I-Real numbers

1.1-Introduction. Numbers are a central element in mathematics. Among the different types of numbers, the set $\mathbb{N} = \{0, 1, 2, ...\}$ of natural numbers, the set $\mathbb{Z} = \mathbb{N} \cup (-\mathbb{N})$, $[m \in (-\mathbb{N}) \Leftrightarrow \exists n \in \mathbb{N};\ m = -n]$ of relative numbers and the set $\mathbb{Q} = \left\{ \frac{p}{q};\ p \in \mathbb{Z},\ q \in \mathbb{N}^* \text{ and } p \wedge q = 1 \right\}$ of rational numbers, ($p \wedge q = 1$ means that p and q are prime to each other). Starting from $\mathbb{Q}$, whose well-known properties are assumed, namely, $(\mathbb{Q}, +, ., \leq)$ is a totally ordered set, the total order relation $\leq$, defined on $\mathbb{Q}$ is compatible with the addition $+$ and the multiplication $\times$, and that $\mathbb{Q}$ is Archimidean i.e., $\forall r \in \mathbb{Q}^*$ there exists $n \in \mathbb{N}^*$, such that $r < n$. The need to introduce a larger set than $\mathbb{Q}$, is motivated by the fact that $\sqrt{2} \notin \mathbb{Q}$. Indeed, if there exists $p \in \mathbb{Z}$ and $q \in \mathbb{N}^*$ with $p \wedge q = 1$, such that $p^2 = 2q^2$, then 2 divides p^2, as the square of an odd number is odd, also 2 divides p, so there exists $p' \in \mathbb{Z}$, such that $p = 2p'$, hence $2p'^2 = q^2$ and therefore 2 divides q, which contradicts $p \wedge q = 1$. Also, the two numbers e and π are not rational. In general, if p is a prime number, then $\sqrt{p}$ is not an rational number,...etc. Such numbers are called irrational numbers. The union of rational numbers and irrational numbers, constitutes the set $\mathbb{R}$ of real numbers. The object of the following section, is to define the set of real numbers by a series of axioms, and to give a second motivation for the introduction of this set.

1.2-Axiomatic definition of real numbers

Since, the set of real numbers, was introduced to complete the set $\mathbb{Q}$ of rational numbers, then we say that x is a real number if either $(x \in \mathbb{Q})$, or ($x \notin \mathbb{Q}$, x is said to be an irrational number). The intuition of their existence is ancient (since Pythagoras and his proof of the irrationality of $\sqrt{2}$). Their rigorous construction, dating from the 19^{iem} century by Cantor and Dedekine. Note that we can define a real number from its decimal development, i.e. a real x can be seen as a relative integer constituting its integer part, separated by a comma, followed by an infinity of digits constituting its decimal part for example: $\pi = 3.1415926536....$ This definition called arithmetic representation of a real number poses a certain number of problems. Also, a real number can be defined as a limit of the so-called Cauchy sequences in $\mathbb{Q}$ (the density of $\mathbb{Q}$ in $\mathbb{R}$). One of the simplest definitions of $\mathbb{R}$ is the following axiomatic definition.

Definition 2.1. The set $\mathbb{R}$ of real numbers, provided with two internal laws: the addition noted $+$, the multiplication noted $\times$. and a comparison relation noted $\leq$ (lower or equal), satisfies the following axioms:

$1-(\mathbb{R}, +, \times)$ is a commutative field.

$*$ The addition is such that $(\mathbb{R}, +)$ is an Abelian group.

$a_1)$ $\forall x, y, z \in \mathbb{R}$, $(x + y) + z = x + (y + z)$. The addition is associative.

$a_2)$ $\forall x, y \in \mathbb{R}$, $x + y = y + x$. The addition is commutative.

$a_3)$ $\forall x \in \mathbb{R}$, $x + 0 = x$. 0 is a neutral element for addition.

$a_4)$ $\forall x \in \mathbb{R}$, $x + (-x) = 0$. Each element x admits a symmetric for the addition noted $-x \in \mathbb{R}$.

