Fractional Analysis
(Lessons and Exercises with Solutions)

Master 2, PDEs & Applications

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Chapter 1

Special functions

1.1 Gamma function

The Gamma function is a special function defined by Leonhard Euler (1707-1783), its objective is to extend the factorial function to the set of complex numbers. Several mathematicians have given particular importance to this function, namely, Adrien-Marie Legendre (1752-1833), Carl Friedrich Gauss...

Definition 1.1. ***Euler's infinite limit****(Euler, 1729 and Gauss, 1811) Letting*

$$\Gamma_n(s) := \frac{1.2.3\ldots n}{s(s+1)(s+2)\ldots(s+n)} n^s, \qquad s \neq 0,\ -1,\ -2,\ \ldots$$

The Gamma function is defined by limit as follows

$$\Gamma(s) = \lim_{n\to\infty} \Gamma_n(s).$$

Examples 1.1. *1. We have* $\Gamma(1) = 1$. *In fact,*

$$\begin{aligned}\Gamma_n(1) &= \frac{1.2.3\ldots n}{1.2.3\ldots n(n+1)} n \\ &= \frac{n}{n+1}.\end{aligned}$$

This implies that

$$\Gamma(1) = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

2. *Note that since*

$$\Gamma_n\left(\frac{16}{3}\right) = \frac{1.2.3\ldots n}{(\frac{16}{3})(\frac{16}{3}+1)\ldots(\frac{16}{3}+n)} n^{\frac{16}{3}},$$

$$\Gamma_n\left(\frac{10}{3}\right) = \frac{1.2.3\ldots n}{(\frac{10}{3})(\frac{10}{3}+1)\ldots(\frac{10}{3}+n)} n^{\frac{10}{3}},$$

we have

$$\begin{aligned} \frac{\Gamma_n\left(\frac{16}{3}\right)}{\Gamma_n\left(\frac{10}{3}\right)} &= \lim_{n\to\infty} \frac{\left(\frac{10}{3}\right)\left(\frac{10}{3}+1\right)}{\left(\frac{10}{3}+1+n\right)\left(\frac{10}{3}+2+n\right)} \\ &= \lim_{n\to\infty} \frac{\frac{130}{9}n^2}{n^2} = \frac{130}{9}. \end{aligned}$$

3. $0! = 1$.
   Since $\Gamma(1) = 0!$ *and since* $\Gamma(1) = 1$ *we obtain* $0! = 1$.

Definition 1.2. ***Weierstrass's infinite product***

$$\frac{1}{\Gamma(x)} = xe^{\gamma x} \prod_{n=1}^{+\infty} (1 + \frac{x}{n}) e^{-\frac{x}{n}}.$$

Lemma 1.1 (Stirling's Formula)**.** *For* $z \in \mathbb{C}$, $|\arg(z)| \leq \delta < \pi$ *and* $|z| \to \infty$, *we have*

$$\frac{1}{\Gamma(z)} = \frac{1}{\sqrt{2\pi}} e^{z} z^{-z+1/2} \left(1 + O\left(z^{-1}\right)\right),$$

that is

$$\Gamma(z) \sim \sqrt{2\pi} e^{-z} z^{z-\frac{1}{2}}.$$

Lemma 1.2. *For $z \in \mathbb{C}$, $|z| \to \infty$, we have*

$$\frac{1}{\Gamma(z)} = \begin{cases} \dfrac{e^{z} z^{-z+1/2}}{\sqrt{2\pi}} \left(1 + O\left(z^{-1}\right)\right), & \text{if} \quad \mathrm{Re}(z) > 1, \\ \sqrt{\dfrac{2}{\pi}} e^{z-1} (1-z)^{-z+1/2} \sin(\pi z) \left(1 + O\left(z^{-1}\right)\right), & \text{if} \quad 0 < \mathrm{Re}(z) < 1. \end{cases}$$

Definition 1.3. ***(Euler's integral)****(Euler, 1730)*

$$\Gamma(x) = \int_{0}^{\infty} e^{-t} t^{x-1} dt, \quad Re(x) > 0. \tag{1.1}$$

Remark 1.1. *For the divergence's raisons of the integral, the definition of the Gamma function by the above integral formula is not applicable for negative values.*

Example 1.1.

$$\Gamma(1) = \int_{0}^{\infty} e^{-t} dt = \left[-e^{-t}\right]_{0}^{\infty} = 1.$$

Theorem 1.1.
1. *The Gamma function(1.1) is defined, continuous on $]0, +\infty[$.*
2. *Moreover, $\Gamma \in C^{\infty}$ and*

$$\forall k \in \mathbb{N}, \quad \forall x > 0, \quad \Gamma^{(k)}(x) = \int_{0}^{+\infty} (\ln t)^{k} e^{-t} t^{x-1} dt.$$

Proof.

