

Equation de Laplace 2D

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Détermination de la température $T(x, y)$ à travers la surface d'une plaque rectangulaire ($a \times b$) dont les 3 extrémités sont soumises à des (C.L.) de Dirichlet et la quatrième à une conditions de Neumann.

$$\frac{\partial^2}{\partial x^2} T(x, y) + \frac{\partial^2}{\partial y^2} T(x, y) = 0$$

Conditions aux limites (C.L.):

$$\begin{aligned} T(x, 0) &= 0, \\ T(x, b) &= 100, \\ T(0, y) &= 0, \\ \frac{\partial}{\partial x} T(a, y) &= 0. \end{aligned}$$

Solution discrétisée par la formulation à 5 points:

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> Restart :
> a := 5 : b := 15 : ndx := 10 : ndy := 30 :
> beta := 1. :
> i_max := ndx + 1; j_max := ndy + 1;
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$$i_{max} := 11$$

$$j_{max} := 31$$

Nombre d'équations:

(1.1)

$$> N := (i_{\max} - 2) \cdot (j_{\max} - 2)$$

$$N := 261$$

(1.2)

Maillage:

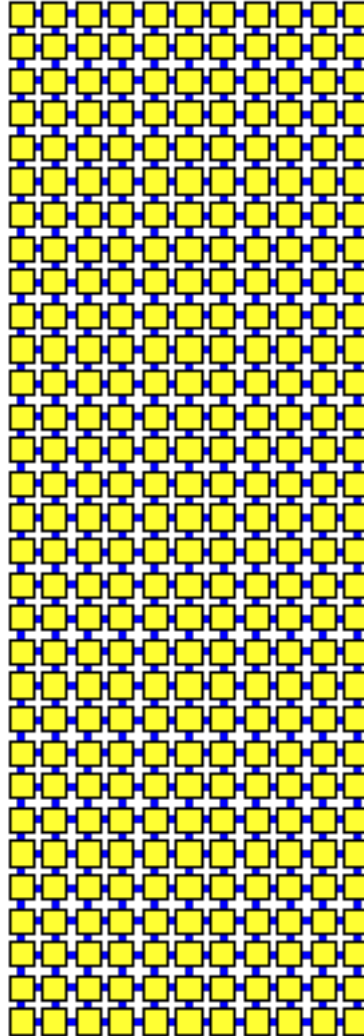
> with(GraphTheory) : with(SpecialGraphs) :

> G := GridGraph(i_{\max}, j_{\max})

G := Graph 1: an undirected unweighted graph with 341 vertices and 640 edge(s)

(1.3)

> DrawGraph(G)



Conditions aux Limites:

> for i from 1 to i_{\max} do $T[i, 1] := 0$ end do;

$$T_{1,1} := 0$$

$$T_{2,1} := 0$$

$$T_{3,1} := 0$$

$$T_{4,1} := 0$$

$$T_{5,1} := 0$$

$$T_{6,1} := 0$$

$$T_{7,1} := 0$$

$$T_{8,1} := 0$$

$$T_{9,1} := 0$$

$$T_{10,1} := 0$$

$$T_{11,1} := 0$$

(1.4)

> for i from 1 to i_{\max} do $T[i, j_{\max}] := 100$ end do;

$T_{1, 31} := 100$

$T_{2, 31} := 100$

$T_{3, 31} := 100$

$T_{4, 31} := 100$

$T_{5, 31} := 100$

$T_{6, 31} := 100$

$T_{7, 31} := 100$

$T_{8, 31} := 100$

$T_{9, 31} := 100$

$T_{10, 31} := 100$

$T_{11, 31} := 100$

(1.5)

> for j from 1 to j_{\max} do $T[1, j] := 0$ end do;

$T_{1, 1} := 0$

$T_{1, 2} := 0$

$T_{1, 3} := 0$

$T_{1, 4} := 0$

$T_{1, 5} := 0$

$T_{1, 6} := 0$

$T_{1, 7} := 0$

$T_{1, 8} := 0$

$T_{1, 9} := 0$

$T_{1, 10} := 0$

$T_{1, 11} := 0$

$T_{1, 12} := 0$

$T_{1, 13} := 0$

$T_{1, 14} := 0$

$T_{1, 15} := 0$

$T_{1, 16} := 0$

$T_{1, 17} := 0$

$T_{1, 18} := 0$

$T_{1, 19} := 0$

$T_{1, 20} := 0$

$T_{1, 21} := 0$

$T_{1, 22} := 0$

$T_{1, 23} := 0$

$T_{1, 24} := 0$

$T_{1, 25} := 0$

$T_{1, 26} := 0$

$T_{1, 27} := 0$

$T_{1, 28} := 0$

$T_{1, 29} := 0$

$$\begin{aligned} T_{1,30} &:= 0 \\ T_{1,31} &:= 0 \end{aligned}$$

(1.6)

