Equation de Convection-Diffusion 1D

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EXEMPLE 1

Détermination de la distribution de x6(x) transportée par convection-diffusion à travers un domaine 1D dont les extrémités sont soumises aux (C.L.):
Schéma Upwind pour le terme convectif.

$$\frac{d}{dx}\left(\rho \cdot u \cdot \phi(x)\right) - \frac{d}{dx}\left(\Gamma \cdot \frac{d}{dx}\phi(x)\right) = 0$$

Conditions aux limites (C.L):

$$\phi(0) = \phi_0 = 1,$$
$$\phi(L) = \phi_L = 0,$$

u = 0.1 m/s , δx=0.2:

```
> Restart : Digits := 4 :
> u := 0.1; L := 1.0; ρ := 1.0; Gam := 0.1; S := 1.0; ndx := 5;
```

$$u := 0.1$$
$$L := 1.0$$
$$\rho := 1.0$$
$$Gam := 0.1$$
$$S := 1.0$$
$$ndx := 5 \qquad \textbf{(1.1)}$$

$$> \delta x := \frac{L}{ndx};$$
$$F := \rho \cdot u \cdot S;$$
$$d := \frac{Gam \cdot S}{\delta x};$$
$$Pe := \frac{F}{d};$$

$$\delta x := 0.2000$$
$$F := 0.100$$
$$d := 0.5000$$
$$Pe := 0.2000 \quad (1.2)$$

$$> i_{max} := ndx;$$
$$i_{max} := 5 \quad (1.3)$$

Nombre d'équations:

$$> Ne := i_{max}$$
$$Ne := 5 \quad (1.4)$$

Abscisses des noeuds:

```
> x[0] := 0;
  for i from 1 to i_max do
      x[i] := δx/2 + (i - 1)·δx;
  end do;
  x[i_max + 1] := L;
```

$$x_0 := 0$$
$$x_1 := 0.1000$$
$$x_2 := 0.3000$$
$$x_3 := 0.5000$$
$$x_4 := 0.7000$$
$$x_5 := 0.9000$$
$$x_6 := 1.0 \quad (1.5)$$

Conditions aux Limites:

```
> ϕ[0] := 1.0;
  ϕ[i_max + 1] := 0;
  ϕ[L] := ϕ[i_max + 1];
```

$$\phi_0 := 1.0$$
$$\phi_6 := 0$$
$$\phi_{1.0} := 0 \quad (1.6)$$

Noeuds internes:

```
> for i from 2 to i_max - 1 do
     Sp[i] := 0;
     Su[i] := 0;
     a_W[i] := d + F;
     a_E[i] := d;
     a_P[i] := a_W[i] + a_E[i] - Sp[i];
  end do;
```

$$Sp_2 := 0$$

$$Su_2 := 0$$
$$a_{W_2} := 0.6000$$
$$a_{E_2} := 0.5000$$
$$a_{P_2} := 1.100$$
$$Sp_3 := 0$$
$$Su_3 := 0$$
$$a_{W_3} := 0.6000$$
$$a_{E_3} := 0.5000$$
$$a_{P_3} := 1.100$$
$$Sp_4 := 0$$
$$Su_4 := 0$$
$$a_{W_4} := 0.6000$$
$$a_{E_4} := 0.5000$$
$$a_{P_4} := 1.100 \qquad \textbf{(1.7)}$$

Noeud gauche:

$> Sp[1] := -(2\cdot d + F);$
$Su[1] := (2\cdot d + F)\cdot \phi[0];$
$a_W[1] := 0;$
$a_E[1] := d;$
$a_P[1] := a_W[1] + a_E[1] - Sp[1];$

$$Sp_1 := -1.100$$
$$Su_1 := 1.100$$
$$a_{W_1} := 0$$
$$a_{E_1} := 0.5000$$
$$a_{P_1} := 1.600 \qquad \textbf{(1.8)}$$

Noeud droit:

$> Sp[i_{max}] := F - 2\cdot d;$
$Su[i_{max}] := (2\cdot d - F)\cdot \phi[i_{max} + 1];$
$a_W[i_{max}] := d + F;$
$a_E[i_{max}] := 0;$
$a_P[i_{max}] := a_W[i_{max}] + a_E[i_{max}] - Sp[i_{max}];$

$$Sp_5 := -0.900$$
$$Su_5 := 0.$$
$$a_{W_5} := 0.6000$$
$$a_{E_5} := 0$$
$$a_{P_5} := 1.500 \qquad \textbf{(1.9)}$$

Equations:

$> k := 1$

(1.1.1)

$$k := 1 \quad (1.1.1)$$

Résolution pour les noeuds internes:

```
> for i from 1 to Ne do
      Eq[k] := a_P[ i]·φ[i] = a_W[ i]·φ[i - 1] + a_E[ i]·φ[i + 1] + Su[ i];
      k := k + 1;
  end do;
```

$$Eq_1 := 1.600\ \phi_1 = 1.100 + 0.5000\ \phi_2$$
$$k := 2$$
$$Eq_2 := 1.100\ \phi_2 = 0.6000\ \phi_1 + 0.5000\ \phi_3$$
$$k := 3$$
$$Eq_3 := 1.100\ \phi_3 = 0.6000\ \phi_2 + 0.5000\ \phi_4$$
$$k := 4$$
$$Eq_4 := 1.100\ \phi_4 = 0.6000\ \phi_3 + 0.5000\ \phi_5$$
$$k := 5$$
$$Eq_5 := 1.500\ \phi_5 = 0.6000\ \phi_4$$
$$k := 6 \quad (1.1.2)$$

Ecriture du système d'équations:

```
> for k from 1 to Ne do Eq[k] end do;
```

$$1.600\ \phi_1 = 1.100 + 0.5000\ \phi_2$$
$$1.100\ \phi_2 = 0.6000\ \phi_1 + 0.5000\ \phi_3$$
$$1.100\ \phi_3 = 0.6000\ \phi_2 + 0.5000\ \phi_4$$
$$1.100\ \phi_4 = 0.6000\ \phi_3 + 0.5000\ \phi_5$$
$$1.500\ \phi_5 = 0.6000\ \phi_4 \quad (1.1.3)$$

```
> Eqs := {seq(Eq[k], k = 1 ..Ne)};
```

$$Eqs := \{1.600\ \phi_1 = 1.100 + 0.5000\ \phi_2,\ 1.100\ \phi_2 = 0.6000\ \phi_1 + 0.5000\ \phi_3,\ 1.100\ \phi_3 = 0.6000\ \phi_2 + 0.5000\ \phi_4,\ 1.100\ \phi_4 = 0.6000\ \phi_3 + 0.5000\ \phi_5,\ 1.500\ \phi_5 = 0.6000\ \phi_4\} \quad (1.1.4)$$

```
> Tmps := [seq(φ[i], i = 1 ..Ne)];
```

$$Tmps := [\phi_1, \phi_2, \phi_3, \phi_4, \phi_5] \quad (1.1.5)$$

```
> SolT := solve(Eqs, Tmps);
```

$$SolT := [[\phi_1 = 0.9348, \phi_2 = 0.7914, \phi_3 = 0.6194, \phi_4 = 0.4129, \phi_5 = 0.1652]] \quad (1.1.6)$$

```
> with(LinearAlgebra) :
```

Forme matricielle:

```
> A, b := GenerateMatrix(Eqs, Tmps)
```

$$A, b := \begin{bmatrix} 1.600 & -0.5000 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.6000 & 1.500 \\ -0.6000 & 1.100 & -0.5000 & 0 & 0 \\ 0 & -0.6000 & 1.100 & -0.5000 & 0 \\ 0 & 0 & -0.6000 & 1.100 & -0.5000 \end{bmatrix}, \begin{bmatrix} 1.100 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (1.1.7)$$

