\frac{x}{n}} \geq |x|e^{\gamma x}\prod_{n\geq 1}\left|1+\frac{x}{n}\right|e^{-\frac{x}{n}},$$

with $x = \mathrm{Re}(z) = 1$*. Note that since*

$$\begin{aligned}\log\prod_{n\geq 1}\left|1+\frac{x}{n}\right|e^{-\frac{x}{n}} = \sum_{n\geq 1}\left(\log\left|1+\frac{x}{n}\right| - \frac{x}{n}\right) = \sum_{n\geq 1}\left(\log\left(1+\frac{x}{n}\right) - \frac{x}{n}\right) \\ \geq -\frac{1}{2}\sum_{n\geq 1}\left(\frac{x}{n}\right)^2 \geq -\frac{x^2}{2}\zeta(2) > -1,\end{aligned}$$

we have

$$|\xi(z)| \geq |x|e^{\gamma x}e^{-3} > \frac{1}{2}.$$

Solution 1.6. *We use integration by parts*

$$\begin{aligned}\Gamma(z+1) &= \int_0^\infty e^{-t}t^z dt \\ &= \left[-t^z e^{-t}\right]_0^{+\infty} + z\int_0^\infty e^{-t}t^{z-1}dt \\ &= z\Gamma(z).\end{aligned}$$

Solution 1.7. 1. *Letting $u = e^{-t}$, so that $t = -\log u$, that is $dt = -\frac{du}{u}$, and therefore*

$$t = 0 \Longrightarrow u = 1,$$

$$t = +\infty \Longrightarrow u = 0.$$

2. *Letting $t = u^2$ we get $u = \sqrt{t}$ that is $dt = 2udu$, moreover*

$$t = 0 \Longrightarrow u = 0,$$

$$t = +\infty \Longrightarrow u = +\infty.$$

Solution 1.8. 1. *We have*

$$I^2 = 4\int_0^{+\infty}\int_0^{+\infty} e^{-(u^2+v^2)}dudv.$$

Evaluate a double integral using a change of variables when we substitute $u = r\cos\theta$, $v = r\sin\theta$, we get

$$\begin{aligned}I^2 &= -2\int_0^{\pi/2}\int_0^{+\infty} -2re^{-r^2}drd\theta \\ &= -2\int_0^{\pi/2}\int_0^{+\infty}\left(e^{-r^2}\right)' drd\theta \\ &= -2\int_0^{\pi/2} -d\theta \\ &= \pi.\end{aligned}$$

Hence

$$I = \sqrt{\pi}.$$

2.

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} e^{-u^2} du$$
$$= \sqrt{\pi}.$$

3.

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$
$$= \frac{\sqrt{\pi}}{2}.$$

$$\Gamma\left(\frac{-1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{\frac{-1}{2}}$$
$$= -2\Gamma\left(\frac{1}{2}\right)$$
$$= -2\sqrt{\pi}.$$

$$\Gamma\left(\frac{-3}{2}\right) = \frac{\Gamma\left(\frac{-1}{2}\right)}{\frac{-3}{2}}$$
$$= \frac{4}{3}\sqrt{\pi}.$$

Solution 1.9. *We will prove this by induction. For n=0, on the one hand, we have*

$$\frac{\sqrt{\pi}\Gamma(1)}{2^0\Gamma(1)} = \sqrt{\pi},$$

on the other hand, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Let $n \in \mathbb{N}$, we assume that