> $k := 1$:

Résolution pour les noeuds internes

> **for** i **from** 2 **to** $i_{\max} - 1$ **do**

for j **from** 2 **to** $j_{\max} - 1$ **do**

$$Eq[k] := T[i+1, j] + T[i-1, j] + \beta^2 \cdot (T[i, j+1] + T[i, j-1]) - 2 \cdot (1 + \beta^2) \cdot T[i, j] = 0;$$

$$Temps[k] := T[i, j];$$

$k := k + 1$

end do;

end do;

Conditions de Neumann:

> **for** j **from** 2 **to** $j_{\max} - 1$ **do**

$$Eq[k] := 2 \cdot (1 + \beta^2) \cdot T[i_{\max}, j] = 2 \cdot T[i_{\max} - 1, j] + \beta^2 \cdot (T[i_{\max}, j - 1] + T[i_{\max}, j + 1]);$$

$$Temps[k] := T[i_{\max}, j];$$

$k := k + 1$:

end do;

> $N := k - 1$:

Ecriture du système d'équations:

> $Eqs := \{seq(Eq[i], i = 1 .. N)\}$:

> $Tmps := [seq(Temps[i], i = 1 .. N)]$;

$$\begin{aligned} Tmps := [& T_{2,2}, T_{2,3}, T_{2,4}, T_{2,5}, T_{2,6}, T_{2,7}, T_{2,8}, T_{2,9}, T_{2,10}, T_{2,11}, T_{2,12}, T_{2,13}, T_{2,14}, \\ & T_{2,15}, T_{2,16}, T_{2,17}, T_{2,18}, T_{2,19}, T_{2,20}, T_{2,21}, T_{2,22}, T_{2,23}, T_{2,24}, T_{2,25}, T_{2,26}, T_{2,27}, \\ & T_{2,28}, T_{2,29}, T_{2,30}, T_{3,2}, T_{3,3}, T_{3,4}, T_{3,5}, T_{3,6}, T_{3,7}, T_{3,8}, T_{3,9}, T_{3,10}, T_{3,11}, T_{3,12}, \\ & T_{3,13}, T_{3,14}, T_{3,15}, T_{3,16}, T_{3,17}, T_{3,18}, T_{3,19}, T_{3,20}, T_{3,21}, T_{3,22}, T_{3,23}, T_{3,24}, T_{3,25}, \\ & T_{3,26}, T_{3,27}, T_{3,28}, T_{3,29}, T_{3,30}, T_{4,2}, T_{4,3}, T_{4,4}, T_{4,5}, T_{4,6}, T_{4,7}, T_{4,8}, T_{4,9}, T_{4,10}, \\ & T_{4,11}, T_{4,12}, T_{4,13}, T_{4,14}, T_{4,15}, T_{4,16}, T_{4,17}, T_{4,18}, T_{4,19}, T_{4,20}, T_{4,21}, T_{4,22}, T_{4,23}, \\ & T_{4,24}, T_{4,25}, T_{4,26}, T_{4,27}, T_{4,28}, T_{4,29}, T_{4,30}, T_{5,2}, T_{5,3}, T_{5,4}, T_{5,5}, T_{5,6}, T_{5,7}, T_{5,8}, \\ & T_{5,9}, T_{5,10}, T_{5,11}, T_{5,12}, T_{5,13}, T_{5,14}, T_{5,15}, T_{5,16}, T_{5,17}, T_{5,18}, T_{5,19}, T_{5,20}, T_{5,21}, \\ & T_{5,22}, T_{5,23}, T_{5,24}, T_{5,25}, T_{5,26}, T_{5,27}, T_{5,28}, T_{5,29}, T_{5,30}, T_{6,2}, T_{6,3}, T_{6,4}, T_{6,5}, T_{6,6}, \\ & T_{6,7}, T_{6,8}, T_{6,9}, T_{6,10}, T_{6,11}, T_{6,12}, T_{6,13}, T_{6,14}, T_{6,15}, T_{6,16}, T_{6,17}, T_{6,18}, T_{6,19}, \\ & T_{6,20}, T_{6,21}, T_{6,22}, T_{6,23}, T_{6,24}, T_{6,25}, T_{6,26}, T_{6,27}, T_{6,28}, T_{6,29}, T_{6,30}, T_{7,2}, T_{7,3}, \\ & T_{7,4}, T_{7,5}, T_{7,6}, T_{7,7}, T_{7,8}, T_{7,9}, T_{7,10}, T_{7,11}, T_{7,12}, T_{7,13}, T_{7,14}, T_{7,15}, T_{7,16}, T_{7,17}, \\ & T_{7,18}, T_{7,19}, T_{7,20}, T_{7,21}, T_{7,22}, T_{7,23}, T_{7,24}, T_{7,25}, T_{7,26}, T_{7,27}, T_{7,28}, T_{7,29}, T_{7,30}, \\ & T_{8,2}, T_{8,3}, T_{8,4}, T_{8,5}, T_{8,6}, T_{8,7}, T_{8,8}, T_{8,9}, T_{8,10}, T_{8,11}, T_{8,12}, T_{8,13}, T_{8,14}, T_{8,15}, \\ & T_{8,16}, T_{8,17}, T_{8,18}, T_{8,19}, T_{8,20}, T_{8,21}, T_{8,22}, T_{8,23}, T_{8,24}, T_{8,25}, T_{8,26}, T_{8,27}, T_{8,28}, \\ & T_{8,29}, T_{8,30}, T_{9,2}, T_{9,3}, T_{9,4}, T_{9,5}, T_{9,6}, T_{9,7}, T_{9,8}, T_{9,9}, T_{9,10}, T_{9,11}, T_{9,12}, T_{9,13}, \\ & T_{9,14}, T_{9,15}, T_{9,16}, T_{9,17}, T_{9,18}, T_{9,19}, T_{9,20}, T_{9,21}, T_{9,22}, T_{9,23}, T_{9,24}, T_{9,25}, T_{9,26}, \\ & T_{9,27}, T_{9,28}, T_{9,29}, T_{9,30}, T_{10,2}, T_{10,3}, T_{10,4}, T_{10,5}, T_{10,6}, T_{10,7}, T_{10,8}, T_{10,9}, T_{10,10}, \\ & T_{10,11}, T_{10,12}, T_{10,13}, T_{10,14}, T_{10,15}, T_{10,16}, T_{10,17}, T_{10,18}, T_{10,19}, T_{10,20}, T_{10,21}, \\ & T_{10,22}, T_{10,23}, T_{10,24}, T_{10,25}, T_{10,26}, T_{10,27}, T_{10,28}, T_{10,29}, T_{10,30}, T_{11,2}, T_{11,3}, \end{aligned}$$

(1.1.1)

$T_{11,4}, T_{11,5}, T_{11,6}, T_{11,7}, T_{11,8}, T_{11,9}, T_{11,10}, T_{11,11}, T_{11,12}, T_{11,13}, T_{11,14}, T_{11,15},$
 $T_{11,16}, T_{11,17}, T_{11,18}, T_{11,19}, T_{11,20}, T_{11,21}, T_{11,22}, T_{11,23}, T_{11,24}, T_{11,25}, T_{11,26},$
 $T_{11,27}, T_{11,28}, T_{11,29}, T_{11,30}$]

> *SolT* := solve(Eqs, Tmps);

