Equation de Laplace 2D

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Détermination de la temperature $T(x, y)$ à travers la surface d'une plaque rectangulaire (a x b) dont les extrémités sont soumises à des (C.L.) de Dirichlet.

$$\frac{\partial^2}{\partial x^2} T(x, y) + \frac{\partial^2}{\partial y^2} T(x, y) = 0$$

Conditions aux limites (C.L):

$$T(x, 0) = 0,$$
$$T(x, b) = 100,$$
$$T(0, y) = 60,$$
$$T(a, y) = 20.$$

Maillage et conditions aux limites:

> $Restart:$

> $a := 1; b := 1; ndx := 3; ndy := 3$

$$a := 1$$
$$b := 1$$
$$ndx := 3$$
$$ndy := 3 \qquad \textbf{(1.1)}$$

> $\Delta x := \frac{a}{ndx}; \Delta y := \frac{b}{ndy}; \beta := \frac{\Delta x}{\Delta y};$

$$\Delta x := \frac{1}{3}$$
$$\Delta y := \frac{1}{3}$$
$$\beta := 1 \quad \textbf{(1.2)}$$

> $i_{\max} := ndx + 1;\ j_{\max} := ndy + 1;$

$$i_{max} := 4$$
$$j_{max} := 4 \quad \textbf{(1.3)}$$

Nombre d'équations:

> $N := (i_{\max} - 2) \cdot (j_{\max} - 2)$

$$N := 4 \quad \textbf{(1.4)}$$

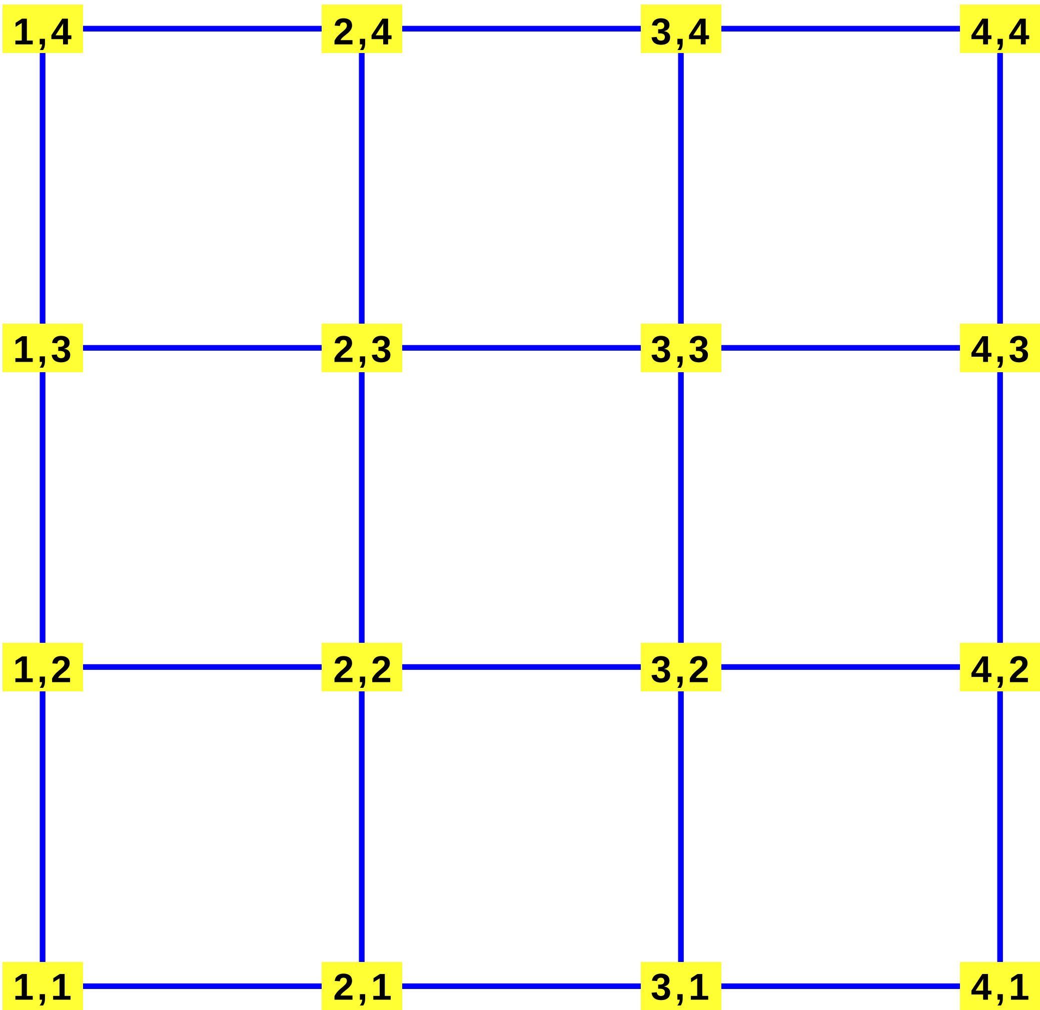
Maillage:

