Equation de Laplace 2D

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Détermination de la temperature $T(x, y)$ à travers la surface d'une plaque rectangulaire (a x b) dont les extrémités sont soumises à des (C.L.) de Dirichlet.

$$\frac{\partial^2}{\partial x^2} T(x, y) + \frac{\partial^2}{\partial y^2} T(x, y) = 0$$

Conditions aux limites (C.L):

$$T(x, 0) = 0,\quad T(x, b) = 100,\quad T(0, y) = 75,\quad T(a, y) = 50.$$

Solution discrétisée (formulation en 5 points):

```
> Restart :
> Digits := 4 :
> i_max := 5; j_max := 5;
```

$$i_{max} := 5$$
$$j_{max} := 5 \qquad \textbf{(1.1)}$$

Nombre d'équations:

```
> N := (i_max - 2)·(j_max - 2)
```

$$N := 9 \qquad \textbf{(1.2)}$$

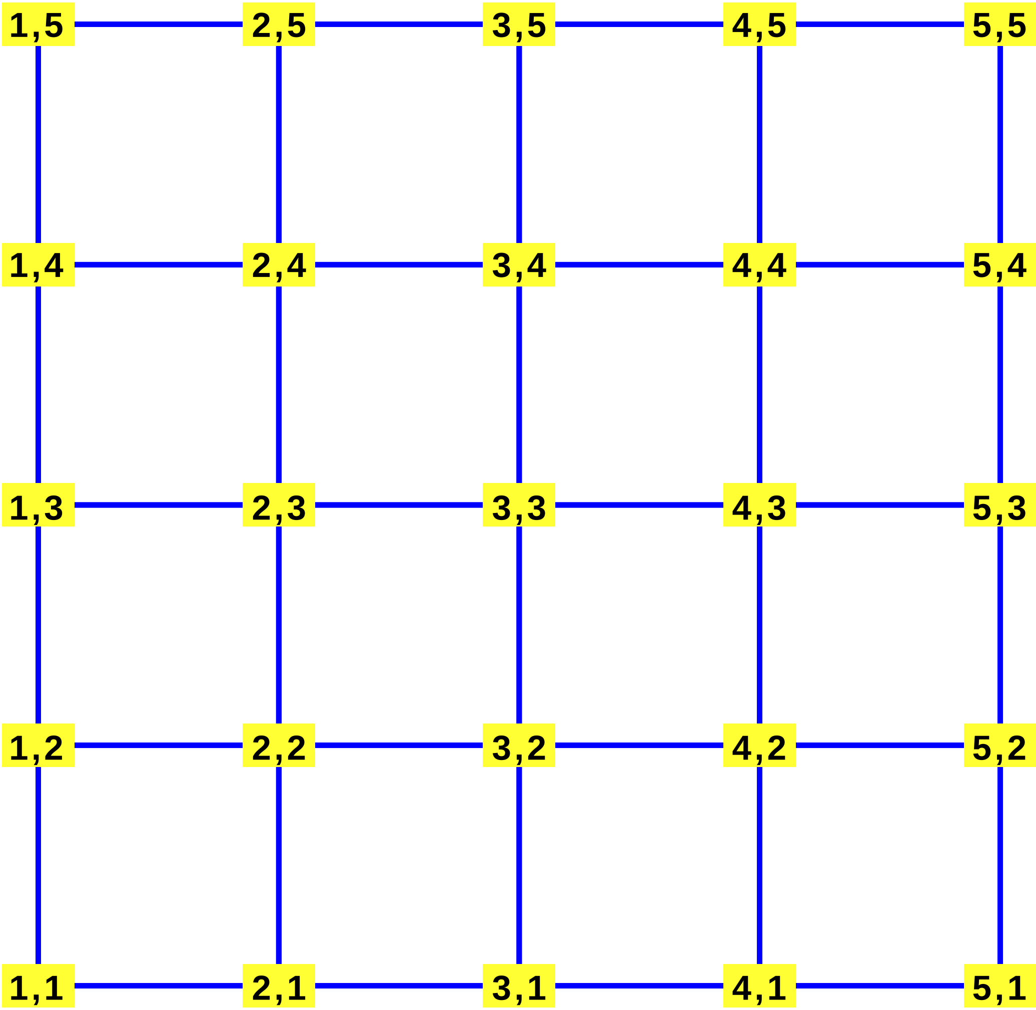
Maillage:

```
> with(GraphTheory) : with(SpecialGraphs) :
> G := GridGraph(i_max, j_max)
```