$$\begin{aligned}
 \text{SolT} := & \left[[T_{2,2} = 0.05677803143, T_{2,3} = 0.1149574082, T_{2,4} = 0.1759746978, T_{2,5} \right. & (1.1.2) \\
 & = 0.2413379364, T_{2,6} = 0.3126649825, T_{2,7} = 0.3917251921, T_{2,8} \\
 & = 0.4804858443, T_{2,9} = 0.5811650928, T_{2,10} = 0.6962937482, T_{2,11} \\
 & = 0.8287890121, T_{2,12} = 0.9820445463, T_{2,13} = 1.160043244, T_{2,14} \\
 & = 1.367502212, T_{2,15} = 1.610064548, T_{2,16} = 1.894560811, T_{2,17} \\
 & = 2.229377027, T_{2,18} = 2.624989879, T_{2,19} = 3.094771740, T_{2,20} \\
 & = 3.656244500, T_{2,21} = 4.333105667, T_{2,22} = 5.158636852, T_{2,23} \\
 & = 6.181706089, T_{2,24} = 7.477920736, T_{2,25} = 9.171727284, T_{2,26} \\
 & = 11.48371761, T_{2,27} = 14.84151983, T_{2,28} = 20.16774959, T_{2,29} \\
 & = 29.71317064, T_{2,30} = 49.73693779, T_{3,2} = 0.1121547175, T_{3,3} \\
 & = 0.2270769035, T_{3,4} = 0.3476034468, T_{3,5} = 0.4767120652, T_{3,6} \\
 & = 0.6175968014, T_{3,7} = 0.7737499417, T_{3,8} = 0.9490530923, T_{3,9} \\
 & = 1.147880779, T_{3,10} = 1.375220888, T_{3,11} = 1.636817754, T_{3,12} \\
 & = 1.939345929, T_{3,13} = 2.290626218, T_{3,14} = 2.699901058, T_{3,15} \\
 & = 3.178195168, T_{3,16} = 3.738801671, T_{3,17} = 4.397957417, T_{3,18} \\
 & = 5.175810749, T_{3,19} = 6.097852579, T_{3,20} = 7.197100595, T_{3,21} \\
 & = 8.517541317, T_{3,22} = 10.11973565, T_{3,23} = 12.09026677, T_{3,24} \\
 & = 14.55824957, T_{3,25} = 17.72527079, T_{3,26} = 21.92162332, T_{3,27} \\
 & = 27.71461210, T_{3,28} = 36.11630791, T_{3,29} = 48.94799516, T_{3,30} \\
 & = 69.23458054, T_{4,2} = 0.1647639351, T_{4,3} = 0.3335920415, T_{4,4} \\
 & = 0.5106501206, T_{4,5} = 0.7003100761, T_{4,6} = 0.9072602164, T_{4,7} \\
 & = 1.136624681, T_{4,8} = 1.394095805, T_{4,9} = 1.686084041, T_{4,10} = 2.019891272, \\
 & T_{4,11} = 2.403915187, T_{4,12} = 2.847895198, T_{4,13} = 3.363214640, T_{4,14} \\
 & = 3.963280632, T_{4,15} = 4.664013397, T_{4,16} = 5.484493286, T_{4,17} \\
 & = 6.447840221, T_{4,18} = 7.582443120, T_{4,19} = 8.923727233, T_{4,20} \\
 & = 10.51676398, T_{4,21} = 12.42022335, T_{4,22} = 14.71249766, T_{4,23} \\
 & = 17.50137577, T_{4,24} = 20.93953999, T_{4,25} = 25.24948298, T_{4,26} \\
 & = 30.76289279, T_{4,27} = 37.97899735, T_{4,28} = 47.63487479, T_{4,29} \\
 & = 60.72792154, T_{4,30} = 78.25338921, T_{5,2} = 0.2133089815, T_{5,3} \\
 & = 0.4318772069, T_{5,4} = 0.6610949179, T_{5,5} = 0.9066179024, T_{5,6} \\
 & = 1.174509307, T_{5,7} = 1.471392760, T_{5,8} = 1.804621405, T_{5,9} = 2.182468309, \\
 & T_{5,10} = 2.614344971, T_{5,11} = 3.111056522, T_{5,12} = 3.685105037, T_{5,13} \\
 & = 4.351056511, T_{5,14} = 5.125993434, T_{5,15} = 6.030084502, T_{5,16} \\
 & = 7.087317857, T_{5,17} = 8.326467060, T_{5,18} = 9.782394278, T_{5,19} \\
 & = 11.49784925, T_{5,20} = 13.52600475, T_{5,21} = 15.93409045, T_{5,22} \\
 & = 18.80865589, T_{5,23} = 22.26319864, T_{5,24} = 26.44905163, T_{5,25} \\
 & = 31.57022837, T_{5,26} = 37.90146750, T_{5,27} = 45.80360973, T_{5,28} \\
 & = 55.71627235, T_{5,29} = 68.07542700, T_{5,30} = 83.05105475, T_{6,2} \\
 & = 0.2565947838, T_{6,3} = 0.5195128869, T_{6,4} = 0.7952344415, T_{6,5}
 \end{aligned}$$