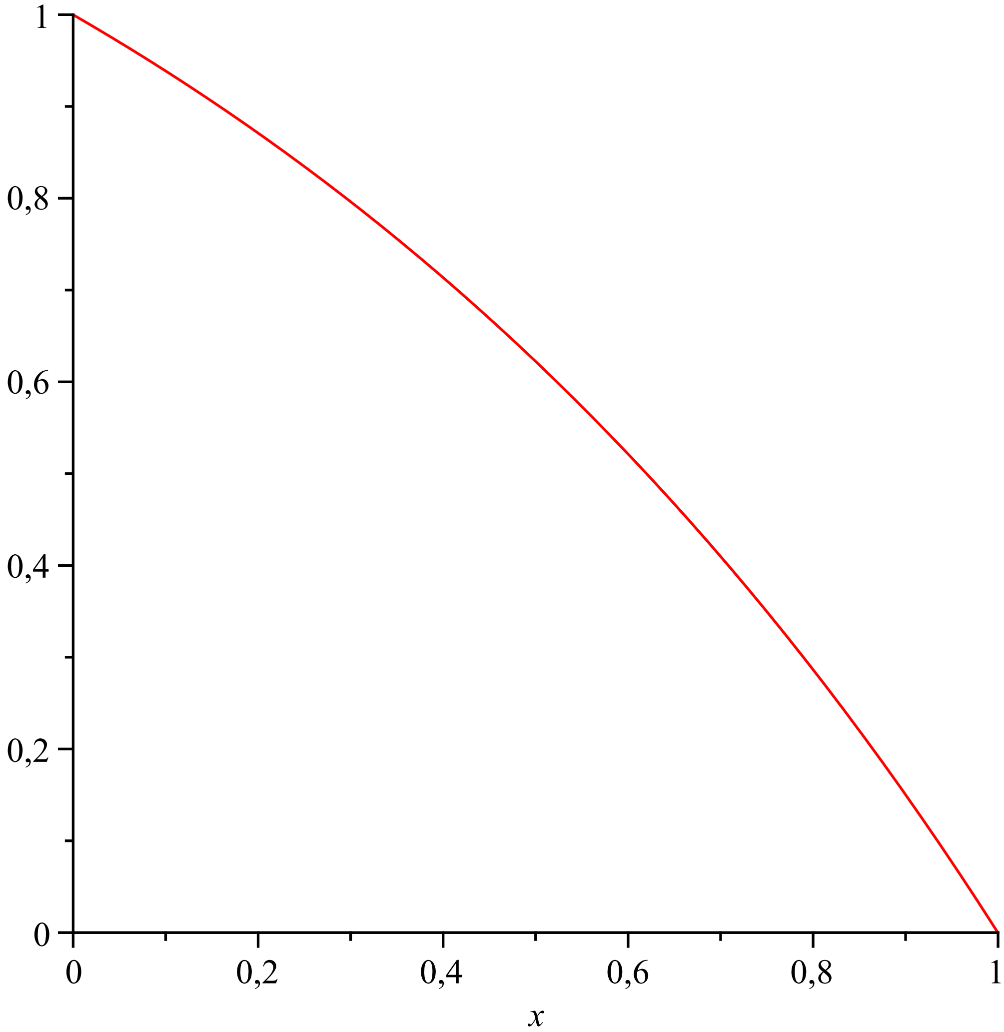
Solution exacte:

> $F(x) := \phi[0] + (\phi[L] - \phi[0]) \cdot \frac{e^{\left(\frac{\rho \cdot u \cdot x}{Gam}\right)} - 1}{e^{\left(\frac{\rho \cdot u \cdot L}{Gam}\right)} - 1};$

$$F := x \rightarrow \phi_0 + \frac{(\phi_L - \phi_0)\left(e^{\frac{\rho u x}{Gam}} - 1\right)}{e^{\frac{\rho u L}{Gam}} - 1} \tag{1.1.8}$$

> $with(plots):$

> $plot(F(x), x = 0..L);$



> **for** i **from** 1 **to** Ne **do**
    $\phi[i] := rhs(SolT_{1,i})$
end do;

$$\phi_1 := 0.9348$$
$$\phi_2 := 0.7914$$
$$\phi_3 := 0.6194$$
$$\phi_4 := 0.4129$$
$$\phi_5 := 0.1652 \tag{1.1.9}$$