$$\Gamma(n+\frac{1}{2}) = \frac{\sqrt{\pi}\Gamma(2n+1)}{2^{2n}\Gamma(n+1)}.$$

We will prove that

$$\Gamma(n+\frac{3}{2}) = \frac{\sqrt{\pi}\Gamma(2n+3)}{2^{2n+2}\Gamma(n+2)}.$$

We have

$$\begin{aligned}
\Gamma(n+\frac{3}{2}) &= (n+\frac{1}{2})\Gamma(n+\frac{1}{2}) \\
&= \frac{\sqrt{\pi}(n+\frac{1}{2})\Gamma(2n+1)}{2^{2n}\Gamma(n+1)} \\
&= \frac{\sqrt{\pi}(2n+1)\Gamma(2n+1)}{2^{2n+1}\Gamma(n+1)} \\
&= \frac{\sqrt{\pi}\Gamma(2n+2)}{2^{2n+1}\Gamma(n+1)} \\
&= \frac{\sqrt{\pi}(2n+2)\Gamma(2n+2)}{2^{2n+1}2(n+1)\Gamma(n+1)} \\
&= \frac{\sqrt{\pi}\Gamma(2n+3)}{2^{2n+2}\Gamma(n+2)}.
\end{aligned}$$

Solution 1.10. *1. We have*

$$\begin{aligned}
\Gamma(s)\Gamma(1-s) &= \int_0^{+\infty}\int_0^{+\infty} u^{s-1}v^{-s}e^{-u}e^{-v}\, dv\, du \\
&= \int_0^{+\infty} dv\, v^{-s}e^{-v}(\int_0^{+\infty} u^{s-1}e^{-u}\, du),
\end{aligned}$$

under the change of variables $w = \frac{u}{v}$, *we obtain*

$$\begin{aligned}
\Gamma(s)\Gamma(1-s) &= \int_0^{+\infty} dv\, e^{-v}(\int_0^{+\infty} u^{s-1}e^{-uv}\, du) \\
&= \int_0^{+\infty} du\, u^{s-1} \int_0^{+\infty} dv\, e^{-v}e^{-uv}) \\
&= \int_0^{+\infty} du\, u^{s-1}(\int_0^{+\infty} dv\, e^{-v(1+u)}) \\
&= \int_0^{+\infty} \frac{u^{s-1}}{1+u}\, du \\
&= \int_0^{1} \frac{u^{s-1}}{1+u}\, du + \int_1^{+\infty} \frac{u^{s-1}}{1+u}\, du.
\end{aligned}$$

Again, under the change of variables $w = \frac{1}{u}$ *in the second integral, we get*

$$\Gamma(s)\Gamma(1-s) = \int_0^{1} \frac{u^{s-1}}{1+u}\, du + \int_0^{1} \frac{u^{-s}}{1+u}\, du.$$

Thus,

$$\Gamma(s)\Gamma(1-s) = \varphi(s) + \varphi(1-s).$$

2. *For all* $n \in \mathbb{N}$, *we have*

$$\frac{1}{1+u} = \sum_{k=0}^{n-1}(-1)^k u^k + \frac{(-1)^n u^n}{1+u}.$$

Hence

$$\begin{aligned}\varphi(s) &= \sum_{k=0}^{n-1}(-1)^k\int_0^1 u^{k+s-1}\,du+(-1)^n\int_0^1\frac{u^{n+s-1}}{1+u}\,du\\ &= \sum_{k=0}^{n-1}\frac{(-1)^k}{k+s}+(-1)^n\int_0^1\frac{u^{n+s-1}}{1+u}\,du.\end{aligned}$$

But

$$0\le\left|\int_0^1\frac{u^{n+s-1}}{1+u}du\right|\le\int_0^1 u^{n+Re(s)-1}\,du=\frac{1}{n+Re(s)}\le\frac{1}{n}\underset{n\to+\infty}{\to}0.$$

Thus,

$$\varphi(s)=\sum_{n=0}^{+\infty}\frac{(-1)^n}{n+s},\quad \textit{pour touts}\in\mathbb{C},\quad \textit{avec}0<Re(s)<1.$$

3. *We use the development of* $\varphi(s)$*, we obtain*

$$\begin{aligned}\Gamma(s)\Gamma(1-s) &= \sum_{n=0}^{+\infty}\frac{(-1)^n}{n+s}+\sum_{n=0}^{+\infty}\frac{(-1)^n}{n+1-s}\\ &= \sum_{n=0}^{+\infty}\frac{(-1)^n}{n+s}+\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n-s}\\ &= \sum_{n=0}^{+\infty}\frac{(-1)^n}{n+s}+\sum_{n=1}^{+\infty}\frac{(-1)^n}{s-n}\\ &= \frac{1}{s}+\sum_{n=1}^{+\infty}\frac{2(-1)^n s}{s^2-n^2}.\end{aligned}$$

