Equation de Diffusion 1D

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Master : Energétique

Matière : Méthodes Numériques Appliquées II

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2010/2011

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EXEMPLE 2

Détermination de la distribution de temperature $T(\mathrm{x})$ à travers une barre de section S, de conductivité thermique w3 et de longueur L dont les extrémités sont soumises à des (C.L.) de Dirichlet et une source de chaleur uniforme q.

$$\frac{\mathrm{d}}{\mathrm{dx}}\left(k\frac{\mathrm{d}}{\mathrm{dx}}T(\mathrm{x})\right)+q=0$$

Conditions aux limites (C.L):

$$T(0)=T_A=100,$$
$$T(\mathrm{L})=T_B=200,$$

Solution

```
> Restart : Digits := 4 :
> L := 0.02; λ := 0.5; S := 1; q := 1000000; ndx := 15;
```

$$L := 0.02$$
$$\lambda := 0.5$$
$$S := 1$$
$$q := 1000000$$
$$ndx := 15 \qquad \textbf{(1.1)}$$

```
> Δx := L/ndx ;
```

$$\Delta x := 0.001333 \quad \textbf{(1.2)}$$

```
> i_max := ndx;
```

$$i_{max} := 15 \quad \textbf{(1.3)}$$

Nombre d'équations:

```
> N := i_max
```

$$N := 15 \quad \textbf{(1.4)}$$

Abscisses des noeuds:

```
> x[0] := 0;
  for i from 1 to N do
      x[i] := Δx/2 + (i - 1)·Δx;
  end do;
  x[N + 1] := L;
```

$$x_0 := 0$$
$$x_1 := 0.0006665$$
$$x_2 := 0.002000$$
$$x_3 := 0.003332$$
$$x_4 := 0.004666$$
$$x_5 := 0.005998$$
$$x_6 := 0.007332$$
$$x_7 := 0.008664$$
$$x_8 := 0.009998$$
$$x_9 := 0.01133$$
$$x_{10} := 0.01267$$
$$x_{11} := 0.01400$$
$$x_{12} := 0.01533$$
$$x_{13} := 0.01667$$
$$x_{14} := 0.01800$$
$$x_{15} := 0.01933$$
$$x_{16} := 0.02 \quad \textbf{(1.5)}$$

Conditions aux Limites:

```
> T[0] := 100;
  T[N + 1] := 200;
```

$$T_0 := 100$$
$$T_{16} := 200 \quad \textbf{(1.6)}$$

Noeuds internes:

```
> for i from 2 to N - 1 do
      Sp[i] := 0;
      Su[i] := q·S·Δx;
      a_W[i] := λ·S/Δx;
      a_E[i] := a_W[i];
      a_P[i] := a_W[i] + a_E[i] - Sp[i];
  end do;
```

$$Sp_2 := 0$$
$$Su_2 := 1333.$$

$$a_{W_2} := 375.1$$
$$a_{E_2} := 375.1$$
$$a_{P_2} := 750.2$$
$$Sp_3 := 0$$
$$Su_3 := 1333.$$
$$a_{W_3} := 375.1$$
$$a_{E_3} := 375.1$$
$$a_{P_3} := 750.2$$
$$Sp_4 := 0$$
$$Su_4 := 1333.$$
$$a_{W_4} := 375.1$$
$$a_{E_4} := 375.1$$
$$a_{P_4} := 750.2$$
$$Sp_5 := 0$$
$$Su_5 := 1333.$$
$$a_{W_5} := 375.1$$
$$a_{E_5} := 375.1$$
$$a_{P_5} := 750.2$$
$$Sp_6 := 0$$
$$Su_6 := 1333.$$
$$a_{W_6} := 375.1$$
$$a_{E_6} := 375.1$$
$$a_{P_6} := 750.2$$
$$Sp_7 := 0$$
$$Su_7 := 1333.$$
$$a_{W_7} := 375.1$$
$$a_{E_7} := 375.1$$
$$a_{P_7} := 750.2$$
$$Sp_8 := 0$$
$$Su_8 := 1333.$$
$$a_{W_8} := 375.1$$
$$a_{E_8} := 375.1$$
$$a_{P_8} := 750.2$$
$$Sp_9 := 0$$
$$Su_9 := 1333.$$
$$a_{W_9} := 375.1$$
$$a_{E_9} := 375.1$$