1. (a) Let us consider the function f

$$f : \mathbb{R} \times \mathbb{R}_{+}^{*} \mapsto \mathbb{R}, \ (x,t) \mapsto e^{-t} t^{(x-1)}, \quad \text{for all} \quad x \in \mathbb{R}.$$

Define the function f_x as follows

$$f_x : \mathbb{R}_{+}^{*} \mapsto \mathbb{R}, \ t \mapsto f(t,x).$$

We note the f_x est continuous, positive on $]0,+\infty[$ therefore, it is locally integrable.
In the neighborhood of zero:

$$e^{-t}t^{(x-1)} \sim t^{(x-1)}.$$

The function f_x is integrable for all $t \in]0,1[$, if and only if, $x \in]0,+\infty[$.
In the neighborhood of infinity:

$$e^{-t}t^{(x-1)} \sim o(\frac{1}{t^2}),$$

hence f_x is integrable on $]1,+\infty[$.

(b) Continuity of function Γ:
Following the theorem of continuity under integral sign, we have

i. $x \mapsto e^{-t}t^{x-1}$ is continuous on $]0,+\infty[$.
ii. Let $a,b \in \mathbb{R}^*$, $0 < a < b < +\infty$:
For all $0 < t < 1$, the function $x \mapsto t^{(x-1)}$ is decreasing on $[a,b]$, so that

$$\forall x \in [a,b] \ : \ 0 < f(t,x) < t^{(a-1)}.$$

For all $t \geq 1$ we have $x \mapsto t^{(x-1)}$ is increasing on $[a,b]$, hence

$$0 < f(t,x) < e^{-t}t^{(b-1)},$$

and hence

$$\varphi :]0,+\infty[\mapsto \mathbb{R}, \quad t \mapsto \varphi(t) = \begin{cases} t^{(a-1)}, & t \in]0,1], \\ e^{-t}t^{(b-1)} & t \in [1,+\infty[\end{cases}$$

is positive, continuous, integrable on $]0,+\infty[$ and

$$\forall (x,t) \in [a,b] \times]0,+\infty[, \quad 0 < f(t,x) < \varphi(t),$$

Thus, Γ is continuous on $]0,+\infty[$.

2 For $k = 1$
By the theorem of differentiation under the integral sign.

(a)

$$\begin{aligned}\frac{\partial}{\partial x}(e^{-t}t^{(x-1)}) &= e^{-t}\frac{\partial}{\partial x}t^{x-1} \\ &= e^{-t}\ln(t)t^{(x-1)}.\end{aligned}$$

(b)

$$|e^{-t}\ln(t)t^{(x-1)}| \leq \begin{cases} |\ln(t)|t^{(a-1)}, & 0 < t < 1, \\ |t^{(b-1)}e^{-t}\ln(t)|, & 1 < t < +\infty. \end{cases}$$

$$\begin{aligned}|e^{-t}\ln(t)t^{(x-1)}| &= [-\ln(t)e^{-t}t^{(x-1)}]_{t\in]0,1[} + [\ln(t)e^{-t}t^{(x-1)}]_{t\in[1,+\infty[} \\ &\leq [-\ln(t)t^{(a-1)}]_{t\in]0,1[} + [\ln(t)e^{-t}t^{(b-1)}]_{t\in[1,+\infty[},\end{aligned}$$

so that φ integrable on $]0,+\infty[$, moreover,

$$\Gamma'(t) = \int_0^{\infty} \ln t e^{-t}t^{x-1}dt.$$

By induction using the same process we get

$$\Gamma^{(k)}(x) = \int_0^{\infty} \ln(t)^k e^{-t}t^{(x-1)}dt.$$

■

Remark 1.2. *We can use the following formula to define $\Gamma(x)$ for non-integer negative values*

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}, \quad x \neq 0,\ -1,\ -2,\ \ldots$$

Example 1.2.

$$\begin{aligned}\Gamma\left(\tfrac{1}{2}\right) &= \sqrt{\pi} \\ &\approx 1.77245\end{aligned}$$

$$\begin{aligned}\Gamma\left(-\frac{3}{2}\right) &= \frac{4}{3}\sqrt{\pi} \\ &\approx 2.36327\end{aligned}$$

$$\begin{aligned}\Gamma\left(-\frac{1}{2}\right) &= -2\sqrt{\pi} \\ &\approx -3.54490\end{aligned}$$

Remark 1.3. *1. Stirling's formula*

$$\Gamma(x) \sim \sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x}, \qquad Re(x) \to +\infty.$$

1.2 Volume of n-dimensional sphere via Gamma function

This section is based on the lectures of Professor Chaur-Chin Chen. We show the importance use of the Gamma function to calculate the volume Vn, of an n-dimensional sphere. For this purpose, let us introduce the Cartesian coordinates $(x_1, x_2, x_3, \cdots, x_n)$, we recall that the sphere is defined by the equation

$$x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 = \rho^2$$

.