(1.3)

G := Graph 1: an undirected unweighted graph with 25 vertices and 40 edge(s) **(1.3)**

> *DrawGraph*(*G*)



Conditions aux Limites:

> **for** i **from** 1 **to** i_{max} **do** $T[i, 1] := 0.$ **end do**;

$$T_{1, 1} := 0.$$
$$T_{2, 1} := 0.$$
$$T_{3, 1} := 0.$$
$$T_{4, 1} := 0.$$
$$T_{5, 1} := 0. \quad \textbf{(1.4)}$$

> **for** i **from** 1 **to** i_{max} **do** $T[i, j_{max}] := 100$ **end do**;

$$T_{1, 5} := 100$$
$$T_{2, 5} := 100$$
$$T_{3, 5} := 100$$
$$T_{4, 5} := 100$$
$$T_{5, 5} := 100 \quad \textbf{(1.5)}$$

> **for** j **from** 1 **to** j_{max} **do** $T[1, j] := 75$ **end do**;

$$T_{1, 1} := 75$$
$$T_{1, 2} := 75$$
$$T_{1, 3} := 75$$

$$T_{1,4} := 75$$
$$T_{1,5} := 75 \qquad (1.6)$$

> **for** j **from** 1 **to** j_{max} **do** $T[i_{max}, j] := 50$ **end do**;

$$T_{5,1} := 50$$
$$T_{5,2} := 50$$
$$T_{5,3} := 50$$
$$T_{5,4} := 50$$
$$T_{5,5} := 50 \qquad (1.7)$$

> $k := 1$

$$k := 1 \qquad (1.1.1)$$

Résolution pour les noeuds internes:

```
> for j from 2 to j_max - 1 do
    for i from 2 to i_max - 1 do
        Eq[k] := T[i + 1, j] + T[i - 1, j] + T[i, j + 1] + T[i, j - 1] - 4 · T[i, j] = 0;
        Temps[k] := T[i, j];
    k := k + 1
   end do;
  end do;
```

Ecriture du système d'équations:

> **for** k **from** 1 **to** N **do** $Eq[k]$ **end do**;

$$T_{3,2} + 75. + T_{2,3} - 4\,T_{2,2} = 0$$
$$T_{4,2} + T_{2,2} + T_{3,3} - 4\,T_{3,2} = 0$$
$$50. + T_{3,2} + T_{4,3} - 4\,T_{4,2} = 0$$
$$T_{3,3} + 75 + T_{2,4} + T_{2,2} - 4\,T_{2,3} = 0$$
$$T_{4,3} + T_{2,3} + T_{3,4} + T_{3,2} - 4\,T_{3,3} = 0$$
$$50 + T_{3,3} + T_{4,4} + T_{4,2} - 4\,T_{4,3} = 0$$
$$T_{3,4} + 175 + T_{2,3} - 4\,T_{2,4} = 0$$
$$T_{4,4} + T_{2,4} + 100 + T_{3,3} - 4\,T_{3,4} = 0$$
$$150 + T_{3,4} + T_{4,3} - 4\,T_{4,4} = 0 \qquad (1.1.2)$$

> $Eqs := \{seq(Eq[i], i = 1..N)\}$:

> $Tmps := [seq(Temps[i], i = 1..N)]$;

$$Tmps := [T_{2,2}, T_{3,2}, T_{4,2}, T_{2,3}, T_{3,3}, T_{4,3}, T_{2,4}, T_{3,4}, T_{4,4}] \qquad (1.1.3)$$

Solution du système:

> $SolT := solve(Eqs, Tmps)$;

$$SolT := [[T_{2,2} = 42.86, T_{3,2} = 33.26, T_{4,2} = 33.93, T_{2,3} = 63.17, T_{3,3} = 56.25, T_{4,3} = 52.46, T_{2,4} = 78.57, T_{3,4} = 76.12, T_{4,4} = 69.64]] \qquad (1.1.4)$$

> *restart*

Méthode matricielle:

> $a := -4;\ b := 1;\ c := 1;$

$$a := -4$$
$$b := 1$$
$$c := 1 \qquad (1.1.5)$$

$$> M := \begin{bmatrix} a & b & 0 & c & 0 & 0 & 0 & 0 & 0 \\ b & a & b & 0 & c & 0 & 0 & 0 & 0 \\ 0 & b & a & 0 & 0 & c & 0 & 0 & 0 \\ c & 0 & 0 & a & b & 0 & c & 0 & 0 \\ 0 & c & 0 & b & a & b & 0 & c & 0 \\ 0 & 0 & c & 0 & b & a & 0 & 0 & c \\ 0 & 0 & 0 & c & 0 & 0 & a & b & 0 \\ 0 & 0 & 0 & 0 & c & 0 & b & a & b \\ 0 & 0 & 0 & 0 & 0 & c & 0 & b & a \end{bmatrix}$$

$$M := \begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix} \tag{1.1.6}$$

$$> R := \begin{bmatrix} -75.0 \\ 0 \\ -50 \\ -75 \\ 0 \\ -50 \\ -175 \\ -100 \\ -150 \end{bmatrix}$$

$$R := \begin{bmatrix} -75.0 \\ 0 \\ -50 \\ -75 \\ 0 \\ -50 \\ -175 \\ -100 \\ -150 \end{bmatrix} \tag{1.1.7}$$

Résolution directe:

> *with*(*LinearAlgebra*) :

> *T* := *LinearSolve*(*M*, *R*, *method*='*LU*')

$$T := \begin{bmatrix} 42.8571428571428542 \\ 33.2589285714285694 \\ 33.9285714285714236 \\ 63.1696428571428541 \\ 56.2499999999999929 \\ 52.4553571428571388 \\ 78.5714285714285694 \\ 76.1160714285714164 \\ 69.6428571428571388 \end{bmatrix} \tag{1.1.8}$$

Résolution par inversion de matrice:

> *MM* := *MatrixInverse*(*M*)

$$MM := \begin{bmatrix} -\frac{67}{224} & -\frac{11}{112} & -\frac{1}{32} & -\frac{11}{112} & -\frac{1}{16} & -\frac{3}{112} & -\frac{1}{32} & -\frac{3}{112} & -\frac{3}{224} \\ -\frac{11}{112} & -\frac{37}{112} & -\frac{11}{112} & -\frac{1}{16} & -\frac{1}{8} & -\frac{1}{16} & -\frac{3}{112} & -\frac{5}{112} & -\frac{3}{112} \\ -\frac{1}{32} & -\frac{11}{112} & -\frac{67}{224} & -\frac{3}{112} & -\frac{1}{16} & -\frac{11}{112} & -\frac{3}{224} & -\frac{3}{112} & -\frac{1}{32} \\ -\frac{11}{112} & -\frac{1}{16} & -\frac{3}{112} & -\frac{37}{112} & -\frac{1}{8} & -\frac{5}{112} & -\frac{11}{112} & -\frac{1}{16} & -\frac{3}{112} \\ -\frac{1}{16} & -\frac{1}{8} & -\frac{1}{16} & -\frac{1}{8} & -\frac{3}{8} & -\frac{1}{8} & -\frac{1}{16} & -\frac{1}{8} & -\frac{1}{16} \\ -\frac{3}{112} & -\frac{1}{16} & -\frac{11}{112} & -\frac{5}{112} & -\frac{1}{8} & -\frac{37}{112} & -\frac{3}{112} & -\frac{1}{16} & -\frac{11}{112} \\ -\frac{1}{32} & -\frac{3}{112} & -\frac{3}{224} & -\frac{11}{112} & -\frac{1}{16} & -\frac{3}{112} & -\frac{67}{224} & -\frac{11}{112} & -\frac{1}{32} \\ -\frac{3}{112} & -\frac{5}{112} & -\frac{3}{112} & -\frac{1}{16} & -\frac{1}{8} & -\frac{1}{16} & -\frac{11}{112} & -\frac{37}{112} & -\frac{11}{112} \\ -\frac{3}{224} & -\frac{3}{112} & -\frac{1}{32} & -\frac{3}{112} & -\frac{1}{16} & -\frac{11}{112} & -\frac{1}{32} & -\frac{11}{112} & -\frac{67}{224} \end{bmatrix} \tag{1.1.9}$$

> *T* := *MatrixVectorMultiply*(*MM*, *R*)

(1.1.10)

$$T := \begin{bmatrix} 42.8571428571428542 \\ 33.2589285714285694 \\ 33.9285714285714306 \\ 63.1696428571428541 \\ 56.2500000000000000 \\ 52.4553571428571388 \\ 78.5714285714285694 \\ 76.1160714285714306 \\ 69.6428571428571388 \end{bmatrix} \tag{1.1.10}$$