$$a_{P_9} := 750.2$$
$$Sp_{10} := 0$$
$$Su_{10} := 1333.$$
$$a_{W_{10}} := 375.1$$
$$a_{E_{10}} := 375.1$$
$$a_{P_{10}} := 750.2$$
$$Sp_{11} := 0$$
$$Su_{11} := 1333.$$
$$a_{W_{11}} := 375.1$$
$$a_{E_{11}} := 375.1$$
$$a_{P_{11}} := 750.2$$
$$Sp_{12} := 0$$
$$Su_{12} := 1333.$$
$$a_{W_{12}} := 375.1$$
$$a_{E_{12}} := 375.1$$
$$a_{P_{12}} := 750.2$$
$$Sp_{13} := 0$$
$$Su_{13} := 1333.$$
$$a_{W_{13}} := 375.1$$
$$a_{E_{13}} := 375.1$$
$$a_{P_{13}} := 750.2$$
$$Sp_{14} := 0$$
$$Su_{14} := 1333.$$
$$a_{W_{14}} := 375.1$$
$$a_{E_{14}} := 375.1$$
$$a_{P_{14}} := 750.2 \qquad \textbf{(1.7)}$$

Noeud gauche:

$$> Sp[1] := -\frac{2\cdot\lambda\cdot S}{\Delta x};$$

$$Su[1] := q\cdot S\cdot\Delta x + \frac{2\cdot\lambda\cdot S}{\Delta x}\cdot T[0];$$

$$a_W[1] := 0;$$

$$a_E[1] := \frac{\lambda\cdot S}{\Delta x};$$

$$a_P[1] := a_W[1] + a_E[1] - Sp[1];$$

$$Sp_1 := -750.2$$
$$Su_1 := 76350.$$
$$a_{W_1} := 0$$

$$a_{E_1} := 375.1$$
$$a_{P_1} := 1125. \quad \textbf{(1.8)}$$

Noeud droit:

$> Sp[N] := -\frac{2\cdot\lambda\cdot S}{\Delta x};$

$Su[N] := q\cdot S\cdot\Delta x + \frac{2\cdot\lambda\cdot S}{\Delta x}\cdot T[N+1];$

$a_W[N] := \frac{\lambda\cdot S}{\Delta x};$

$a_E[N] := 0;$

$a_P[N] := a_W[N] + a_E[N] - Sp[N];$

$>$

$$Sp_{15} := -750.2$$
$$Su_{15} := 1.513\ 10^5$$
$$a_{W_{15}} := 375.1$$
$$a_{E_{15}} := 0$$
$$a_{P_{15}} := 1125. \quad \textbf{(1.9)}$$

Equations:

$> k := 1$

$$k := 1 \quad \textbf{(1.1.1)}$$

Résolution pour les noeuds internes:

```
> for i from 1 to N do
      Eq[k] := a_P[i]·T[i] = a_W[i]·T[i-1] + a_E[i]·T[i+1] + Su[i];
      k := k + 1;
  end do;
```

$$Eq_1 := 1125.\ T_1 = 76350. + 375.1\ T_2$$
$$k := 2$$
$$Eq_2 := 750.2\ T_2 = 375.1\ T_1 + 375.1\ T_3 + 1333.$$
$$k := 3$$
$$Eq_3 := 750.2\ T_3 = 375.1\ T_2 + 375.1\ T_4 + 1333.$$
$$k := 4$$
$$Eq_4 := 750.2\ T_4 = 375.1\ T_3 + 375.1\ T_5 + 1333.$$
$$k := 5$$
$$Eq_5 := 750.2\ T_5 = 375.1\ T_4 + 375.1\ T_6 + 1333.$$
$$k := 6$$
$$Eq_6 := 750.2\ T_6 = 375.1\ T_5 + 375.1\ T_7 + 1333.$$
$$k := 7$$
$$Eq_7 := 750.2\ T_7 = 375.1\ T_6 + 375.1\ T_8 + 1333.$$
$$k := 8$$
$$Eq_8 := 750.2\ T_8 = 375.1\ T_7 + 375.1\ T_9 + 1333.$$
$$k := 9$$
$$Eq_9 := 750.2\ T_9 = 375.1\ T_8 + 375.1\ T_{10} + 1333.$$
$$k := 10$$
$$Eq_{10} := 750.2\ T_{10} = 375.1\ T_9 + 375.1\ T_{11} + 1333.$$