Proposition 1.1. *The volume V_2 of the 2-dimensional sphere with radius ρ is given by*

$$V_2 = \pi\rho^2.$$

Proof. Letting

$$\begin{aligned} x_1 =& r\cos\theta, \quad 0 \leq \theta \leq 2\pi, \\ x_2 =& r\sin\theta, \quad 0 \leq r \leq \rho. \end{aligned}$$

We have

$$\begin{aligned} J_2 &= \frac{\partial(x_1,x_2)}{\partial(r,\theta)} \\ &= \begin{vmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_2}{\partial r} \\ \frac{\partial x_1}{\partial \theta} & \frac{\partial x_2}{\partial \theta} \end{vmatrix} \\ &= r. \end{aligned}$$

Thus, we obtain

$$V_2 = \int_0^\rho \int_0^{2\pi} J_2 dr d\theta = \int_0^\rho \int_0^{2\pi} r dr d\theta = \pi\rho^2.$$

■

Proposition 1.2. *The volume V_3 of the 3-dimensional sphere of radius ρ is given by*

$$V_3 = \frac{4\pi\rho^3}{3}.$$

Proof. For

$$\begin{aligned} x_1 =& r\cos\theta_1 \cos\theta_2, \quad 0 \leq \theta_2 \leq 2\pi \\ x_2 =& r\cos\theta_1 \sin\theta_2, \quad -\frac{\pi}{2} \leq \theta_1 \leq \frac{\pi}{2} \\ x_3 =& r\sin\theta_1, \quad 0 \leq r \leq \rho. \end{aligned}$$

We get

$$\begin{aligned} J_3 &= \frac{\partial(x_1,x_2,x_3)}{\partial(r,\theta_1,\theta_2)} \\ &= \begin{vmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_2}{\partial r} & \frac{\partial x_3}{\partial r} \\ \frac{\partial x_1}{\partial \theta_1} & \frac{\partial x_2}{\partial \theta_1} & \frac{\partial x_3}{\partial \theta_1} \\ \frac{\partial x_1}{\partial \theta_2} & \frac{\partial x_2}{\partial \theta_2} & \frac{\partial x_3}{\partial \theta_2} \end{vmatrix} \\ &= r^2\cos\theta_1, \end{aligned}$$

and therefore

$$\begin{aligned} V_3 &= \int_0^{\rho}\int_{-\pi/2}^{\pi/2}\int_0^{2\pi} J_3 dr d\theta_1 d\theta_2 \\ &= \int_0^{\rho}\int_{-\pi/2}^{\pi/2}\int_0^{2\pi} r^2\cos\theta_1 dr d\theta_1 d\theta_2 \\ &= \frac{4\pi\rho^3}{3}. \end{aligned}$$

■

Theorem 1.2. *The volume V_n of the n-dimensional sphere of radius ρ is given by*

$$V_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}\cdot\rho^n.$$

Proof. Let us consider the following polar coordinates

$$\begin{aligned} x_1 =& r\cos\theta_1\cos\theta_2\cdot\cdots\cdot\cos\theta_{n-2}\cos\theta_{n-1}, \quad 0\le\theta_{n-1}\le 2\pi \\ x_{n-1} =& r\cos\theta_1\sin\theta_2, \quad -\frac{\pi}{2}\le\theta_1\le\frac{\pi}{2} \\ x_n =& r\sin\theta_1, \quad 0\le r\le\rho, \end{aligned}$$

and for $2 \leq k \leq n-2$,

$$x_k = r\cos\theta_1\cos\theta_2\cdots\cos\theta_{n-k}\sin\theta_{n-k+1}, \quad -\frac{\pi}{2} \leq \theta_{n-k} \leq \frac{\pi}{2}.$$

This implies that

$$J_n = r^{n-1}c_1^{n-1}c_2^{n-2}\cdots c_{n-1}^{1}s_1s_2\cdots s_{n-1}\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ -t_1 & -t_1 & -t_1 & \cdots & \frac{1}{t_1} \\ -t_2 & -t_2 & -t_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -t_{n-2} & -t_{n-2} & \frac{1}{t_{n-2}} & \cdots & 0 \\ -t_{n-1} & \frac{1}{t_{n-1}} & 0 & \cdots & 0 \end{vmatrix},$$

with $t_i = \frac{\sin\theta_i}{\cos\theta_i}$.