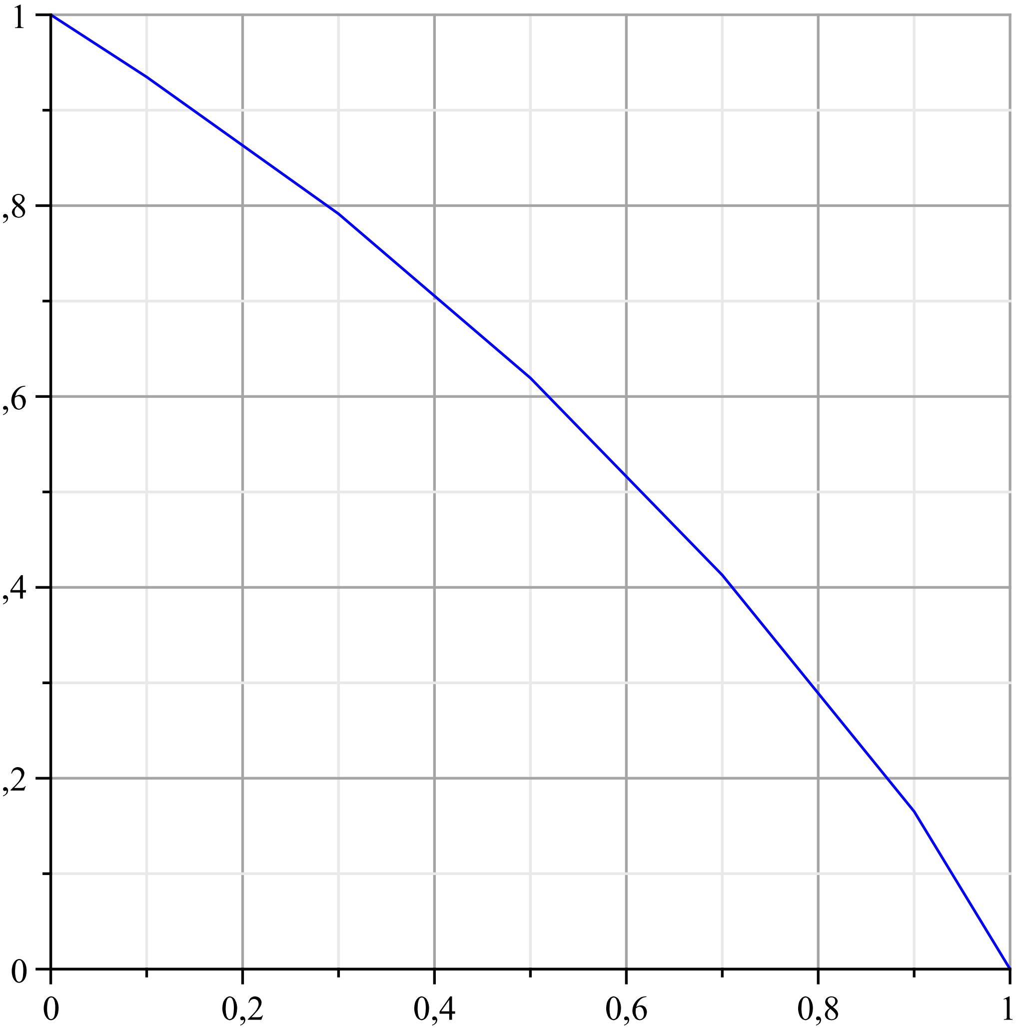
> $lpN := [\,seq([x[i], \phi[i]], i = 0..i_{max} + 1)\,]$

(1.1.10)

$lpN := [[0, 1.0], [0.1000, 0.9348], [0.3000, 0.7914], [0.5000, 0.6194], [0.7000, 0.4129], [0.9000, 0.1652], [1.0, 0]]$ **(1.1.10)**

Courbe Numérique:

> *listplot(lpN, color = blue, gridlines = true)*

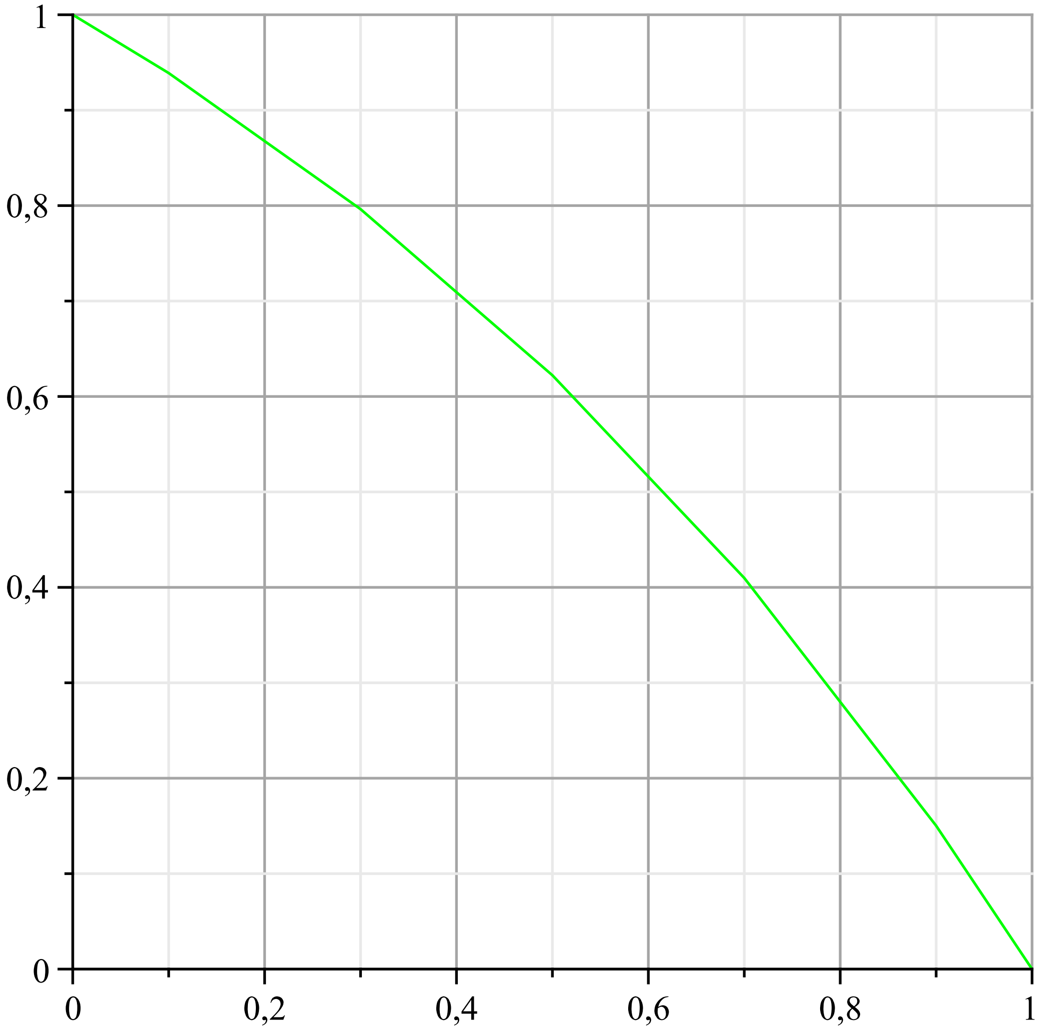


> $lpT := [seq([x[i], F(x[i])], i = 0..i_{max} + 1)]$

$lpT := [[0, 1.0], [0.1000, 0.9389], [0.3000, 0.7963], [0.5000, 0.6222], [0.7000, 0.4098], [0.9000, 0.1502], [1.0, 0.]]$ **(1.1.11)**