$$k := 11$$
$$Eq_{11} := 750.2\ T_{11} = 375.1\ T_{10} + 375.1\ T_{12} + 1333.$$
$$k := 12$$
$$Eq_{12} := 750.2\ T_{12} = 375.1\ T_{11} + 375.1\ T_{13} + 1333.$$
$$k := 13$$
$$Eq_{13} := 750.2\ T_{13} = 375.1\ T_{12} + 375.1\ T_{14} + 1333.$$
$$k := 14$$
$$Eq_{14} := 750.2\ T_{14} = 375.1\ T_{13} + 375.1\ T_{15} + 1333.$$
$$k := 15$$
$$Eq_{15} := 1125.\ T_{15} = 375.1\ T_{14} + 1.513\ 10^{5}$$
$$k := 16 \qquad \textbf{(1.1.2)}$$

Ecriture du système d'équations:

> **for** k **from** 1 **to** N **do** $Eq[k]$ **end do**;

$$1125.\ T_1 = 76350. + 375.1\ T_2$$
$$750.2\ T_2 = 375.1\ T_1 + 375.1\ T_3 + 1333.$$
$$750.2\ T_3 = 375.1\ T_2 + 375.1\ T_4 + 1333.$$
$$750.2\ T_4 = 375.1\ T_3 + 375.1\ T_5 + 1333.$$
$$750.2\ T_5 = 375.1\ T_4 + 375.1\ T_6 + 1333.$$
$$750.2\ T_6 = 375.1\ T_5 + 375.1\ T_7 + 1333.$$
$$750.2\ T_7 = 375.1\ T_6 + 375.1\ T_8 + 1333.$$
$$750.2\ T_8 = 375.1\ T_7 + 375.1\ T_9 + 1333.$$
$$750.2\ T_9 = 375.1\ T_8 + 375.1\ T_{10} + 1333.$$
$$750.2\ T_{10} = 375.1\ T_9 + 375.1\ T_{11} + 1333.$$
$$750.2\ T_{11} = 375.1\ T_{10} + 375.1\ T_{12} + 1333.$$
$$750.2\ T_{12} = 375.1\ T_{11} + 375.1\ T_{13} + 1333.$$
$$750.2\ T_{13} = 375.1\ T_{12} + 375.1\ T_{14} + 1333.$$
$$750.2\ T_{14} = 375.1\ T_{13} + 375.1\ T_{15} + 1333.$$
$$1125.\ T_{15} = 375.1\ T_{14} + 1.513\ 10^{5} \qquad \textbf{(1.1.3)}$$

> $Eqs := \{seq(Eq[k], k = 1..N)\}$;