> *with(GraphTheory) : with(SpecialGraphs) :*

> $G := GridGraph(i_{\max}, j_{\max})$

G := Graph 2: an undirected unweighted graph with 16 vertices and 24 edge(s) **(1.5)**

> *DrawGraph(G)*



Conditions aux Limites:

> $Tb := 0.;\ Tg := 60;\ Td := 20;\ Th := 100.;$

$$Tb := 0.$$
$$Tg := 60$$
$$Td := 20$$
$$Th := 100. \quad \textbf{(1.6)}$$

> for i from 2 to $i_{max} - 1$ do $T[i, 1] := Tb$ end do; $T[1, 1] := 0.5\cdot(Tb + Tg)$; $T[i_{max}, 1] := 0.5\cdot(Tb + Td)$;

$$T_{2,1} := 0.$$
$$T_{3,1} := 0.$$
$$T_{1,1} := 30.0$$
$$T_{4,1} := 10.0 \quad \textbf{(1.7)}$$

> for i from 2 to $i_{max} - 1$ do $T[i, j_{max}] := Th$ end do; $T[1, j_{max}] := 0.5\cdot(Tg + Th)$; $T[i_{max}, j_{max}] := 0.5\cdot(Td + Th)$;

$$T_{2,4} := 100.$$
$$T_{3,4} := 100.$$
$$T_{1,4} := 80.0$$
$$T_{4,4} := 60.0 \quad \textbf{(1.8)}$$

> for j from 2 to $j_{max} - 1$ do $T[1, j] := Tg$ end do;

$$T_{1,2} := 60$$
$$T_{1,3} := 60 \quad \textbf{(1.9)}$$

> for j from 2 to $j_{max} - 1$ do $T[i_{max}, j] := Td$ end do;

$$T_{4,2} := 20$$
$$T_{4,3} := 20 \quad \textbf{(1.10)}$$

> $k := 1$:

Solution discrétisée par la formulation à 5 points:

Résolution pour les noeuds internes:

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> for j from 2 to j_max - 1 do
    for i from 2 to i_max -1 do
       Eq[k] := T[i+1, j] + T[i-1, j] + β^2·(T[i, j+1] + T[i, j-1]) - 2·(1 + β^2)
   ·T[i, j] = 0;
       Temps[k] := T[i, j];
    k := k + 1
    end do;
  end do;
```

Ecriture du système d'équations:

> for k from 1 to N do $Eq[k]$ end do;

$$T_{3,2} + 60 + T_{2,3} - 4\,T_{2,2} = 0$$
$$20 + T_{2,2} + T_{3,3} - 4\,T_{3,2} = 0$$
$$T_{3,3} + 160. + T_{2,2} - 4\,T_{2,3} = 0$$
$$120. + T_{2,3} + T_{3,2} - 4\,T_{3,3} = 0 \quad \textbf{(2.1)}$$

> $Eqs := \{seq(Eq[i], i = 1..N)\}$:

> $Tmps := [seq(Temps[i], i = 1..N)]$;

$$Tmps := [T_{2,2}, T_{3,2}, T_{2,3}, T_{3,3}] \quad \textbf{(2.2)}$$

> $SolT := solve(Eqs, Tmps)$;

$$SolT := [[T_{2,2} = 37.50000000, T_{3,2} = 27.50000000, T_{2,3} = 62.50000000, T_{3,3} = 52.50000000]] \quad \textbf{(2.3)}$$

> $Solution := evalf(SolT)$;

(2.4)

$Solution := [[T_{2,2} = 37.50000000, T_{3,2} = 27.50000000, T_{2,3} = 62.50000000, T_{3,3} = 52.50000000]]$ **(2.4)**

> $Sys := [seq(Eq[i], i = 1..N)];$

$Sys := [T_{3,2} + 60 + T_{2,3} - 4\,T_{2,2} = 0, 20 + T_{2,2} + T_{3,3} - 4\,T_{3,2} = 0, T_{3,3} + 160. + T_{2,2} - 4\,T_{2,3} = 0, 120. + T_{2,3} + T_{3,2} - 4\,T_{3,3} = 0]$ **(2.5)**

> $Var := [seq(Temps[i], i = 1..N)];$

$$Var := [T_{2,2}, T_{3,2}, T_{2,3}, T_{3,3}] \quad \textbf{(2.6)}$$

> $with(LinearAlgebra):$

> $A, b := GenerateMatrix(Sys, Var);$

$$A, b := \begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix}, \begin{bmatrix} -60 \\ -20 \\ -160. \\ -120. \end{bmatrix} \quad \textbf{(2.7)}$$

Solution discrétisée par la formulation à 9 points:

Résolution pour les noeuds internes:

> **for** j **from** 2 **to** $j_{max} - 1$ **do**

for i **from** 2 **to** $i_{max} - 1$ **do**

$$Eq[k] := T[i+1, j+1] + T[i+1, j-1] + T[i-1, j+1] + T[i-1, j-1] + 2\cdot\frac{5-\beta^2}{1+\beta^2}\cdot(T[i+1, j] + T[i-1, j]) + 2\cdot\frac{5\cdot\beta^2 - 1}{1+\beta^2}\cdot(T[i, j+1] + T[i, j-1]) - 20\cdot T[i, j] = 0;$$

$Temps[k] := T[i, j];$

$k := k + 1$

end do;

end do;

Ecriture du système d'équations:

> **for** k **from** 1 **to** N **do** $Eq[k]$ **end do**;

$$T_{3,3} + 330.0 + 4\,T_{3,2} + 4\,T_{2,3} - 20\,T_{2,2} = 0$$
$$110.0 + T_{2,3} + 4\,T_{2,2} + 4\,T_{3,3} - 20\,T_{3,2} = 0$$
$$880.0 + T_{3,2} + 4\,T_{3,3} + 4\,T_{2,2} - 20\,T_{2,3} = 0$$
$$660.0 + T_{2,2} + 4\,T_{2,3} + 4\,T_{3,2} - 20\,T_{3,3} = 0 \quad \textbf{(3.1)}$$

> $Eqs := \{seq(Eq[i], i = 1..N)\}:$

> $Tmps := [seq(Temps[i], i = 1..N)];$

$$Tmps := [T_{2,2}, T_{3,2}, T_{2,3}, T_{3,3}] \quad \textbf{(3.2)}$$

> $SolT := solve(Eqs, Tmps);$

$SolT := [[T_{2,2} = 37.14285714, T_{3,2} = 26.66666667, T_{2,3} = 63.33333333, T_{3,3} = 52.85714286]]$ **(3.3)**

> $Solution := evalf(SolT);$

$Solution := [[T_{2,2} = 37.14285714, T_{3,2} = 26.66666667, T_{2,3} = 63.33333333, T_{3,3} = 52.85714286]]$ **(3.4)**

> $Sys := [seq(Eq[i], i = 1..N)];$

$Sys := [T_{3,3} + 330.0 + 4\,T_{3,2} + 4\,T_{2,3} - 20\,T_{2,2} = 0, 110.0 + T_{2,3} + 4\,T_{2,2} + 4\,T_{3,3}$ **(3.5)**

$-20\ T_{3,\,2}=0,\ 880.0+T_{3,\,2}+4\ T_{3,\,3}+4\ T_{2,\,2}-20\ T_{2,\,3}=0,\ 660.0+T_{2,\,2}+4\ T_{2,\,3}$
$+4\ T_{3,\,2}-20\ T_{3,\,3}=0\,]$

> *Var* := [*seq*(*Temps*[i], i = 1 ..N)];

$$Var := [T_{2,\,2},\ T_{3,\,2},\ T_{2,\,3},\ T_{3,\,3}] \tag{3.6}$$

> *with*(*LinearAlgebra*) :

> *A*, *b* := *GenerateMatrix*(*Sys*, *Var*);

$$A, b := \begin{bmatrix} -20 & 4 & 4 & 1 \\ 4 & -20 & 1 & 4 \\ 4 & 1 & -20 & 4 \\ 1 & 4 & 4 & -20 \end{bmatrix}, \begin{bmatrix} -330.0 \\ -110.0 \\ -880.0 \\ -660.0 \end{bmatrix} \tag{3.7}$$

>