4. *For* $t=0$*, we have*

$$1=\frac{\sin\pi s}{\pi}\left[\frac{1}{s}+\sum_{n=1}^{+\infty}\frac{2(-1)^n s}{s^2-n^2}\right],$$

so that

$$\begin{aligned}\Gamma(s)\Gamma(1-s) &= \frac{1}{s}+\sum_{n=1}^{+\infty}\frac{2(-1)^n s}{s^2-n^2}\\ &= \frac{\pi}{\sin \pi s}.\end{aligned}$$

Solution 1.11. *It is easy to show that*

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi},$$

$$\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)=\frac{2\pi\sqrt{3}}{3},$$

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)=\pi\sqrt{2}.$$

Solution 1.12. *1. We will prove this by induction.*

For $n=0$, we have

$$J_0(z)=\int_0^1 s^{z-1}\mathrm{d}s=\left[\frac{s^z}{z}\right]_0^1=\frac{1}{z}.$$

Let $n\geq 0$, assume that

$$J_n(z)=\frac{n!}{z(z+1)\dots(z+n)}.$$

We have

$$\begin{aligned}J_{n+1}(z) &= \int_0^1 s^{z-1}(1-s)^{n+1}\mathrm{d}s\\ &= \left[\frac{s^z}{z}(1-s)^{n+1}\right]_0^1+\int_0^1\frac{s^z}{z}(n+1)(1-s)^n\mathrm{d}s\\ &= 0+\frac{n+1}{z}J_n(z+1)\\ &= \frac{(n+1)n!}{z(z+1)\dots(z+n)(z+n+1)}.\end{aligned}$$

2. *Consider the sequence of functions*

$$g_n(t) = \left(1 - \frac{t}{n}\right)^n t^{z-1} \chi_{]0;n[}(t).$$

It follows

(a) $g_n(t) \underset{n \to +\infty}{\longrightarrow} e^{-t} t^{z-1} \chi_{]0;+\infty[}.$

(b) *Since*

$$1 - u \leq e^{-u} \quad \textit{for all} \quad u \in [0;1],$$

we get

$$|g_n(t)| \leq e^{-t} t^{\mathrm{Re}\,(z)-1}.$$

Following the Lebesgue's theorem,

$$\Gamma(z) = \lim_{n \to +\infty} \int_0^n t^{z-1} \left(1 - \frac{t}{n}\right)^n \mathrm{d}t.$$

For $t = ns$*, we obtain,*

$$\Gamma(z) = \lim_{n \to +\infty} n^z \int_0^1 s^{z-1} (1-s)^n \,\mathrm{d}t = \lim_{n \to +\infty} n^z J_n(z).$$

Solution 1.13. 1. *We have*

$$\begin{aligned} B(y,x) &= \frac{\Gamma(y)\Gamma(x)}{\Gamma(y+x)} \\ &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \\ &= B(x,y) \end{aligned}$$

2. *We consider the following change of variables* $u = \sin^2\theta$*, so that* $dt = 2\cos\theta\sin\theta$*, and that,*

$$\begin{aligned} B(x,y) &= 2\int_0^{\frac{\pi}{2}} (\sin^2\theta)^{(x-1)} \, (\cos^2\theta)^{(x-1)} \sin\theta\cos\theta \; d\theta \\ &= 2\int_0^{\frac{\pi}{2}} \sin^{(2x-1)}\theta \; \cos^{(2y-1)}\theta \; d\theta. \end{aligned}$$

Solution 1.14. *1. We have*

$$\begin{aligned} B(x+1,y) &= \frac{\Gamma(x+1)\Gamma(y)}{\Gamma(x+y+1)} \\ &= \frac{x\Gamma(x)\Gamma(y)}{(x+y)\Gamma(x+y)} \\ &= \frac{x}{x+y}B(x,y). \end{aligned}$$

