Equation de Diffusion 2D

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Dr. Laïd MESSAOUDI

Département de Mécanique

Université de Batna

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EXAMEN

Détermination de la distribution de temperature $T(\mathrm{x})$ à travers une plaque de largeur L, de hauteur H et d'éapisseur e et de conductivité thermique k soumise aux (C.L.) suivantes.

$$\frac{\mathrm{d}}{\mathrm{dx}}\left(k\frac{\mathrm{d}}{\mathrm{dx}}T(\mathrm{x},\mathrm{y})\right)+\frac{\mathrm{d}}{\mathrm{dy}}\left(k\frac{\mathrm{d}}{\mathrm{dy}}T(\mathrm{x},\mathrm{y})\right)=0$$

Conditions aux limites (C.L):

$$\begin{aligned} T(0,y)&=T_0=100,\\ T(\mathrm{L},\mathrm{y})&=T_0=100,\\ -kA\frac{\mathrm{d}}{\mathrm{dx}}T(\mathrm{x},0)&=q=\frac{600\ kW}{m^2},\\ -kA\frac{\mathrm{d}}{\mathrm{dx}}T(\mathrm{x},\mathrm{H})&=q=\frac{600\ kW}{m^2}, \end{aligned}$$

Solution

```
> Restart : Digits := 4 :
> L := 1.0; H := 0.4; e := 0.001; λ := 1000; δx := 0.2; δy := 0.1;
```

$$L := 1.0$$
$$H := 0.4$$
$$e := 0.001$$

$$\lambda := 1000$$
$$\delta x := 0.2$$
$$\delta y := 0.1 \quad \textbf{(1.1)}$$

$> ndx := \frac{L}{\delta x}\ ;\ ndy := \frac{H}{\delta y}; Se := \delta y \cdot e; Ss := \delta x \cdot e;$

$$ndx := 5.000$$
$$ndy := 4.000$$
$$Se := 0.0001$$
$$Ss := 0.0002 \quad \textbf{(1.2)}$$

$> i_{max} := \text{round}(ndx);\ j_{max} := \text{round}(ndy);$

$$i_{max} := 5$$
$$j_{max} := 4 \quad \textbf{(1.3)}$$

Nombre d'équations:

$> Ne := i_{max} \cdot j_{max}$

$$Ne := 20 \quad \textbf{(1.4)}$$

Abscisses des noeuds:

$> x[0] := 0;$

for i **from** 1 **to** i_{max} **do**

$$x[i] := \frac{\delta x}{2} + (i-1) \cdot \delta x\ ;$$

end do;

$x[\ i_{max} + 1\ ] := L;$

$$x_0 := 0$$
$$x_1 := 0.1000$$
$$x_2 := 0.3000$$
$$x_3 := 0.5000$$
$$x_4 := 0.7000$$
$$x_5 := 0.9000$$
$$x_6 := 1.0 \quad \textbf{(1.5)}$$

Ordonnées des noeuds:

$> y[0] := 0;$

for j **from** 1 **to** j_{max} **do**

$$y[j] := \frac{\delta y}{2} + (j-1) \cdot \delta y\ ;$$

end do;

$y[\ j_{max} + 1\ ] := H;$

$$y_0 := 0$$
$$y_1 := 0.05000$$
$$y_2 := 0.1500$$
$$y_3 := 0.2500$$
$$y_4 := 0.3500$$
$$y_5 := 0.4 \quad \textbf{(1.6)}$$

Conditions aux Limites:

$> T0 := 100;\ q := 600000;$

$$T0 := 100$$
$$q := 600000 \quad \textbf{(1.7)}$$

Noeuds internes:

```
> for j from 2 to j_max - 1 do
  for i from 2 to i_max - 1 do
    Sp[i,j] := 0;
    Su[i,j] := 0;
    a_W[i,j] := λ·Se / δx;
    a_E[i,j] := a_W[i,j];
    a_S[i,j] := λ·Ss / δy;
    a_N[i,j] := a_S[i,j];
    a_P[i,j] := a_W[i,j] + a_E[i,j] + a_S[i,j] + a_N[i,j] - Sp[i,j];
  end do;
  end do;
```