Subtracting each column from the preceding one, we get

$$\begin{aligned} J_n &= r^{n-1}c_1^{n-1}c_2^{n-2}\cdots c_{n-1}^{1}s_1s_2\cdots s_{n-1}\prod_{j=1}^{n-1}(t_j+\frac{1}{t_j}) \\ &= r^{n-1}c_1^{n-1}c_2^{n-2}\cdots c_{n-1}^{1}s_1s_2\cdots s_{n-1}\prod_{j=1}^{n-1}(\frac{1}{s_j\cdot c_j}) \\ &= r^{n-1}c_1^{n-2}c_2^{n-3}\cdots c_{n-3}^{2}c_{n-2}^{1} \\ &= r^{n-1}\cos^{n-2}\theta_1\cos^{n-3}\theta_2\cdots\cos^{2}\theta_{n-3}\cos^{1}\theta_{n-2} \end{aligned}$$

Thus,

$$\begin{aligned} V_n &= \int_0^{\rho}\int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\cdots\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}(J_n)d\theta_1 d\theta_2\cdots d\theta_{n-2}d\theta_{n-1}dr \\ &= \int_0^{\rho}\int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\cdots\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}[r^{n-1}\cos^{n-2}\theta_1\cos^{n-3}\theta_2\cdots\cos\theta_{n-2}]d\theta_1 d\theta_2 \\ &= \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}\cdot\rho^n. \end{aligned}$$

■

Remark 1.4. *1. The surface S_n for a sphere with radius r in $\mathbb{R}^n$ is given by*

$$S_n = \frac{2\Gamma(\frac{1}{2})^n r^{n-1}}{\Gamma(\frac{n}{2})}.$$

1.3 Incomplete Gamma functions

Definition 1.4. *The complete Gamma function Γ can be generalized to the upper incomplete Gamma function and the lower incomplete one respectively as follows:*

$$\gamma(a,z) = \int_0^z e^{-t}t^{a-1}dt, \qquad Re(z) > 0,$$
$$\Gamma(a,z) = \int_z^{\infty} e^{-t}t^{a-1}dt, \qquad Re(z) > 0.$$

Remark 1.5. *We note that*

1. $\Gamma(a) = \lim_{x\to\infty}\gamma(a,x)$;
2. $\Gamma(a,x) = \Gamma(a) - \gamma(a,x)$;
3. $\Gamma(a,0) = \Gamma(a)$;

4. $\Gamma(a+1,z) = a\Gamma(a,z) + z^a e^{-z}$;

5. $\Gamma(a+1,z) = a\Gamma(a,z) + z^a e^{-z}$;

6. $\gamma(a+1,z) = a\gamma(a,z) - z^a e^{-z}$.

Proposition 1.3. *The recursive formula for the serial*

$$\gamma(a,z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{z^{a+n}}{a+n}$$

is given by

$$\gamma(a,z) = z^a \sum_{n=0}^{\infty} \frac{\alpha_n}{a+n}, \quad \text{with} \quad \alpha_0 = 1 \quad \text{and} \quad \alpha_n = \alpha_{n-1} \frac{(-z)}{n}.$$

Proof. We have

$$\gamma(s,z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{z^{s+n}}{s+n} = z^s \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{z^n}{s+n} = z^s \sum_{n=0}^{\infty} \frac{\alpha_n}{s+n},$$

for some constant g and $\mathrm{Re}(z) + g + \frac{1}{2} > 0$

$$b_i(g) = \frac{\sqrt{2} e^{g+\frac{1}{2}}}{\pi} \sum_{n=0}^{N} B_{i,n} \sum_{l=0}^{N} C_{n,l} \Gamma\left(l + \frac{1}{2}\right) \left(l + g + \frac{1}{2}\right)^{-\left(l+\frac{1}{2}\right)} e^l.$$

$$B_{i,j} := \begin{cases} 0, & i > j, \\ \frac{1}{2}, & i = j = 0, \\ 1, & i = 0, j > 0, \\ \frac{(-1)^{j-i+1}(i+j-1)!}{(j-i)!((n-1)!)^2}, & \text{otherwise}, \end{cases}$$

and

$$C_{i,j} := \begin{cases} 0, & i < j, \\ 1, & i = 0, j = 0, \\ \frac{(-1)^{i-j} i(i+j)! 4^j}{(i+1)(2j)!(i-j)!}, & \text{otherwise}. \end{cases}$$

■

For more details see [2].

1.4 Beta function

Definition 1.5. *(Euler 1730 & Jacques Binet 1839, (1786-1856)) The Beta function $B(.,.)$ or the Beta integral is the name used for the Eulerian integral of the first kind. The Beta function is usually defined by*

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad Re(x)>0, Re(y)>0.$$

In the following proposition we introduce the relationship between the Gamma function and the Beta function.

Proposition 1.4.