2. Also, we have

$$\begin{aligned} B(x,y+1) &= B(y+1,x) \\ &= \frac{y}{y+x}B(y,x) \\ &= \frac{y}{x+y}B(x,y). \end{aligned}$$

3. It follows that

$$\begin{aligned} B(x,y) &= B((x-1)+1,y) \\ &= \frac{x-1}{x+y-1}B(x-1,y). \end{aligned}$$

Solution 1.15. *We use the change of variables $s = t^5$, so that,*

$$dt = \frac{1}{5}s^{-\frac{4}{5}}ds.$$

Thus, we obtain

$$\begin{aligned}
\int_0^1 \frac{tdt}{\sqrt{1-t^5}} &= \frac{1}{5}\int_0^1 s^{\frac{1}{5}}(1-s)^{-\frac{1}{2}}s^{-\frac{4}{5}}ds \\
&= \frac{1}{5}\int_0^1 s^{-\frac{3}{5}}(1-s)^{-\frac{1}{2}}ds \\
&= \frac{1}{5}\int_0^1 s^{\frac{2}{5}-1}(1-s)^{\frac{1}{2}-1}ds \\
&= \frac{1}{5}B(\frac{2}{5},\frac{1}{2}).
\end{aligned}$$

Solution 1.16. *We use the change of variables $s = t^4$, hence*

$$dt = \frac{1}{4}s^{-\frac{3}{4}}ds,$$

and hance

$$\begin{aligned}
\int_0^1 \frac{tdt}{\sqrt{1-t^5}} &= \frac{1}{4}\int_0^1 s^{\frac{1}{2}}(1-s)^{-\frac{1}{2}}s^{-\frac{3}{4}}ds \\
&= \frac{1}{4}\int_0^1 s^{-\frac{1}{4}}(1-s)^{-\frac{1}{2}}ds \\
&= \frac{1}{5}\int_0^1 s^{\frac{3}{4}-1}(1-s)^{\frac{1}{2}-1}ds \\
&= \frac{1}{4}B(\frac{3}{4},\frac{1}{2}).
\end{aligned}$$

Solution 1.17. *1. We have*

$$\begin{aligned}
B(\frac{1}{2},\frac{1}{2}) &= \int_0^1 s^{-\frac{1}{2}}(1-s)^{-\frac{1}{2}}ds \\
&= 2\int_0^1 \frac{du}{\sqrt{1-u^2}} \quad \textit{under the change of variables } s = u^2 \\
&= 2\arcsin 1 \\
&= \pi.
\end{aligned}$$

But

$$\begin{aligned} B(\frac{1}{2},\frac{1}{2}) &= \frac{\Gamma\left(\frac{1}{2}\right)\left(\Gamma\frac{1}{2}\right)}{\Gamma(1)} \\ &= \Gamma^2\left(\frac{1}{2}\right). \end{aligned}$$

Thus

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

2. *We have*

$$\begin{aligned} \Gamma\left(\frac{1}{2}\right) &= \int_0^{+\infty} e^{-s}s^{-\frac{1}{2}}ds \\ &= 2\int_0^{+\infty} e^{-u^2}du \quad \text{under the change of variables } s = u^2 \\ &= \int_{-\infty}^{+\infty} e^{-u^2}du. \end{aligned}$$

This implies that

$$\int_{-\infty}^{+\infty} e^{-u^2}du = \sqrt{\pi}.$$

Solution 1.18. *It is evident that*

$$\begin{aligned} E_{1,2}(z) &= \sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(k+2)} \\ &= \frac{1}{z}\sum_{k=0}^{+\infty}\frac{z^{k+1}}{(k+1)!} \\ &= \frac{e^z-1}{z}, \end{aligned}$$

$$\begin{aligned} E_{1,3}(z) &= \sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(k+3)} \\ &= \frac{1}{z^2}\sum_{k=0}^{+\infty}\frac{z^{k+2}}{(k+2)!} \\ &= \frac{e^z-z-1}{z^2}, \end{aligned}$$

and

$$E_{2,1}(z) = \sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(2k+1)} = \cosh z.$$