Noeuds gauches:

```
> for j from 2 to j_max - 1 do
  Sp[1,j] := - 2·λ·Se / δx;
  Su[1 ,j] := 2·λ·Se / δx · T0;
  a_W[1,j] := 0;
  a_E[1,j] := λ·Se / δx;
  a_S[1,j] := λ·Ss / δy;
  a_N[1,j] := a_S[1,j];
  a_P[1,j] := a_W[1,j] + a_E[1,j] + a_S[1,j] + a_N[1,j] - Sp[1,j];
  end do;
```

$$Sp_{1,2} := -1.000$$
$$Su_{1,2} := 100.0$$
$$a_{W_{1,2}} := 0$$
$$a_{E_{1,2}} := 0.5000$$
$$a_{S_{1,2}} := 2.000$$
$$a_{N_{1,2}} := 2.000$$
$$a_{P_{1,2}} := 5.500$$
$$Sp_{1,3} := -1.000$$
$$Su_{1,3} := 100.0$$
$$a_{W_{1,3}} := 0$$
$$a_{E_{1,3}} := 0.5000$$
$$a_{S_{1,3}} := 2.000$$
$$a_{N_{1,3}} := 2.000$$
$$a_{P_{1,3}} := 5.500$$

(1.8)

Noeuds droits:

> **for** j **from** 2 **to** $j_{max} - 1$ **do**

$$Sp[i_{max}, j] := -\frac{2 \cdot \lambda \cdot Se}{\delta x};$$

$$Su[i_{max}, j] := \frac{2 \cdot \lambda \cdot Se}{\delta x} \cdot T0;$$

$$a_W[i_{max}, j] := \frac{\lambda \cdot Se}{\delta x};$$

$$a_E[i_{max}, j] := 0;$$

$$a_S[i_{max}, j] := \frac{\lambda \cdot Ss}{\delta y};$$

$$a_N[i_{max}, j] := a_S[i_{max}, j];$$

$$a_P[i_{max}, j] := a_W[i_{max}, j] + a_E[i_{max}, j] + a_S[i_{max}, j] + a_N[i_{max}, j] - Sp[i_{max}, j];$$

end do;

$$Sp_{5,2} := -1.000$$
$$Su_{5,2} := 100.0$$
$$a_{W_{5,2}} := 0.5000$$
$$a_{E_{5,2}} := 0$$
$$a_{S_{5,2}} := 2.000$$
$$a_{N_{5,2}} := 2.000$$
$$a_{P_{5,2}} := 5.500$$
$$Sp_{5,3} := -1.000$$
$$Su_{5,3} := 100.0$$
$$a_{W_{5,3}} := 0.5000$$
$$a_{E_{5,3}} := 0$$
$$a_{S_{5,3}} := 2.000$$
$$a_{N_{5,3}} := 2.000$$
$$a_{P_{5,3}} := 5.500 \qquad \textbf{(1.9)}$$

Noeuds bas:

> **for** i **from** 2 **to** $i_{max} - 1$ **do**

$$Sp[i, 1] := 0;$$

$$Su[i, 1] := q \cdot Ss;$$

$$a_W[i, 1] := \frac{\lambda \cdot Se}{\delta x};$$

$$a_E[i, 1] := a_W[i, 1];$$

$$a_S[i, 1] := 0;$$

$$a_N[i, 1] := \frac{\lambda \cdot Ss}{\delta y};$$

$$a_P[i, 1] := a_W[i, 1] + a_E[i, 1] + a_S[i, 1] + a_N[i, 1] - Sp[i, 1];$$

end do;

$$Sp_{2,1} := 0$$

$$Su_{2,1} := 120.0$$
$$a_{W_{2,1}} := 0.5000$$
$$a_{E_{2,1}} := 0.5000$$
$$a_{S_{2,1}} := 0$$
$$a_{N_{2,1}} := 2.000$$
$$a_{P_{2,1}} := 3.000$$
$$Sp_{3,1} := 0$$
$$Su_{3,1} := 120.0$$
$$a_{W_{3,1}} := 0.5000$$
$$a_{E_{3,1}} := 0.5000$$
$$a_{S_{3,1}} := 0$$
$$a_{N_{3,1}} := 2.000$$
$$a_{P_{3,1}} := 3.000$$
$$Sp_{4,1} := 0$$
$$Su_{4,1} := 120.0$$
$$a_{W_{4,1}} := 0.5000$$
$$a_{E_{4,1}} := 0.5000$$
$$a_{S_{4,1}} := 0$$
$$a_{N_{4,1}} := 2.000$$
$$a_{P_{4,1}} := 3.000 \qquad \textbf{(1.10)}$$