$$Eqs := \{1125.\ T_1 = 76350. + 375.1\ T_2, 750.2\ T_2 = 375.1\ T_1 + 375.1\ T_3 + 1333., 750.2\ T_3 = 375.1\ T_2 + 375.1\ T_4 + 1333., 750.2\ T_4 = 375.1\ T_3 + 375.1\ T_5 + 1333., 750.2\ T_5 = 375.1\ T_4 + 375.1\ T_6 + 1333., 750.2\ T_6 = 375.1\ T_5 + 375.1\ T_7 + 1333., 750.2\ T_7 = 375.1\ T_6 + 375.1\ T_8 + 1333., 750.2\ T_8 = 375.1\ T_7 + 375.1\ T_9 + 1333., 750.2\ T_9 = 375.1\ T_8 + 375.1\ T_{10} + 1333., 750.2\ T_{10} = 375.1\ T_9 + 375.1\ T_{11} + 1333., 750.2\ T_{11} = 375.1\ T_{10} + 375.1\ T_{12} + 1333., 750.2\ T_{12} = 375.1\ T_{11} + 375.1\ T_{13} + 1333., 750.2\ T_{13} = 375.1\ T_{12} + 375.1\ T_{14} + 1333., 750.2\ T_{14} = 375.1\ T_{13} + 375.1\ T_{15} + 1333., 1125.\ T_{15} = 375.1\ T_{14} + 1.513\ 10^{5}\} \qquad \textbf{(1.1.4)}$$

> $Tmps := [seq(T[i], i = 1..N)]$;