Solution 1.19. *We have*

$$\begin{aligned} E_{\alpha,\beta}(x) &= \sum_{k=0}^{+\infty}\frac{x^k}{\Gamma(\alpha k+\beta)} \\ &= \sum_{k=-1}^{+\infty}\frac{x^{k+1}}{\Gamma(\alpha(k+1)+\beta)} \\ &= \sum_{k=-1}^{+\infty}\frac{xx^k}{\Gamma(\alpha k+(\alpha+\beta))} \\ &= \frac{1}{\Gamma(\beta)}+x\sum_{k=0}^{+\infty}\frac{x^k}{\Gamma(\alpha k+\alpha+\beta)} \\ &= \frac{1}{\Gamma(\beta)}+xE_{\alpha,\alpha+\beta}(x). \end{aligned}$$

Solution 1.20. *Proceeding as before, we obtain*

$$\begin{aligned}\beta E_{\alpha,\beta+1}(x)+\alpha x\frac{d}{dx}[E_{\alpha,\beta+1}(x)] &= \sum_{k=0}^{+\infty}\frac{\beta x^k}{\Gamma(\alpha k+\beta+1)}+\sum_{k=0}^{+\infty}\alpha x\frac{d}{dx}\frac{x^k}{\Gamma(\alpha k+\beta+1)}\\ &= \sum_{k=0}^{+\infty}\frac{(\alpha k+\beta)x^k}{\Gamma(\alpha k+\beta+1)}\\ &= \sum_{k=0}^{+\infty}\frac{x^k}{\Gamma(\alpha k+\beta)}\\ &= E_{\alpha,\beta}(x).\end{aligned}$$

Solution 1.21. *We remark that*

$$\vartheta_\alpha(x):=\sum_{k=1}^{+\infty}\frac{(-1)^{k+1}x^{\alpha k}}{\Gamma(\alpha k+1)}.$$

Thus, we obtain

$$\begin{aligned}\frac{d}{dx}\vartheta_\alpha(x) &= \frac{d}{dx}\sum_{k=1}^{+\infty}\frac{(-1)^{k+1}\alpha k x^{\alpha k-1}}{\Gamma(\alpha k+1)}\\ &= \sum_{k=1}^{+\infty}\frac{(-1)^{k+1}x^{\alpha k-1}}{\Gamma(\alpha k)}\\ &= \sum_{k=0}^{+\infty}\frac{(-1)^{k}x^{\alpha k+\alpha-1}}{\Gamma(\alpha k)}\\ &= x^{\alpha-1}\sum_{k=0}^{+\infty}\frac{(-1)^{k}x^{\alpha k}}{\Gamma(\alpha k)}\\ &= x^{\alpha-1}E_{\alpha,\alpha}(-x^\alpha).\end{aligned}$$

Solution 1.22. *We have*

$$\begin{aligned} E_{\beta,\gamma}^{\lambda,p}(z) &= \sum_{n=0}^{+\infty}\left[\frac{1}{\Gamma(\lambda)}\int_0^{+\infty} t^{\lambda+n-1}e^{-t-\frac{p}{t}}dt\right]\frac{1}{\Gamma(\lambda)\Gamma(\beta n+\gamma)}\frac{z^n}{n!} \\ &= \frac{1}{[\Gamma(\lambda)]^2}\int_0^{+\infty} t^{\lambda-1}e^{-t-\frac{p}{t}}\sum_{n=0}^{+\infty}\frac{(tz)^n}{\Gamma(\beta n+\gamma)n!}dt. \\ &= \frac{1}{[\Gamma(\lambda)]^2}\int_0^{+\infty} t^{\lambda-1}e^{-t-\frac{p}{t}}E_{\beta,\gamma}(tz)dt. \end{aligned}$$