Noeuds hauts:

```
> for i from 2 to i_max - 1 do
  Sp[i, j_max] := 0;
  Su[i, j_max] := q·Ss;
  a_W[i, j_max] := λ·Se/δx;
  a_E[i, j_max] := a_W[i, j_max];
  a_S[i, j_max] := λ·Ss/δy;
  a_N[i, j_max] := 0;
  a_P[i, j_max] := a_W[i, j_max] + a_E[i, j_max] + a_S[i, j_max] + a_N[i, j_max] - Sp[i, j_max];
  end do;
```

$$Sp_{2,4} := 0$$
$$Su_{2,4} := 120.0$$
$$a_{W_{2,4}} := 0.5000$$
$$a_{E_{2,4}} := 0.5000$$
$$a_{S_{2,4}} := 2.000$$
$$a_{N_{2,4}} := 0$$

$$a_{P_{2,4}} := 3.000$$
$$Sp_{3,4} := 0$$
$$Su_{3,4} := 120.0$$
$$a_{W_{3,4}} := 0.5000$$
$$a_{E_{3,4}} := 0.5000$$
$$a_{S_{3,4}} := 2.000$$
$$a_{N_{3,4}} := 0$$
$$a_{P_{3,4}} := 3.000$$
$$Sp_{4,4} := 0$$
$$Su_{4,4} := 120.0$$
$$a_{W_{4,4}} := 0.5000$$
$$a_{E_{4,4}} := 0.5000$$
$$a_{S_{4,4}} := 2.000$$
$$a_{N_{4,4}} := 0$$
$$a_{P_{4,4}} := 3.000 \qquad \textbf{(1.11)}$$

Noeud (1,1):

$$> Sp[1,1] := -\frac{2\cdot\lambda\cdot Se}{\delta x};$$

$$Su[1,1] := \frac{2\cdot\lambda\cdot Se}{\delta x}\cdot T0 + q\cdot Ss;$$

$$a_W[1,1] := 0;$$

$$a_E[1,1] := \frac{\lambda\cdot Se}{\delta x};$$

$$a_S[1,1] := 0;$$

$$a_N[1,1] := \frac{\lambda\cdot Ss}{\delta y};$$

$$a_P[1,1] := a_W[1,1] + a_E[1,1] + a_S[1,1] + a_N[1,1] - Sp[1,1];$$

$$Sp_{1,1} := -1.000$$
$$Su_{1,1} := 220.0$$
$$a_{W_{1,1}} := 0$$
$$a_{E_{1,1}} := 0.5000$$
$$a_{S_{1,1}} := 0$$
$$a_{N_{1,1}} := 2.000$$
$$a_{P_{1,1}} := 3.500 \qquad \textbf{(1.12)}$$

Noeud (imax,1):

$$> Sp[i_{max},1] := -\frac{2\cdot\lambda\cdot Se}{\delta x};$$

$$Su[i_{max},1] := \frac{2\cdot\lambda\cdot Se}{\delta x}\cdot T0 + q\cdot Ss;$$

$$a_W[\,i_{max},1\,] := \frac{\lambda \cdot Se}{\delta x};$$
$$a_E[\,i_{max},1\,] := 0;$$
$$a_S[\,i_{max},1\,] := 0;$$
$$a_N[\,i_{max},1\,] := \frac{\lambda \cdot Ss}{\delta y};$$
$$a_P[\,i_{max},1\,] := a_W[\,i_{max},1\,] + a_E[\,i_{max},1\,] + a_S[\,i_{max},1\,] + a_N[\,i_{max},1\,] - Sp[\,i_{max},1\,];$$

$$Sp_{5,1} := -1.000$$
$$Su_{5,1} := 220.0$$
$$a_{W_{5,1}} := 0.5000$$
$$a_{E_{5,1}} := 0$$
$$a_{S_{5,1}} := 0$$
$$a_{N_{5,1}} := 2.000$$
$$a_{P_{5,1}} := 3.500 \qquad \textbf{(1.13)}$$