$*$ The multiplication is such that $(\mathbb{R}^*, \times)$ is an Abelian group, $(\mathbb{R}^* = \mathbb{R} \setminus \{0\})$.

a_5) $\forall x, y, z \in \mathbb{R}$, $(x \times y) \times z = x \times (y \times z)$. Multiplication is associative.

a_6) $\forall x, y \in \mathbb{R}$, $x \times y = y \times x$. Multiplication is commutative.

a_7) $\forall x \in \mathbb{R}$, $x \times 1 = x$. 1 is a neutral element for multiplication.

a_8) $\forall x \in \mathbb{R}^*$, $x \times x^{-1} = 1$. Each element x admits the reverse for multiplication, noted $x^{-1} \in \mathbb{R}$, x^{-1} is also noted $\dfrac{1}{x}$.

$*$ Multiplication is distributive with respect to addition:

a_9) $\forall x, y, z \in \mathbb{R}$, $x \times (y + z) = (x \times y) + (x \times z)$.

$2-$ $(\mathbb{R}, \leq)$ is completely ordered.

a_{10}) $\forall x \in \mathbb{R}$, $x \leq x$. The comparison relationship is reflexive.

a_{11}) $\forall x, y, z \in \mathbb{R}$, if ($x \leq y$ and $y \leq z$) then $x \leq z$. The comparison relationship is transitive.

a_{12}) $\forall x, y \in \mathbb{R}$, if ($x \leq y$ and $y \leq x$) then $x = y$. The comparison relationship is antisymmetric.

a_{13}) $\forall x, y \in \mathbb{R}$, $x \leq y$ or $y \leq x$. The comparison relation is a total order relation.

For every $x, y \in \mathbb{R}$, we write $x \leq y$ or equivalently $y \geq x$ (y is upper than or equal to x), and the relation ($x \leq y$; $x \neq y$) is written $x < y$ (x less than y).

A real number x is said to be positive if $0 < x$, the set of positive real numbers is denoted by $\mathbb{R}_+^*$. x is said to be negative if $x < 0$, the set of negative real numbers is denoted by $\mathbb{R}_-^*$. For every $x, y \in \mathbb{R}$, we write $x - y$ instead of $x + (-y)$ and xy instead of $x \times y$.

$3-$Compatibility of the relation $\leq$ with addition and multiplication.

a_{14}) $\forall x, y, x', y' \in \mathbb{R}$, satisfying ($x \leq y$ and $x' \leq y'$), we have $x + x' \leq y + y'$. The relation $\leq$ is compatible with addition.

a_{15}) $\forall x, y, x', y' \in \mathbb{R}^*$, checking ($x \leq y$ and $x' \leq y'$), we have $xx' \leq yy'$. The relation $\leq$ is compatible with multiplication.

As a consequence, for every x, y in $\mathbb{R}$, if $x \leq y$ then $-y \leq -x$ and for every x, y in $\mathbb{R}^*$, if $x \leq y$ then $y^{-1} \leq x^{-1}$.

1.3. Intervals, absolute value, bounded parts

Definition 2.2. A non-empty part E in $\mathbb{R}$ is an interval if, $\forall x, y \in E$ satisfying $x < y$, there exists $z \in E$ such that $x < z < y$.

If a, b and x_0 are three real numbers such that: $a < x_0 < b$. The unbounded open intervals of $\mathbb{R}$ are: $]-\infty, a[$, $]b, +\infty[$, $\mathbb{R} =]-\infty, +\infty[$, and the open bounded interval of $\mathbb{R}$ is $]a, b[$. The unbounded closed intervals of $\mathbb{R}$ are: $]-\infty, a]$, $[b, +\infty[$, $\mathbb{R} =]-\infty, +\infty[$ and the closed bounded interval of $\mathbb{R}$ is $[a, b]$. Neither open nor closed bounded intervals of $\mathbb{R}$ are $]a, b]$, $[a, b[$. In the case where $a = b$, $[a, a] = \{a\}$ and $]a, a[= \phi$. The numbers a and b are called the limits of the interval and $b - a$ is its length. The total order relation makes it possible to define the absolute value function in $\mathbb{R}$.

Definition 2.3. The absolute value in $\mathbb{R}$, is an application noted $|.|$, defined from $\mathbb{R}$ to $\mathbb{R}_+$ by: $\forall x \in \mathbb{R}, |x| = \begin{cases} x, & \text{if } 0 \leq x; \\ -x, & \text{if } x < 0. \end{cases}$ As a direct consequence we have: $\forall x \in \mathbb{R}$, $x \leq |x|$, if $\alpha \in \mathbb{R}_+$ (fixed), then for every $x \in \mathbb{R}$, $|x| \leq \alpha$ iff $-\alpha \leq x \leq \alpha$, (iff means, if and only if).