$$Tmps := [T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9, T_{10}, T_{11}, T_{12}, T_{13}, T_{14}, T_{15}] \qquad \textbf{(1.1.5)}$$

> $SolT := solve(Eqs, Tmps)$;

$$SolT := [[T_1 = 116.7, T_2 = 146.5, T_3 = 172.7, T_4 = 195.3, T_5 = 214.4, T_6 = 230.0, T_7 = 242.0, T_8 = 250.4, T_9 = 255.3, T_{10} = 256.6, T_{11} = 254.4, T_{12} = 248.6, T_{13} = 239.3, T_{14} = 226.4, T_{15} = 210.0]] \qquad \textbf{(1.1.6)}$$

```
> with(LinearAlgebra) :
> A, b := GenerateMatrix(Eqs, Tmps)
```

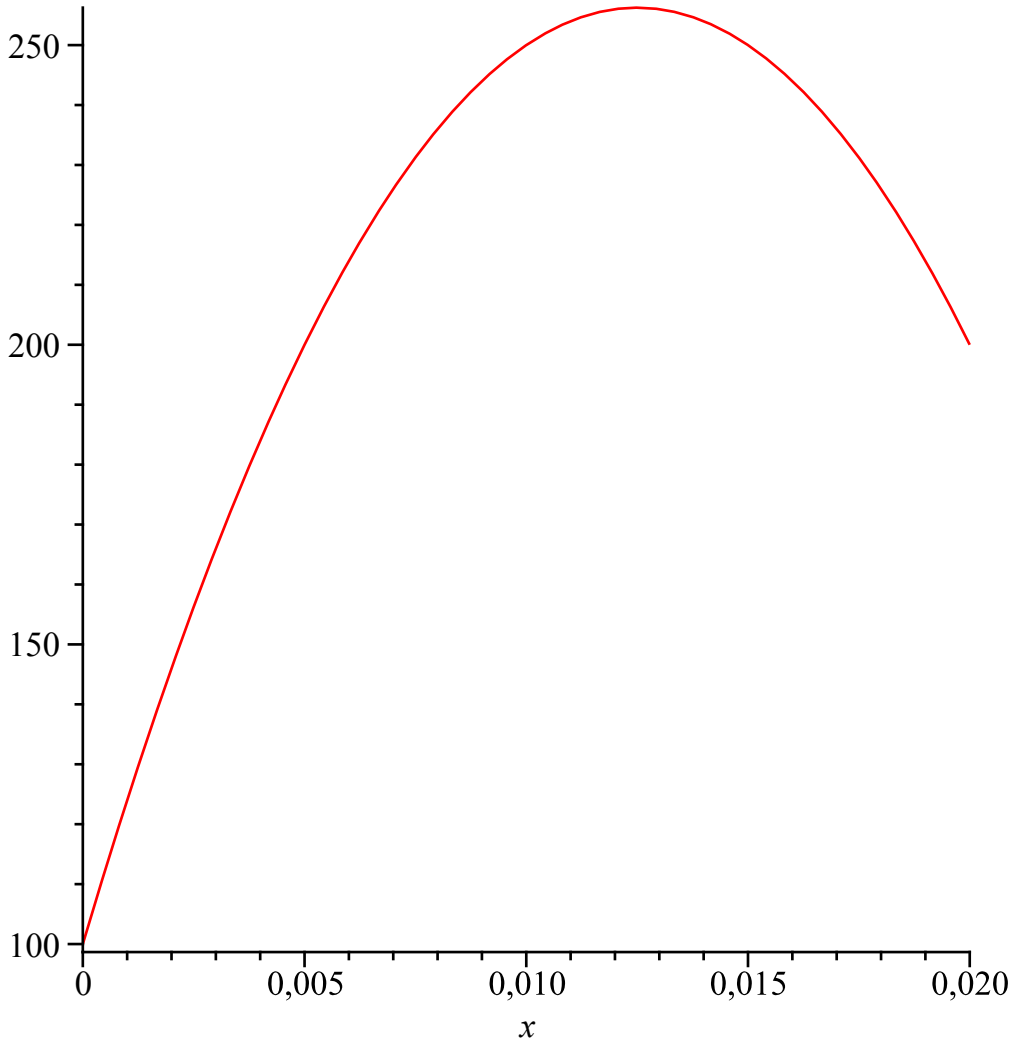
$$A, b := \begin{bmatrix} 15 \text{ x } 15 \text{ Matrix} \\ \text{Data Type: anything} \\ \text{Storage: rectangular} \\ \text{Order: Fortran\_order} \end{bmatrix}, \begin{bmatrix} 1 \text{ .. } 15 \text{ Vector}_{\text{column}} \\ \text{Data Type: anything} \\ \text{Storage: rectangular} \\ \text{Order: Fortran\_order} \end{bmatrix} \tag{1.1.7}$$