Noeud (1,jmax):

$$> Sp[1,j_{max}] := -\frac{2 \cdot \lambda \cdot Se}{\delta x};$$
$$Su[1,j_{max}] := \frac{2 \cdot \lambda \cdot Se}{\delta x} \cdot T0 + q \cdot Ss;$$
$$a_W[1,j_{max}] := 0;$$
$$a_E[1,j_{max}] := \frac{\lambda \cdot Se}{\delta x};$$
$$a_S[1,j_{max}] := \frac{\lambda \cdot Ss}{\delta y};$$
$$a_N[1,j_{max}] := 0;$$
$$a_P[1,j_{max}] := a_W[1,j_{max}] + a_E[1,j_{max}] + a_S[1,j_{max}] + a_N[1,j_{max}] - Sp[1,j_{max}];$$

$$Sp_{1,4} := -1.000$$
$$Su_{1,4} := 220.0$$
$$a_{W_{1,4}} := 0$$
$$a_{E_{1,4}} := 0.5000$$
$$a_{S_{1,4}} := 2.000$$
$$a_{N_{1,4}} := 0$$
$$a_{P_{1,4}} := 3.500 \qquad \textbf{(1.14)}$$

Noeud (imax,jmax):

$$> Sp[i_{max},j_{max}] := -\frac{2 \cdot \lambda \cdot Se}{\delta x};$$
$$Su[i_{max},j_{max}] := \frac{2 \cdot \lambda \cdot Se}{\delta x} \cdot T0 + q \cdot Ss;$$
$$a_W[i_{max},j_{max}] := \frac{\lambda \cdot Se}{\delta x};$$
$$a_E[i_{max},j_{max}] := 0;$$

$$a_S[i_{max}, j_{max}] := \frac{\lambda \cdot Ss}{\delta y};$$
$$a_N[i_{max}, j_{max}] := 0;$$
$$a_P[i_{max}, j_{max}] := a_W[i_{max}, j_{max}] + a_E[i_{max}, j_{max}] + a_S[i_{max}, j_{max}] + a_N[i_{max}, j_{max}] - Sp[i_{max}, j_{max}];$$

$$Sp_{5,4} := -1.000$$
$$Su_{5,4} := 220.0$$
$$a_{W_{5,4}} := 0.5000$$
$$a_{E_{5,4}} := 0$$
$$a_{S_{5,4}} := 2.000$$
$$a_{N_{5,4}} := 0$$
$$a_{P_{5,4}} := 3.500 \quad \textbf{(1.15)}$$

Equations:

> $k := 1$

$$k := 1 \quad \textbf{(1.1.1)}$$

Résolution pour les noeuds internes:

```
> for j from 1 to j_max do
  for i from 1 to i_max do
        Eq[k] := a_P[i,j]·T[i,j] = a_W[i,j]·T[i-1,j] + a_E[i,j]·T[i+1,j] + a_S[i,j]
    ·T[i,j-1] + a_N[i,j]·T[i,j+1] + Su[i,j];
        k := k + 1;
  end do;
  end do;
```