$= 1.090557308, T_{6,6} = 1.412766350, T_{6,7} = 1.769815648, T_{6,8} = 2.170528745,$
 $T_{6,9} = 2.624822821, T_{6,10} = 3.143963780, T_{6,11} = 3.740860895, T_{6,12}$
 $= 4.430411916, T_{6,13} = 5.229912934, T_{6,14} = 6.159552090, T_{6,15}$
 $= 7.243013321, T_{6,16} = 8.508226578, T_{6,17} = 9.988315884, T_{6,18}$
 $= 11.72281768, T_{6,19} = 13.75927074, T_{6,20} = 16.15531533, T_{6,21}$
 $= 18.98147781, T_{6,22} = 22.32483679, T_{6,23} = 26.29371129, T_{6,24}$
 $= 31.02323952, T_{6,25} = 36.68091136, T_{6,26} = 43.46913910, T_{6,27}$
 $= 51.61770173, T_{6,28} = 61.35117787, T_{6,29} = 72.80645938, T_{6,30}$
 $= 85.87540278, T_{7,2} = 0.2935572667, T_{7,3} = 0.5943451154, T_{7,4}$
 $= 0.9097726533, T_{7,5} = 1.247610538, T_{7,6} = 1.616183139, T_{7,7} = 2.024574736,$
 $T_{7,8} = 2.482855107, T_{7,9} = 3.002330448, T_{7,10} = 3.595826435, T_{7,11}$
 $= 4.278011362, T_{7,12} = 5.065768799, T_{7,13} = 5.978631217, T_{7,14}$
 $= 7.039288673, T_{7,15} = 8.274190115, T_{7,16} = 9.714259249, T_{7,17}$
 $= 11.39575222, T_{7,18} = 13.36128983, T_{7,19} = 15.66110069, T_{7,20}$
 $= 18.35450800, T_{7,21} = 21.51166868, T_{7,22} = 25.21550217, T_{7,23}$
 $= 29.56357022, T_{7,24} = 34.66928378, T_{7,25} = 40.66103847, T_{7,26}$
 $= 47.67647582, T_{7,27} = 55.84688021, T_{7,28} = 65.26427803, T_{7,29}$
 $= 75.92382986, T_{7,30} = 87.64409699, T_{8,2} = 0.3232891676, T_{8,3}$
 $= 0.6545376548, T_{8,4} = 1.001900518, T_{8,5} = 1.373929053, T_{8,6} = 1.779780930,$
 $T_{8,7} = 2.229445049, T_{8,8} = 2.733986499, T_{8,9} = 3.305817429, T_{8,10}$
 $= 3.959000149, T_{8,11} = 4.709589319, T_{8,12} = 5.576020702, T_{8,13}$
 $= 6.579554462, T_{8,14} = 7.744781270, T_{8,15} = 9.100199215, T_{8,16}$
 $= 10.67886809, T_{8,17} = 12.51914392, T_{8,18} = 14.66548871, T_{8,19}$
 $= 17.16933419, T_{8,20} = 20.08994730, T_{8,21} = 23.49518676, T_{8,22}$
 $= 27.46193298, T_{8,23} = 32.07578364, T_{8,24} = 37.42928690, T_{8,25}$
 $= 43.61748293, T_{8,26} = 50.72884548, T_{8,27} = 58.82906528, T_{8,28}$
 $= 67.93522418, T_{8,29} = 77.98048503, T_{8,30} = 88.77715531, T_{9,2}$
 $= 0.3450617489, T_{9,3} = 0.6986158181, T_{9,4} = 1.069362711, T_{9,5} = 1.466424224,$
 $T_{9,6} = 1.899566479, T_{9,7} = 2.379438033, T_{9,8} = 2.917828410, T_{9,9}$
 $= 3.527952620, T_{9,10} = 4.224767413, T_{9,11} = 5.025325062, T_{9,12}$
 $= 5.949170227, T_{9,13} = 7.018784658, T_{9,14} = 8.260082729, T_{9,15}$
 $= 9.702957390, T_{9,16} = 11.38186996, T_{9,17} = 13.33646665, T_{9,18}$
 $= 15.61218692, T_{9,19} = 18.26080008, T_{9,20} = 21.34076024, T_{9,21}$
 $= 24.91719808, T_{9,22} = 29.06125935, T_{9,23} = 33.84834444, T_{9,24}$
 $= 39.35459728, T_{9,25} = 45.65076085, T_{9,26} = 52.79235790, T_{9,27}$
 $= 60.80531123, T_{9,28} = 69.66706838, T_{9,29} = 79.28573078, T_{9,30}$
 $= 89.48403923, T_{10,2} = 0.3583420101, T_{10,3} = 0.7255011573, T_{10,4}$
 $= 1.110510285, T_{10,5} = 1.522838654, T_{10,6} = 1.972622730, T_{10,7}$
 $= 2.470912194, T_{10,8} = 3.029936488, T_{10,9} = 3.663397228, T_{10,10}$
 $= 4.386791821, T_{10,11} = 5.217773289, T_{10,12} = 6.176550488, T_{10,13}$
 $= 7.286331214, T_{10,14} = 8.573807600, T_{10,15} = 10.06967765, T_{10,16}$
 $= 11.80918772, T_{10,17} = 13.83266580, T_{10,18} = 16.18599223, T_{10,19}$
 $= 18.92091895, T_{10,20} = 22.09509550, T_{10,21} = 25.77158596, T_{10,22}$

