Equation de Laplace 2D

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Détermination de la temperature $T(x, y)$ à travers la surface d'une plaque rectangulaire (a x b) dont les extrémités sont soumises à des (C.L.) de Dirichlet.

$$\frac{\partial^2}{\partial x^2} T(x, y) + \frac{\partial^2}{\partial y^2} T(x, y) = 0$$

Conditions aux limites (C.L):

$$T(x, 0) = 0,$$
$$T(x, b) = 100,$$
$$T(0, y) = 75,$$
$$T(a, y) = 50.$$

Solution discrétisée (formulation en 9 points):

```
> Restart:
> a := 1; b := 1; ndx := 4; ndy := 4
```

$$a := 1$$
$$b := 1$$
$$ndx := 4$$
$$ndy := 4$$

(1.1)

```
> Δx := a/ndx ; Δy := b/ndy; β := Δx/Δy;
```

$$\Delta x := \frac{1}{4}$$
$$\Delta y := \frac{1}{4}$$

(1.2)

$$\beta := 1 \tag{1.2}$$

> $i_{max} := ndx + 1;\ j_{max} := ndy + 1;$

$$i_{max} := 5$$
$$j_{max} := 5 \tag{1.3}$$

Nombre d'équations:

> $N := (i_{max} - 2) \cdot (j_{max} - 2)$

$$N := 9 \tag{1.4}$$

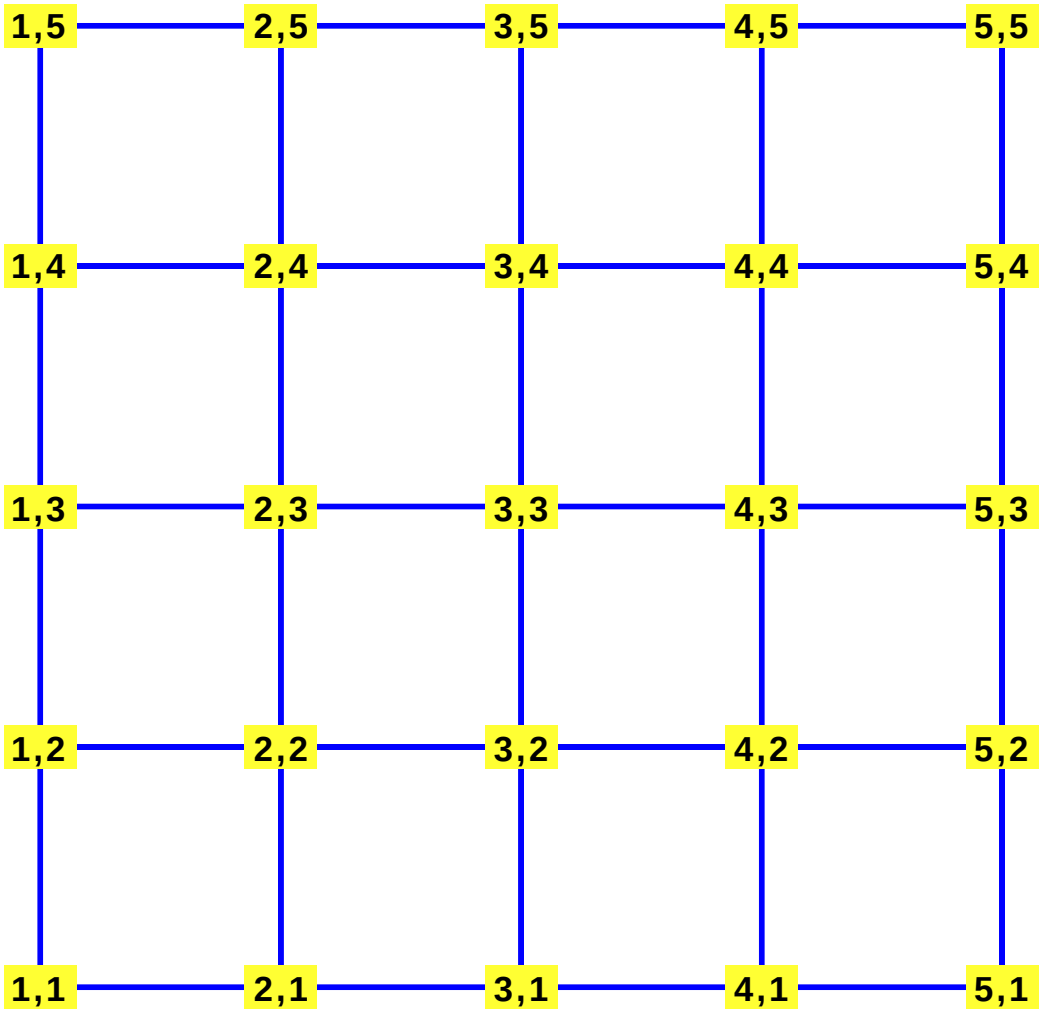
Maillage:

> $with(GraphTheory) : with(SpecialGraphs) :$

> $G := GridGraph(i_{max}, j_{max})$

$$G := Graph\ 1:\ an\ undirected\ unweighted\ graph\ with\ 25\ vertices\ and\ 40\ edge(s) \tag{1.5}$$

> $DrawGraph(G)$



Conditions aux Limites:

> $Tb := 0.;\ Tg := 75;\ Td := 50;\ Th := 100.;$

$$Tb := 0.$$
$$Tg := 75$$
$$Td := 50$$
$$Th := 100. \tag{1.6}$$

> **for** i **from** 2 **to** $i_{max} - 1$ **do** $T[i, 1] := Tb$ **end do**; $T[1, 1] := 0.5 \cdot (Tb + Tg);\ T[i_{max}, 1] := 0.5 \cdot (Tb + Td);$

$$T_{2,1} := 0.$$

$$T_{3,1} := 0.$$
$$T_{4,1} := 0.$$
$$T_{1,1} := 37.5$$
$$T_{5,1} := 25.0 \quad \textbf{(1.7)}$$

> **for** i **from** 2 **to** $i_{max} - 1$ **do** $T[i, j_{max}] := Th$ **end do**; $T[1, j_{max}] := 0.5 \cdot (Tg + Th)$; $T[i_{max}, j_{max}] := 0.5 \cdot (Td + Th)$;

$$T_{2,5} := 100.$$
$$T_{3,5} := 100.$$
$$T_{4,5} := 100.$$
$$T_{1,5} := 87.5$$
$$T_{5,5} := 75.0 \quad \textbf{(1.8)}$$

> **for** j **from** 2 **to** $j_{max} - 1$ **do** $T[1, j] := Tg$ **end do**;

$$T_{1,2} := 75$$
$$T_{1,3} := 75$$
$$T_{1,4} := 75 \quad \textbf{(1.9)}$$

> **for** j **from** 2 **to** $j_{max} - 1$ **do** $T[i_{max}, j] := Td$ **end do**;

$$T_{5,2} := 50$$
$$T_{5,3} := 50$$
$$T_{5,4} := 50 \quad \textbf{(1.10)}$$

> $k := 1$

$$k := 1 \quad \textbf{(1.1.1)}$$

Résolution pour les noeuds internes:

```
> for j from 2 to j_max - 1 do
    for i from 2 to i_max - 1 do
      Eq[k] := T[i+1, j+1] + T[i+1, j-1] + T[i-1, j+1] + T[i-1, j-1]
      + 2·((5 - β^2)/(1 + β^2))·(T[i+1, j] + T[i-1, j]) + 2·((5·β^2 - 1)/(1 + β^2))·(T[i, j+1] + T[i, j
      - 1]) - 20·T[i, j] = 0;
      Temps[k] := T[i, j];
    k := k + 1
    end do;
  end do;
```

Ecriture du système d'équations:

> **for** k **from** 1 **to** N **do** $Eq[k]$ **end do**;

$$T_{3,3} + 412.5 + 4\,T_{3,2} + 4\,T_{2,3} - 20\,T_{2,2} = 0$$
$$T_{4,3} + T_{2,3} + 4\,T_{4,2} + 4\,T_{2,2} + 4\,T_{3,3} - 20\,T_{3,2} = 0$$
$$275.0 + T_{3,3} + 4\,T_{3,2} + 4\,T_{4,3} - 20\,T_{4,2} = 0$$
$$T_{3,4} + T_{3,2} + 450 + 4\,T_{3,3} + 4\,T_{2,4} + 4\,T_{2,2} - 20\,T_{2,3} = 0$$
$$T_{4,4} + T_{4,2} + T_{2,4} + T_{2,2} + 4\,T_{4,3} + 4\,T_{2,3} + 4\,T_{3,4} + 4\,T_{3,2} - 20\,T_{3,3} = 0$$
$$300 + T_{3,4} + T_{3,2} + 4\,T_{3,3} + 4\,T_{4,4} + 4\,T_{4,2} - 20\,T_{4,3} = 0$$
$$962.5 + T_{3,3} + 4\,T_{3,4} + 4\,T_{2,3} - 20\,T_{2,4} = 0$$
$$600. + T_{4,3} + T_{2,3} + 4\,T_{4,4} + 4\,T_{2,4} + 4\,T_{3,3} - 20\,T_{3,4} = 0$$