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Proof. We have

$$\begin{aligned}\Gamma(x)\Gamma(y) &= \int_0^\infty e^{-s}s^{(x-1)}ds\int_0^\infty e^{-t}t^{(y-1)}dt\\ &= \int_0^\infty\int_0^\infty e^{-(s+t)}s^{(x-1)}t^{(y-1)}dsdt.\end{aligned}$$

We use the change of variables

$$\begin{cases} u=s+t, \\ v=\dfrac{s}{s+t}, \end{cases} \Longrightarrow \begin{cases} s=uv, \\ t=u(1-v), \end{cases}$$

with the domain of integration $D=\{u,\ v\in\mathbb{R},\ u>0\ et\ 0<v<1\}$, we get

$$\begin{aligned}
\Gamma(x)\Gamma(y) &= \int\int_D (uv)^{(x-1)}[u(1-v)]^{(y-1)}e^{-u}|-u|dudv \\
&= \int\int_D u^{(x-1+y)}v^{(x-1)}(1-v)^{(y-1)}e^{-u}dudv \\
&= \int_0^\infty [\int_0^1 u^{(x-1+y)}v^{(x-1)}(1-v)^{(y-1)}e^{-u}du]dv \\
&= \int_0^\infty u^{(x-1+y)}e^{-u}du \int_0^1 v^{(x-1)}(1-v)^{(y-1)}dv \\
&= \Gamma(x+y)\int_0^1 v^{(x-1)}(1-v)^{(y-1)}dv \\
&= \Gamma(x+y)B(x,y).
\end{aligned}$$

■

Examples 1.2. *It is easy to see that*

1.

$$B(1,1) = 1.$$

2.

$$B(\frac{1}{3},\frac{2}{3}) = \frac{2\sqrt{3}}{3}\pi.$$

3.

$$B(\frac{1}{4},\frac{3}{4}) = \pi\sqrt{2}.$$

1.5 Mittag-Leffler function

The Mittag-Leffler function is an important special function with generalize of the exponential, it has many applications in mathematics sciences namely in fractional calculus. The Mittag-Leffler function is defind by

Definition 1.6. *(Mittag-Leffler, 1903)*

$$E_\alpha(x) = \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C} \textit{ and } \quad \mathrm{Re}(\alpha) > 0.$$

Definition 1.7. *(Wiman, 1905)*

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \quad \mathrm{Re}(\alpha) > 0 \quad \textit{and} \quad \mathrm{Re}(\beta) > 0.$$

The following results are inspired from [1].

Theorem 1.3. *The following integral formula holds:*

$$x^{\beta-1} E_{\alpha,\beta}(x^\alpha) = x^{\beta-1} E_{2\alpha,\beta}(x^{2\alpha}) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} E_{2\alpha,\beta}(\tau^{2\alpha}) \tau^{\beta-1} \, \mathrm{d}\tau \quad (\beta >$$

Proof. Letting

$$\gamma_\alpha(x,\tau) := \left[1 + \frac{(x-\tau)^\alpha}{\Gamma(\alpha+1)}\right].$$

On the one hand

$$\begin{aligned} \int_0^x E_{2\alpha,\beta}(\tau^{2\alpha}) \tau^{\beta-1} \gamma_\alpha(x,\tau) \, \mathrm{d}\tau &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(2k\alpha+\beta)} \int_0^x \tau^{2k\alpha+\beta-1} \gamma_\alpha(x,\tau) \, \mathrm{d}\tau \\ &= x^\beta \sum_{k=0}^{\infty} \left[\frac{x^{2k\alpha}}{\Gamma(2k\alpha+\beta+1)} + \frac{x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+\beta+} \right. \\ &= x^\beta \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+\beta+1)} \\ &= x^\beta E_{\alpha,\beta+1}(x^\alpha). \end{aligned}$$

On the other hand, using the definition and by term-by-term integration, we get

$$\int_0^x E_{\alpha,\beta}(\lambda \tau^\alpha) \tau^{\beta-1} \mathrm{d}\tau = x^\beta E_{\alpha,\beta+1}(\lambda x^\alpha), \quad \beta > 0.$$

Thus,

$$\int_0^x E_{2\alpha,\beta}(\tau^{2\alpha})\tau^{\beta-1}\left\{1+\frac{(x-\tau)^\alpha}{\Gamma(\alpha+1)}\right\}\mathrm{d}\tau=\int_0^x E_{\alpha,\beta}(\tau^\alpha)\tau^{\beta-1}\mathrm{d}\tau,\quad \beta>0.$$

By differentiation with respect to x, we get the desired result. ■

Let ς_1 and ς_2 two complex parameters. Define the following function

$$J_{\nu,\alpha,\beta,\varsigma_1,\varsigma_2}(\delta):=\int_0^\delta \tau^{\beta-1}E_{\alpha,\beta}(\varsigma_1\tau^\alpha)\,(\delta-\tau)^{\nu-1}E_{\alpha,\nu}(\varsigma_2(\delta-\tau)^\alpha)\,\mathrm{d}\tau,$$