Solution exacte:

```
> F(x) := ( (T[N+1] - T[0])/L + q/(2·λ) · (L - x) ) · x + T[0];
```

$$F := x \rightarrow \left(\frac{T_{N+1} - T_0}{L} + \frac{1}{2} \frac{q\,(L-x)}{\lambda} \right) x + T_0 \tag{1.1.8}$$

```
> with(plots) :
> plot(F(x), x = 0..L);
```




```
> for i from 1 to N do
      T[i] := rhs(SolT_{1, i})
  end do;
```

$$T_1 := 116.7$$
$$T_2 := 146.5$$
$$T_3 := 172.7$$

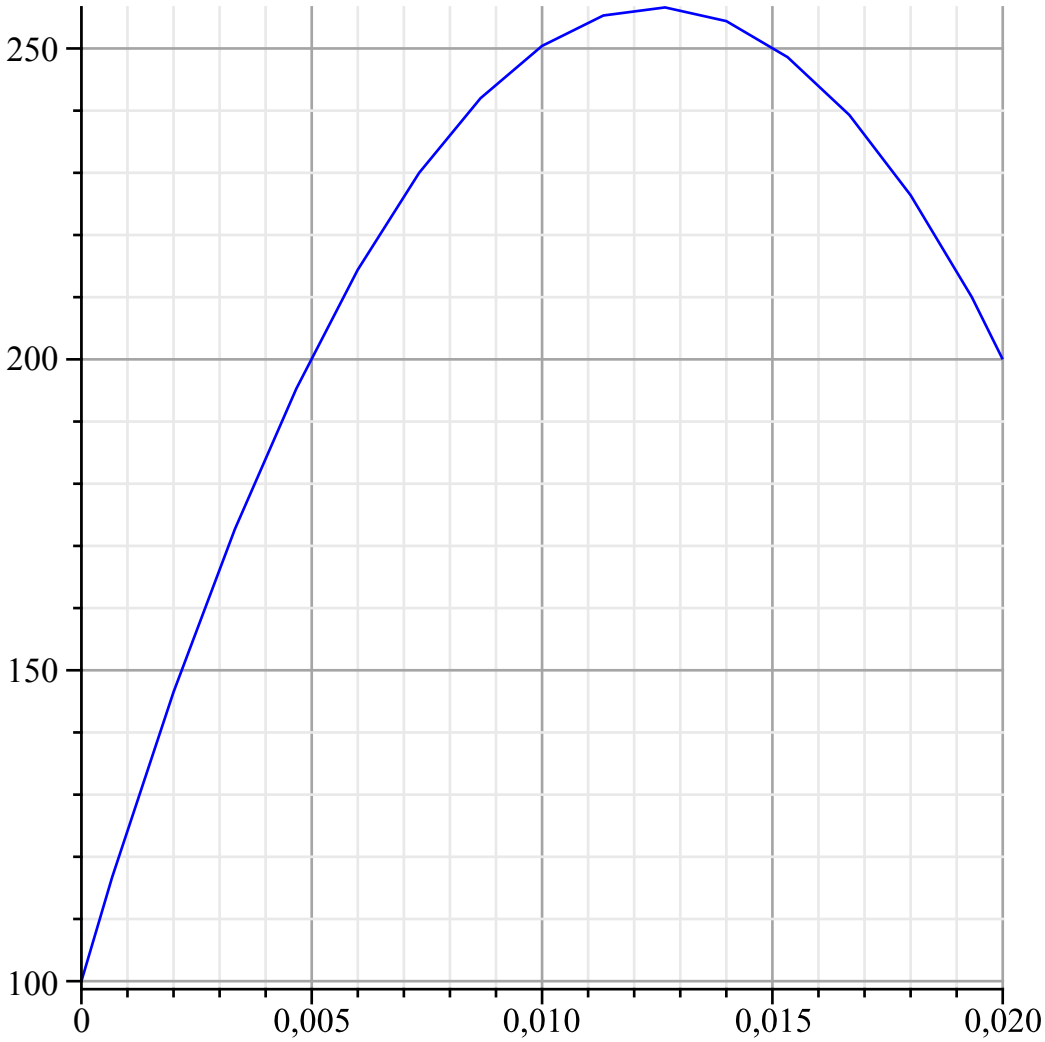
$$T_4 := 195.3$$
$$T_5 := 214.4$$
$$T_6 := 230.0$$
$$T_7 := 242.0$$
$$T_8 := 250.4$$
$$T_9 := 255.3$$
$$T_{10} := 256.6$$
$$T_{11} := 254.4$$
$$T_{12} := 248.6$$
$$T_{13} := 239.3$$
$$T_{14} := 226.4$$
$$T_{15} := 210.0 \quad \textbf{(1.1.9)}$$

> $lpN := [\, seq([x[i], T[i]], i = 0..N + 1)\,]$

$lpN := [[0, 100], [0.0006665, 116.7], [0.002000, 146.5], [0.003332, 172.7], [0.004666, 195.3], [0.005998, 214.4], [0.007332, 230.0], [0.008664, 242.0], [0.009998, 250.4], [0.01133, 255.3], [0.01267, 256.6], [0.01400, 254.4], [0.01533, 248.6], [0.01667, 239.3], [0.01800, 226.4], [0.01933, 210.0], [0.02, 200]]$ **(1.1.10)**

Courbe Numérique:

> $listplot(lpN, color = blue, gridlines = true)$

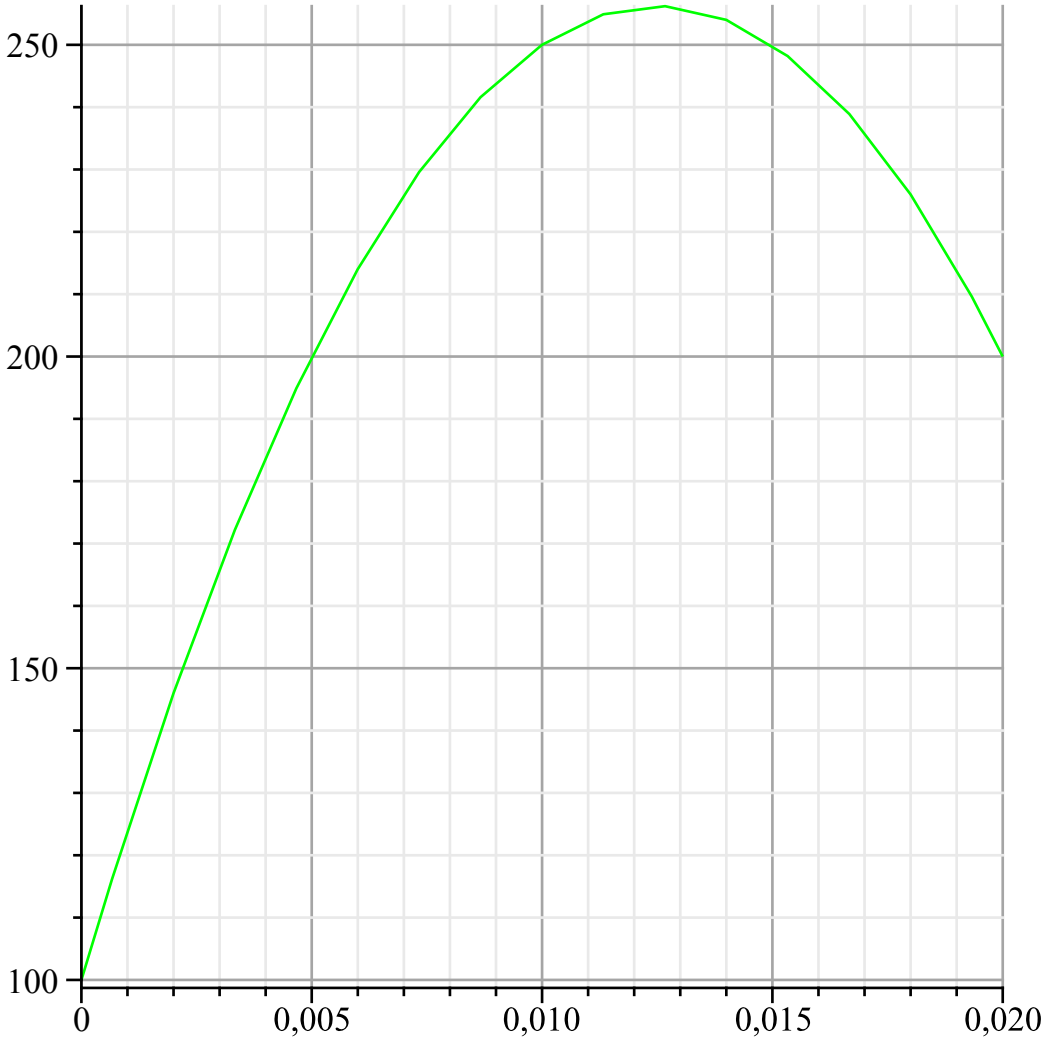


> $lpT := [seq([x[i], F(x[i])], i = 0..N + 1)]$

$lpT := [[0, 100.], [0.0006665, 116.2], [0.002000, 146.0], [0.003332, 172.2], [0.004666, 194.9], [0.005998, 214.0], [0.007332, 229.6], [0.008664, 241.6], [0.009998, 250.0], [0.01133, 254.9], [0.01267, 256.2], [0.01400, 254.0], [0.01533, 248.2], [0.01667, 238.9], [0.01800, 226.0], [0.01933, 209.6], [0.02, 200.0]]$ **(1.1.11)**