Ecriture du système d'équations:

```
> for k from 1 to Ne do Eq[k] end do;
```

$$3.500\,T_{1,1} = 220.0 + 0.5000\,T_{2,1} + 2.000\,T_{1,2}$$
$$3.000\,T_{2,1} = 0.5000\,T_{1,1} + 0.5000\,T_{3,1} + 120.0 + 2.000\,T_{2,2}$$
$$3.000\,T_{3,1} = 0.5000\,T_{2,1} + 0.5000\,T_{4,1} + 120.0 + 2.000\,T_{3,2}$$
$$3.000\,T_{4,1} = 0.5000\,T_{3,1} + 0.5000\,T_{5,1} + 120.0 + 2.000\,T_{4,2}$$
$$3.500\,T_{5,1} = 0.5000\,T_{4,1} + 220.0 + 2.000\,T_{5,2}$$
$$5.500\,T_{1,2} = 100.0 + 0.5000\,T_{2,2} + 2.000\,T_{1,1} + 2.000\,T_{1,3}$$
$$5.000\,T_{2,2} = 0.5000\,T_{1,2} + 0.5000\,T_{3,2} + 2.000\,T_{2,1} + 2.000\,T_{2,3}$$
$$5.000\,T_{3,2} = 0.5000\,T_{2,2} + 0.5000\,T_{4,2} + 2.000\,T_{3,1} + 2.000\,T_{3,3}$$
$$5.000\,T_{4,2} = 0.5000\,T_{3,2} + 0.5000\,T_{5,2} + 2.000\,T_{4,1} + 2.000\,T_{4,3}$$
$$5.500\,T_{5,2} = 0.5000\,T_{4,2} + 100.0 + 2.000\,T_{5,1} + 2.000\,T_{5,3}$$
$$5.500\,T_{1,3} = 100.0 + 0.5000\,T_{2,3} + 2.000\,T_{1,2} + 2.000\,T_{1,4}$$
$$5.000\,T_{2,3} = 0.5000\,T_{1,3} + 0.5000\,T_{3,3} + 2.000\,T_{2,2} + 2.000\,T_{2,4}$$
$$5.000\,T_{3,3} = 0.5000\,T_{2,3} + 0.5000\,T_{4,3} + 2.000\,T_{3,2} + 2.000\,T_{3,4}$$
$$5.000\,T_{4,3} = 0.5000\,T_{3,3} + 0.5000\,T_{5,3} + 2.000\,T_{4,2} + 2.000\,T_{4,4}$$
$$5.500\,T_{5,3} = 0.5000\,T_{4,3} + 100.0 + 2.000\,T_{5,2} + 2.000\,T_{5,4}$$
$$3.500\,T_{1,4} = 220.0 + 0.5000\,T_{2,4} + 2.000\,T_{1,3}$$
$$3.000\,T_{2,4} = 0.5000\,T_{1,4} + 0.5000\,T_{3,4} + 2.000\,T_{2,3} + 120.0$$

$$3.000\,T_{3,4} = 0.5000\,T_{2,4} + 0.5000\,T_{4,4} + 2.000\,T_{3,3} + 120.0$$
$$3.000\,T_{4,4} = 0.5000\,T_{3,4} + 0.5000\,T_{5,4} + 2.000\,T_{4,3} + 120.0$$
$$3.500\,T_{5,4} = 0.5000\,T_{4,4} + 220.0 + 2.000\,T_{5,3} \quad \textbf{(1.1.2)}$$

```
> Eqs := {seq(Eq[k], k = 1 ..Ne)};
```