Theorem 1.4. *The following integral formula holds for some* $\beta>0$ *and* $\nu>0$

$$J_{\nu,\alpha,\beta,\varsigma_1,\varsigma_2}(\delta)=\frac{\varsigma_1 E_{\alpha,\beta+\nu}(\delta^\alpha\varsigma_1)-\varsigma_2 E_{\alpha,\beta+\nu}(\delta^\alpha\varsigma_2)}{\varsigma_1-\varsigma_2}\,\delta^{\beta+\nu-1}$$

Proof. We have

$$\begin{aligned}
J_{\nu,\alpha,\beta,\varsigma_1,\varsigma_2}(\delta) &= \sum_{n,m=0}^{\infty}\frac{\varsigma_1^n\varsigma_2^m}{\Gamma(n\alpha+\beta)\Gamma(m\alpha+\nu)}\int_0^\delta \tau^{n\alpha+\beta-1}(\delta-\tau)^{m\alpha+\nu-1}\mathrm{d}\tau\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\varsigma_1^n\varsigma_2^m\delta^{(n+m)\alpha+\beta+\nu-1}}{\Gamma((m+n)\alpha+\beta+\nu)}\\
&= \delta^{\beta+\nu-1}\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{\varsigma_1^n(\varsigma_2)^{k-n}\,\delta^{k\alpha}}{\Gamma(k\alpha+\beta+\nu)}\\
&= \delta^{\beta+\nu-1}\sum_{k=0}^{\infty}\frac{\varsigma_2^k\,\delta^{k\alpha}}{\Gamma(k\alpha+\beta+\nu)}\sum_{n=0}^{k}\left(\frac{\varsigma_1}{\varsigma_2}\right)^n\\
&= \frac{\delta^{\beta+\nu-1}}{\varsigma_1-\varsigma_2}\sum_{k=0}^{\infty}\frac{\delta^{k\alpha}(\varsigma_1^{k+1}-\varsigma_2^{k+1})}{\Gamma(k\alpha+\beta+\nu)}.
\end{aligned}$$

By the definition of the generalized Mittag-Leffler function we obtain the desired result. ■

Denotting by $(\delta)_n$ the following Pochhammer symbol

$$(\delta)_n = \delta(\delta+1)\dots(\delta+n-1), \qquad \delta \neq 0.$$

Now, we give the second generalized Mittag-Leffler function

Definition 1.8.

$$E^{\delta}_{\beta,\gamma}(x) = \sum_{n=0}^{+\infty} \frac{(\delta)_n x^n}{\Gamma(\beta n+\gamma) n!}, \qquad \textit{such that} \quad \alpha,\, \beta,\, \gamma \in \mathbb{C} \quad \textit{and} \quad Re(\beta) > 0,$$

Let us consider the $B_p(x,y)$ the generalized Beta function defined by

$$B_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{\frac{-p}{t(t-1)}} dt.$$

We give the third generalized Mittag-Leffler function

Definition 1.9. *(Ozarslan, Yasar 2013)*

$$E^{\gamma,c}_{\alpha,\beta}(x,p) = \sum_{n=0}^{+\infty} \frac{B_p(\gamma+n, c-n)}{B_p(\gamma, c-n)} \frac{(c)_n x^n}{\Gamma(\beta n+\gamma) n!}, \qquad p > 0, \qquad Re(c) > 0, \quad Re(\gamma) >$$

Examples 1.3.

$$\begin{aligned} E_1(x) &= \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(k+1)} \\ &= e^x. \end{aligned}$$

$$\begin{aligned} E_2(x) &= \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(2k+1)} \\ &= \cos(x^2). \end{aligned}$$

$$\begin{aligned} E_{1,1}(x) &= \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(k+1)} \\ &= \sum_{k=0}^{+\infty} \frac{x^k}{k!} \\ &= e^x. \end{aligned}$$

Remark 1.6.

$$E_\beta(x) = E_{\beta,1}(x).$$

$$E_{\beta,1}(x) = E^1_{\beta,1}(x).$$

$$E_{\beta,\gamma}(x) = E^1_{\beta,\gamma}(x).$$

1.6 Mellin-Ross function

The Mellin-Ross function $E_t(\nu,a)$ appears in the fractional integral of the exponential function of type e^{at}, it is related to the incomplete Gamma function and the Mittag-Leffler function. Letting

$$\begin{aligned} \gamma^*(a,x) &:= \frac{x^{-a}}{\Gamma(a)}\gamma(a,x) \\ &= \frac{1}{\Gamma(a)x^2}\int_0^x e^{-t}t^{a-1}. \end{aligned}$$

Definition 1.10. *The Mellin-Ross function is defined by*

$$E_t(\nu,a) = t^\nu e^{at}\gamma^*(\nu,t),$$

we can also write

$$E_t(\nu,a) = t^\nu \sum_{k=0}^{+\infty} \frac{(at)^k}{\Gamma(\nu+k+1)} = t^\nu E_{1,1+\nu}(at).$$

1.7 Exercises

Exercise 1.1. 1. *Prove that*

$$\Gamma(x+1) = x\Gamma(x), \quad x > 0.$$

2. *Deduce that*

$$\Gamma(n+1) = n! \quad \textit{for all } n \in \mathbb{N}.$$

3. *Calculate* $\Gamma(5)$ *and* $\Gamma(7)$.