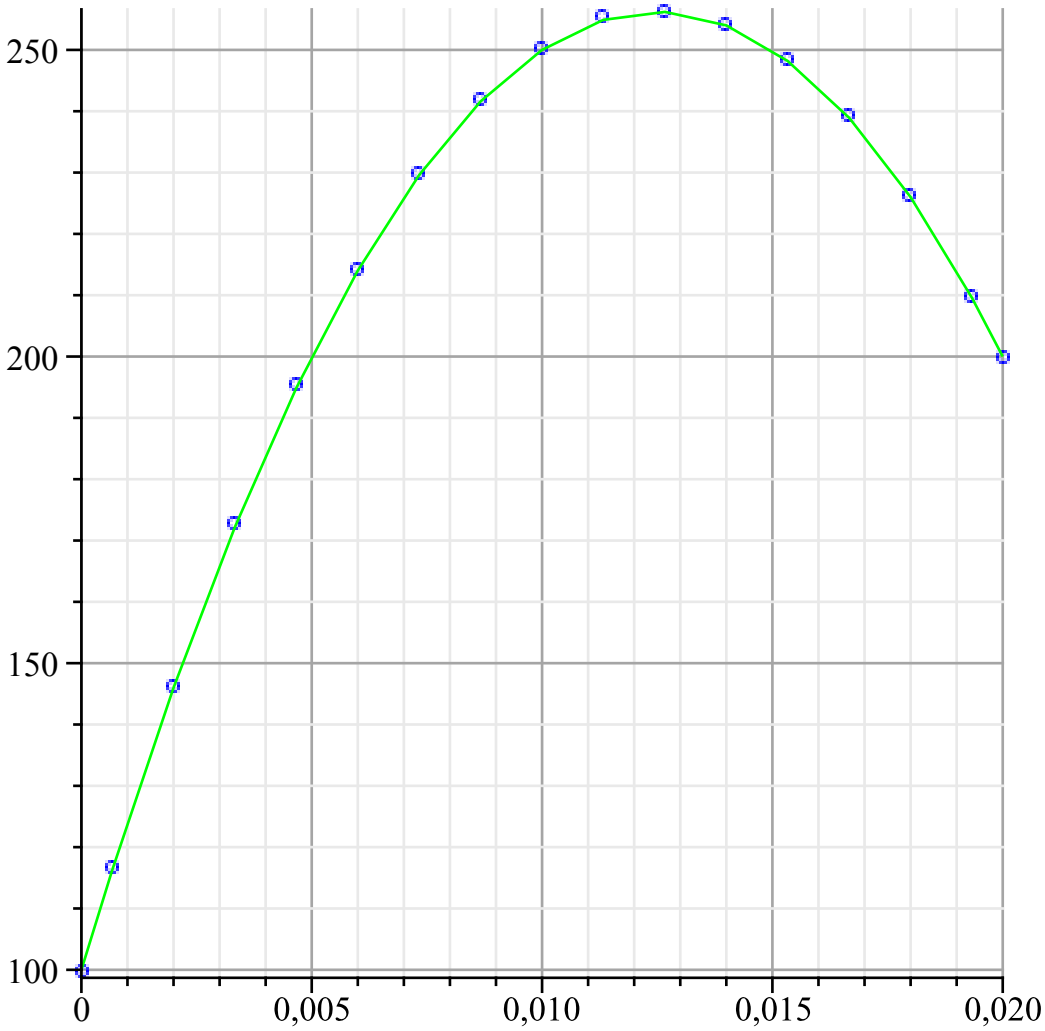
Courbe Théorique avec une liste de points:

> *listplot(lpT, color = green, gridlines = true)*



Tracé des deux courbes ensembles:

> *multiple(listplot, [lpN, color = blue, style = point, symbol = circle], [lpT, color = green, style = line], color = black, gridlines = true)*



> Erreur relative:

```
> for i from 1 to N do
  x[i];
  T[i];
  F(x[i]);
  (T[i] - F(x[i]))/F(x[i]) * 100
  end do
```

0.0006665
116.7
116.2
0.4303
0.002000
146.5
146.0
0.3425
0.003332
172.7
172.2
0.2904
0.004666
195.3
194.9
0.2052
0.005998

214.4
214.0
0.1869
0.007332
230.0
229.6
0.1742
0.008664
242.0
241.6
0.1656
0.009998
250.4
250.0
0.1600
0.01133
255.3
254.9
0.1569
0.01267
256.6
256.2
0.1561
0.01400
254.4
254.0
0.1575
0.01533
248.6
248.2
0.1612
0.01667
239.3
238.9
0.1674
0.01800
226.4
226.0
0.1770
0.01933
210.0
209.6
0.1908

(1.1.12)

>