$$Eqs := \{3.500\,T_{1,1} = 220.0 + 0.5000\,T_{2,1} + 2.000\,T_{1,2},\ 5.500\,T_{1,2} = 100.0 + 0.5000\,T_{2,2} + 2.000\,T_{1,1} + 2.000\,T_{1,3},\ 5.500\,T_{1,3} = 100.0 + 0.5000\,T_{2,3} + 2.000\,T_{1,2} + 2.000\,T_{1,4},\ 3.500\,T_{1,4} = 220.0 + 0.5000\,T_{2,4} + 2.000\,T_{1,3},\ 3.000\,T_{2,1} = 0.5000\,T_{1,1} + 0.5000\,T_{3,1} + 120.0 + 2.000\,T_{2,2},\ 5.000\,T_{2,2} = 0.5000\,T_{1,2} + 0.5000\,T_{3,2} + 2.000\,T_{2,1} + 2.000\,T_{2,3},\ 5.000\,T_{2,3} = 0.5000\,T_{1,3} + 0.5000\,T_{3,3} + 2.000\,T_{2,2} + 2.000\,T_{2,4},\ 3.000\,T_{2,4} = 0.5000\,T_{1,4} + 0.5000\,T_{3,4} + 2.000\,T_{2,3} + 120.0,\ 3.000\,T_{3,1} = 0.5000\,T_{2,1} + 0.5000\,T_{4,1} + 120.0 + 2.000\,T_{3,2},\ 5.000\,T_{3,2} = 0.5000\,T_{2,2} + 0.5000\,T_{4,2} + 2.000\,T_{3,1} + 2.000\,T_{3,3},\ 5.000\,T_{3,3} = 0.5000\,T_{2,3} + 0.5000\,T_{4,3} + 2.000\,T_{3,2} + 2.000\,T_{3,4},\ 3.000\,T_{3,4} = 0.5000\,T_{2,4} + 0.5000\,T_{4,4} + 2.000\,T_{3,3} + 120.0,\ 3.000\,T_{4,1} = 0.5000\,T_{3,1} + 0.5000\,T_{5,1} + 120.0 + 2.000\,T_{4,2},\ 5.000\,T_{4,2} = 0.5000\,T_{3,2} + 0.5000\,T_{5,2} + 2.000\,T_{4,1} + 2.000\,T_{4,3},\ 5.000\,T_{4,3} = 0.5000\,T_{3,3} + 0.5000\,T_{5,3} + 2.000\,T_{4,2} + 2.000\,T_{4,4},\ 3.000\,T_{4,4} = 0.5000\,T_{3,4} + 0.5000\,T_{5,4} + 2.000\,T_{4,3} + 120.0,\ 3.500\,T_{5,1} = 0.5000\,T_{4,1} + 220.0 + 2.000\,T_{5,2},\ 5.500\,T_{5,2} = 0.5000\,T_{4,2} + 100.0 + 2.000\,T_{5,1} + 2.000\,T_{5,3},\ 5.500\,T_{5,3} = 0.5000\,T_{4,3} + 100.0 + 2.000\,T_{5,2} + 2.000\,T_{5,4},\ 3.500\,T_{5,4} = 0.5000\,T_{4,4} + 220.0 + 2.000\,T_{5,3}\} \quad \textbf{(1.1.3)}$$

```
> SolT := solve(Eqs);
```

$$SolT := \{T_{1,1} = 262.2,\ T_{1,2} = 237.8,\ T_{1,3} = 237.8,\ T_{1,4} = 262.2,\ T_{2,1} = 444.7,\ T_{2,2} = 415.3,\ T_{2,3} = 415.3,\ T_{2,4} = 444.7,\ T_{3,1} = 504.9,\ T_{3,2} = 475.1,\ T_{3,3} = 475.1,\ T_{3,4} = 504.9,\ T_{4,1} = 444.7,\ T_{4,2} = 415.3,\ T_{4,3} = 415.3,\ T_{4,4} = 444.7,\ T_{5,1} = 262.2,\ T_{5,2} = 237.8,\ T_{5,3} = 237.8,\ T_{5,4} = 262.2\} \quad \textbf{(1.1.4)}$$

```
> Tmps := [seq(lhs(SolT_i), i = 1 ..Ne)];
```

$$Tmps := [T_{1,1}, T_{1,2}, T_{1,3}, T_{1,4}, T_{2,1}, T_{2,2}, T_{2,3}, T_{2,4}, T_{3,1}, T_{3,2}, T_{3,3}, T_{3,4}, T_{4,1}, T_{4,2}, T_{4,3}, T_{4,4}, T_{5,1}, T_{5,2}, T_{5,3}, T_{5,4}] \quad \textbf{(1.1.5)}$$

```
> with(LinearAlgebra) :
```

Forme matricielle:

```
> A, b := GenerateMatrix(Eqs, Tmps)
```

$$A, b := \begin{bmatrix} 20\ x\ 20\ Matrix \\ Data\ Type:\ anything \\ Storage:\ rectangular \\ Order:\ Fortran\_order \end{bmatrix}, \begin{bmatrix} 1\ ..\ 20\ Vector_{column} \\ Data\ Type:\ anything \\ Storage:\ rectangular \\ Order:\ Fortran\_order \end{bmatrix} \quad \textbf{(1.1.6)}$$

```
> b[8]
```

$$120.0 \quad \textbf{(1.1.7)}$$

```
> `?`
```

$$`?` \quad \textbf{(1.1.8)}$$

```
>
```