4. *Calculate* $\frac{\Gamma(\frac{22}{5})}{\Gamma(\frac{17}{5})}$.

5. *Calculate* $\frac{\Gamma(\frac{11}{4})}{\Gamma(\frac{23}{4})}$.

Exercise 1.2. 1. *Prove that*

$$\Gamma_n(x) = \frac{e^{x(\log n - 1 - 1/2 - \ldots - 1/n)} e^{x + x/2 + \ldots + x/n}}{x(1+x)(1+x/2)\ldots(1+x/n)}.$$

2. *Deduce that*

$$\Gamma_n(x) = e^{x(\log n - 1 - 1/2 - \ldots - 1/n)} \frac{1}{x} \frac{e^x}{1+x} \frac{e^{x/2}}{1+x/2} \times \ldots \times \frac{e^{x/n}}{1+x/n}.$$

3. *Prove that the sequence* $\gamma_n := 1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n} - \ln n$ *is decreasing.*

4. *Let us consider the Euler-Mascheron's constant* γ *given by*

$$\gamma := \lim_{n\to\infty} \left[1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n} - \ln n \right] = 0.57721\,56649\,01532\,86060\,\ldots.$$

Deduce the value of $\Gamma(x)$.

Exercise 1.3. *Let us consider the function* ψ *given by*

$$\psi(x) := \frac{\Gamma(x)}{\Gamma'(x)}.$$

It is well known that

$$\frac{1}{\psi(x)} = -\gamma - \frac{1}{x} + \sum_{n\geq 1}\left(\frac{1}{n} - \frac{1}{x+n}\right).$$

Prove that

$$\left(\frac{1}{\psi(x)}\right)' = \sum_{n\geq 0}\frac{1}{(x+n)^2}.$$

Exercise 1.4. *Prove that there are two constants* c_1 *and* c_2 *such that, for any positive real* R *large enough, we have*

$$\left|\frac{1}{\Gamma(z)}\right| \leq \begin{cases} c_1 R^R & \text{if} \quad \mathrm{Re}\,(z) > 1 \\ c_2 (R+1)^R & \text{if} \quad 0 < \mathrm{Re}\,(z) < 1. \end{cases}$$

Exercise 1.5. *Let us consider the function* $\xi(z) = 1/\Gamma(z)$*,* $z \in [1-i, 1+i]$. *Prove that*

$$|\xi(z)| \geq |x| e^{\gamma x} e^{-3} > \frac{1}{2}.$$

Exercise 1.6. *Using the integral formula to prove that*

$$\Gamma(z+1) = z\Gamma(z), \quad \text{for all } z \in \mathbb{C} \text{ with } Re(z) > 0.$$

Exercise 1.7. *Prove that for all* $x > 0$

1.
$$\Gamma(x) = \int_0^1 (-\log(u))^{x-1} du.$$

2.
$$\Gamma(x) = 2\int_0^{+\infty} e^{-u^2} u^{2x-1} du.$$

Exercise 1.8. 1. *Calculate the integral*

$$I = 2\int_0^{+\infty} e^{-u^2}\,du.$$

2. *Deduce the value of* $\Gamma\left(\frac{1}{2}\right)$.

3. *Deduce* $\Gamma(\frac{3}{2})$, $\Gamma(-\frac{1}{2})$ *and* $\Gamma(-\frac{3}{2})$.

Exercise 1.9. *Prove that*

$$\Gamma(n+\frac{1}{2}) = \frac{\sqrt{\pi}\,\Gamma(2n+1)}{2^{2n}\Gamma(n+1)}, \quad \text{for all} \quad n \in \mathbb{N}.$$

Exercise 1.10. *Let us consider the following function*

$$\varphi(s) := \int_0^1 \frac{u^{s-1}}{1+u}\,du.$$

1. *Prove that*

$$\Gamma(s)\Gamma(1-s) = \varphi(s) + \varphi(1-s).$$

2. *Prove that*

$$\varphi(s) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n+s}, \quad \text{pour tout} \quad s \in \mathbb{C}, \quad \text{with} \quad 0 < Re(s) < 1.$$

3. *Give the power series expansion for the function* $\Gamma(s)\Gamma(1-s)$.