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= 30.01756189, T10,23 = 34.90173752, T10,24 = 40.48999690, T10,25
= 46.83860531, T10,26 = 53.98451404, T10,27 = 61.93275337, T10,28
= 70.64200733, T10,29 = 80.01133047, T10,30 = 89.87327084, T11,2
= 0.3628051341, T11,3 = 0.7345365162, T11,4 = 1.124338616, T11,5
= 1.541797379, T11,6 = 1.997173591, T11,7 = 2.501651525, T11,8
= 3.067608121, T11,9 = 3.708907982, T11,10 = 4.441229353, T11,11
= 5.282425788, T11,12 = 6.252927219, T11,13 = 7.376182113, T11,14
= 8.679138802, T11,15 = 10.19275790, T11,16 = 11.95253748, T11,17
= 13.99901659, T11,18 = 16.37819726, T11,19 = 19.14178800, T11,20
= 22.34711685, T11,21 = 26.05648838, T11,22 = 30.33566475, T11,23
= 35.25104682, T11,24 = 40.86504752, T11,25 = 47.22914943, T11,26
= 54.37433959, T11,27 = 62.29918086, T11,28 = 70.95687711, T11,29
= 80.24431292, T11,30 = 89.99771365 ]]

```

```
> LT := [seq(T1,j, j = 1 .. jmax), seq(rhs(SolT1,i), i = 1 .. N) ] :
```

```
> with(plots) :
```

```
> for i from 1 to imax - 2 do Ns[i] := i *  $\frac{N - 29}{i_{\max} - 2}$  end do:
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```
> GTemps := [ [seq(T1,j, j = 1 .. jmax) ], [ T2,1, seq(rhs(SolT1,i), i = 1 .. Ns1), T2,jmax ],
[ T3,1, seq(rhs(SolT1,i), i = Ns1 + 1 .. Ns2), T3,jmax ], [ T4,1, seq(rhs(SolT1,i), i
= Ns2 + 1 .. Ns3), T4,jmax ], [ T5,1, seq(rhs(SolT1,i), i = Ns3 + 1 .. Ns4), T5,jmax ],
[ T6,1, seq(rhs(SolT1,i), i = Ns4 + 1 .. Ns5), T6,jmax ], [ T7,1, seq(rhs(SolT1,i), i
= Ns5 + 1 .. Ns6), T7,jmax ], [ T8,1, seq(rhs(SolT1,i), i = Ns6 + 1 .. Ns7), T8,jmax ],
[ T9,1, seq(rhs(SolT1,i), i = Ns7 + 1 .. Ns8), T9,jmax ], [ T10,1, seq(rhs(SolT1,i), i
= Ns8 + 1 .. Ns9), T10,jmax ], [ T11,1, seq(rhs(SolT1,j), j = N - 28 .. N), T11,jmax ] ] :
```

Tracé des isothermes:

```
> listcontplot(GTemps, title
= "Countour des températures: Formulation 5 point - CL de Neumann", axes
= boxed, gridlines = true, thickness = 2, coloring = [blue, green], contours = [5,
10, 20, 30, 40, 50, 60, 70, 80, 90])
```

Countour des températures: Formulation 5 point - CL de Neumann