(1.1.2)

$$825.0 + T_{3,3} + 4\,T_{3,4} + 4\,T_{4,3} - 20\,T_{4,4} = 0 \tag{1.1.2}$$

```
> Eqs := {seq(Eq[i], i = 1 ..N)} :
> Tmps := [seq(Temps[i], i = 1 ..N)];
```

$$Tmps := [T_{2,2}, T_{3,2}, T_{4,2}, T_{2,3}, T_{3,3}, T_{4,3}, T_{2,4}, T_{3,4}, T_{4,4}] \tag{1.1.3}$$

```
> SolT := solve(Eqs, Tmps);
```

$$SolT := [[T_{2,2} = 42.59510870,\ T_{3,2} = 32.26284585,\ T_{4,2} = 33.49184783,\ T_{2,3} = 63.52519763,\ T_{3,3} = 56.25000000,\ T_{4,3} = 52.38389328,\ T_{2,4} = 79.00815217,\ T_{3,4} = 76.82806324,\ T_{4,4} = 69.90489130]] \tag{1.1.4}$$

```
> Solution := evalf(SolT);
```

$$Solution := [[T_{2,2} = 42.59510870,\ T_{3,2} = 32.26284585,\ T_{4,2} = 33.49184783,\ T_{2,3} = 63.52519763,\ T_{3,3} = 56.25000000,\ T_{4,3} = 52.38389328,\ T_{2,4} = 79.00815217,\ T_{3,4} = 76.82806324,\ T_{4,4} = 69.90489130]] \tag{1.1.5}$$

```
> Sys := [seq(Eq[i], i = 1 ..N)];
```

$$Sys := [T_{3,3} + 412.5 + 4\,T_{3,2} + 4\,T_{2,3} - 20\,T_{2,2} = 0,\ T_{4,3} + T_{2,3} + 4\,T_{4,2} + 4\,T_{2,2} + 4\,T_{3,3} - 20\,T_{3,2} = 0,\ 275.0 + T_{3,3} + 4\,T_{3,2} + 4\,T_{4,3} - 20\,T_{4,2} = 0,\ T_{3,4} + T_{3,2} + 450 + 4\,T_{3,3} + 4\,T_{2,4} + 4\,T_{2,2} - 20\,T_{2,3} = 0,\ T_{4,4} + T_{4,2} + T_{2,4} + T_{2,2} + 4\,T_{4,3} + 4\,T_{2,3} + 4\,T_{3,4} + 4\,T_{3,2} - 20\,T_{3,3} = 0,\ 300 + T_{3,4} + T_{3,2} + 4\,T_{3,3} + 4\,T_{4,4} + 4\,T_{4,2} - 20\,T_{4,3} = 0,\ 962.5 + T_{3,3} + 4\,T_{3,4} + 4\,T_{2,3} - 20\,T_{2,4} = 0,\ 600. + T_{4,3} + T_{2,3} + 4\,T_{4,4} + 4\,T_{2,4} + 4\,T_{3,3} - 20\,T_{3,4} = 0,\ 825.0 + T_{3,3} + 4\,T_{3,4} + 4\,T_{4,3} - 20\,T_{4,4} = 0] \tag{1.1.6}$$

```
> Var := [seq(Temps[i], i = 1 ..N)];
```

$$Var := [T_{2,2}, T_{3,2}, T_{4,2}, T_{2,3}, T_{3,3}, T_{4,3}, T_{2,4}, T_{3,4}, T_{4,4}] \tag{1.1.7}$$

```
> with(LinearAlgebra) :
> A, b := GenerateMatrix(Sys, Var);
```

$$A, b := \begin{bmatrix} -20 & 4 & 0 & 4 & 1 & 0 & 0 & 0 & 0 \\ 4 & -20 & 4 & 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & -20 & 0 & 1 & 4 & 0 & 0 & 0 \\ 4 & 1 & 0 & -20 & 4 & 0 & 4 & 1 & 0 \\ 1 & 4 & 1 & 4 & -20 & 4 & 1 & 4 & 1 \\ 0 & 1 & 4 & 0 & 4 & -20 & 0 & 1 & 4 \\ 0 & 0 & 0 & 4 & 1 & 0 & -20 & 4 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 & 4 & -20 & 4 \\ 0 & 0 & 0 & 0 & 1 & 4 & 0 & 4 & -20 \end{bmatrix}, \begin{bmatrix} -412.5 \\ 0 \\ -275.0 \\ -450 \\ 0 \\ -300 \\ -962.5 \\ -600. \\ -825.0 \end{bmatrix} \tag{1.1.8}$$