4. *Letting*

$$a_n := \frac{2\sin \pi s}{\pi} . \frac{(-1)^n s}{s^2 - n^2}, \quad n \in \mathbb{N}.$$

Use the formula

$$\begin{aligned} \forall t \in]0,1[\quad \cos(st) &= \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos nt \\ &= \frac{\sin \pi s}{\pi s} + \sum_{n=1}^{+\infty} \frac{2\sin \pi s}{\pi}\frac{(-1)^n s}{s^2 - n^2}\cos nt \end{aligned}$$

to prove that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \quad s \in \mathbb{C} \quad \text{such that} \quad 0 < Re(s) < 1.$$

Exercise 1.11. *Using the formula*

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \quad x \geq 1$$

calculate $\Gamma\left(\frac{1}{2}\right)$, $\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})$ *and* $\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})$.

Exercise 1.12. *Let us consider the following integral*

$$J_n(z) = \int_0^1 s^{z-1}(1-s)^n \, \mathrm{d}s, \quad \mathrm{Re}(z) > 0.$$

1. *Prove that for all* $n \in \mathbb{N}$, *we have*

$$J_n(z) = \frac{n!}{z(z+1)\ldots(z+n)}.$$

2. *Prove that*

$$\Gamma(z) = \lim_{n \to +\infty} n^z J_n(z).$$

Exercise 1.13. *Let* $x, y \in \mathbb{C}$ *with* $Re(x) > 0$, $Re(y) > 0$. *Prove that*

1.

$$B(x,y) = B(y,x).$$

2.

$$B(x,y) = 2\int_0^{\frac{\pi}{2}} \sin^{(2x-1)}\theta \, \cos^{(2y-1)}\theta \, d\theta.$$

Exercise 1.14. *Let* $x, y \in \mathbb{C}$ *with* $Re(x) > 0$, $Re(y) > 0$.

1. *Prove that*

$$B(x+1,y) = \frac{x}{x+y}B(x,y).$$

2. *Deduce* $B(x, y+1)$.

3. *Deduce a relationship between the* $B(x,y)$ *and* $B(x-1,y)$.

Exercise 1.15. *Prove that*

$$\int_0^1 \frac{t\,dt}{\sqrt{1-t^5}} = \frac{1}{5}B(\frac{2}{5},\frac{1}{2}).$$

Exercise 1.16. *Prove that*

$$\int_0^1 \frac{t^2dt}{\sqrt{1-t^4}} = \frac{1}{4}B(\frac{3}{4},\frac{1}{2}).$$

Exercise 1.17. 1. *Calculate* $B(\frac{1}{2}$ *and* $\frac{1}{2})$.

2. *Deduce*

$$\int_{-\infty}^{+\infty} e^{-u^2}du.$$

Exercise 1.18. *Calculate* $E_{1,2}(z)$, $E_{1,3}(z)$ *and* $E_{2,1}(z)$, $z \in \mathbb{C}$, $\quad \mathrm{Re}\,(z) > 0$.

Exercise 1.19. *Prove that*

$$\beta E_{\alpha,\beta}(x) = xE_{\alpha,\alpha+\beta}(x) + \frac{1}{\Gamma(\beta)}.$$

Exercise 1.20. *Prove that*

$$E_{\alpha,\beta+1}(x) + \alpha x\frac{d}{dx}[E_{\alpha,\beta+1}(x)] = E_{\alpha,\beta}(x)$$

Exercise 1.21. *Let us consider the function* ϑ_α *given by*

$$\vartheta_\alpha(x) := 1 - E(-x^\alpha), \quad 0 < \alpha \leq 1, \quad x > 0.$$

Prove that

$$\frac{d}{dx}\vartheta_\alpha(x) = x^{\alpha-1}E_{\alpha,\alpha}(-x^\alpha).$$

Exercise 1.22. *Consider the generalized Pochhammer symbol defined by*

$$(\lambda,p)_n = \frac{1}{\Gamma(\lambda)}\int_0^{+\infty} t^{\lambda+k-1}e^{-t-\frac{p}{t}}dt,$$

and consider the generalization

$$E_{\beta,\gamma}^{\lambda,p}(x) = \sum_{n=0}^{+\infty} \frac{(\lambda;p)_n x^n}{\Gamma(\lambda)\Gamma(\beta n+\gamma)n!},$$

such that $Re(p) > 0, \quad Re(\lambda+n) > 0$ *for* $p = 0$.
Prove that

$$E_{\beta,\gamma}^{\lambda,p}(z) = \frac{1}{[\Gamma(\lambda)]^2} \int_0^{+\infty} t^{\lambda-1} e^{-t-\frac{p}{t}} E_{\beta,\gamma}(tz)dt.$$

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