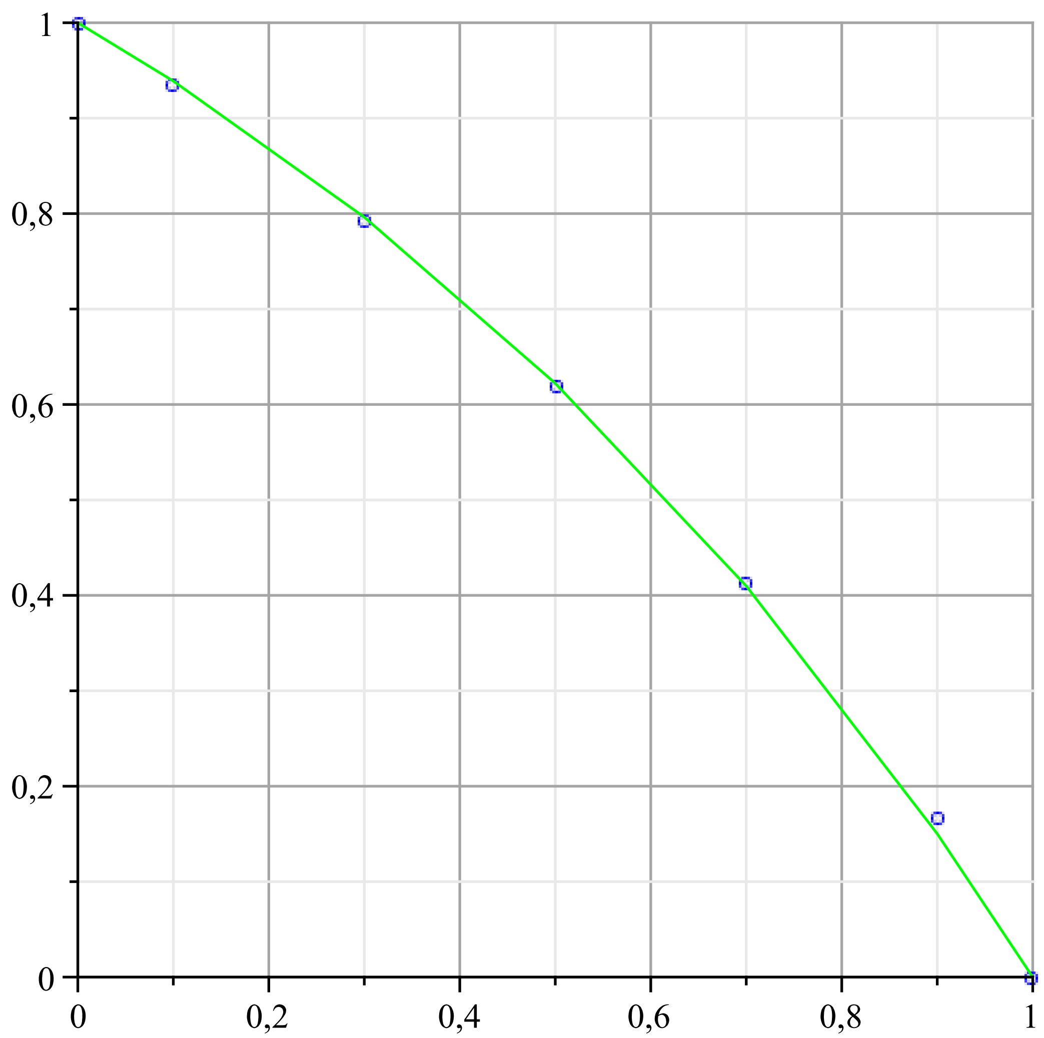
Courbe Théorique avec une liste de points:

> *listplot(lpT, color = green, gridlines = true)*



Tracé des deux courbes ensembles:

> *multiple*(*listplot*, [*lpN*, *color* = *blue*, *style* = *point*, *symbol* = *circle*], [*lpT*, *color* = *green*, *style* = *line*], *color* = *black*, *gridlines* = *true*)



Erreur relative:

$>$ **for** i **from** 1 **to** Ne **do**

$x[i]$;

$\phi[i]$;

$F(x[i])$;

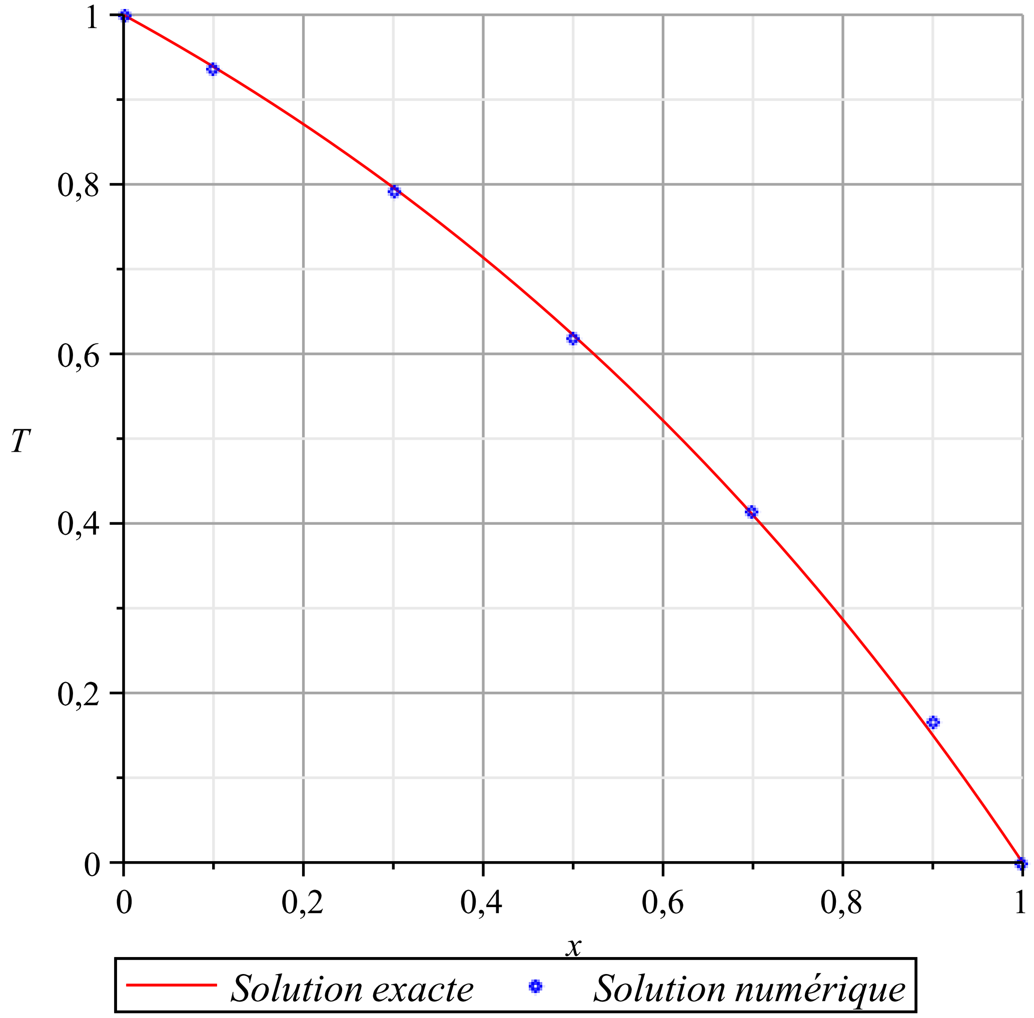
$$\frac{\phi[i]-F(x[i])}{F(x[i])}\cdot 100$$

end do

0.1000
0.9348
0.9389
-0.4367
0.3000
0.7914
0.7963
-0.6153
0.5000
0.6194
0.6222
-0.4500
0.7000
0.4129
0.4098
0.7565

$$\begin{array}{c} 0.9000 \\ 0.1652 \\ 0.1502 \\ 9.987 \end{array} \quad (1.1.12)$$

> $P1 := plot(F(x), x = 0 .. L, labels = [x, T], legend = Solution\ exacte)$:
> $P2 := plot(lpN, color = blue, style = point, legend = Solution\ numérique)$:
> $display(\{P1, P2\}, gridlines = true);$



> ?

`?` (1.1.13)

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