Proposition 2.1. The following are true, for every $x, y \in \mathbb{R}$:

1) $x \in \mathbb{R}$, $|x| = 0$ iff $x = 0$.
2) $|xy| = |x|\,|y|$.
3) $|x+y| \leq |x| + |y|$, (triangular inequality).
4) $||x| - |y|| \leq |x-y|$.

Proof.

1) evident.

2) if x and y have the same sign, then $|xy| = xy$. In the case where $x, y \in \mathbb{R}_+$, $|x| = x$ and $|y| = y$, and in the case where $x, y \in \mathbb{R}_-$, $|x| = -x$ and $|y| = -y$, so in both cases $|x|\,|y| = xy$. If x and y are of different signs, then $|xy| = -xy$. In the case where for example $x \in \mathbb{R}_+$ and $y \in \mathbb{R}_-$, $|x| = x$ and $|y| = -y$, then $|x|\,|y| = x(-y) = -xy$.

3) Since, from, 2) $\forall z \in \mathbb{R}$, $|z|^2 = z^2$, then for any $x, y \in \mathbb{R}$, we have $|x+y|^2 = (x+y)^2 = |x|^2 + 2xy + |y|^2 \leq |x|^2 + 2\,|x|\,|y| + |y|^2$
$= (|x| + |y|)^2$, so $|x+y| \leq |x| + |y|$.

4) We demonstrate in the same way that: $\forall x, y \in \mathbb{R}$, $||x| - |y|| \leq |x+y|$, and by replacing y by $-y$ in the last inequality, we get the result.

Definition 2.4. Let E be a non-empty part of $\mathbb{R}$. We say that:

i) E is bounded above, if there is a real number M such that, $\forall x \in E$, $x \leq M$, in this case M is called an upper bound of E.

ii) E is bounded below, if there is a real number m such that, $\forall x \in E$, $m \leq x$, in this case m is called a lower bound of E.

iii) E is bounded, if E is both bounded above and below. Equivalently E is bounded, iff there exists $\alpha \in \mathbb{R}_+$, such that $\forall x \in E$, $|x| \leq \alpha$.

Remark 2.1

a) If M is an upper bound of E, any element greater than M is also an upper bound of E. When E is bounded above, the least upper bound of E is called the supremum of E, and denoted by $supE$, or $\max E$ if it belongs to E. The $supE$ when it exists, it is unique.

b) If m is a lower bound of E, any element less than m is also a lower bound of E. When E is bounded below, the first lower bound of E is called the infimum of E and denoted by $\inf E$, or $\min E$ if it belongs to E. The $\inf E$ when it exists, it is unique.

a) In the case where a non-empty part E of $\mathbb{R}$ is bounded, $[\inf E, \sup E]$ is the smallest closed interval containing E.

Let us end the axiomatic definition of $\mathbb{R}$, by the following.

4)$-$Axiom of the upper bound

a_{16}) Any non empty, bounded above (respectively bounded below) part of $\mathbb{R}$, has an supremum (respectively an infimum).

Remark 2.2. If $x, y \in \mathbb{R}$ such that $x < y + \epsilon$, $\forall \epsilon > 0$, then $x \leq y$. Indeed, suppose that $x > y$ then for $\varepsilon = x - y$, we have $x < y + x - y = x$, a contradiction.

Proposition 2.1. Let E be a bounded part of $\mathbb{R}$, M_0 and m_0 two real numbers, then:

1) $M_0 = supE$ iff $\begin{cases} i)\ \forall x \in \mathbb{R},\ x \leq M_0; \\ \quad ii)\ \forall \varepsilon > 0,\ \text{there exists } x_\varepsilon \in E,\ \text{such that } M_0 - \varepsilon < x_\varepsilon. \end{cases}$

2) $m_0 = \inf E$ iff $\begin{cases} i)\ \forall x \in \mathbb{R},\ m_0 \leq x; \\ \quad ii)\ \forall \varepsilon > 0,\ \text{there exists } x_\varepsilon \in E,\ \text{such that } x_\varepsilon < m_0 + \varepsilon. \end{cases}$

Proof.

1) Since M_0 is the an upper bound of E, then $i)$ $\forall x \in E$, $x \leq M_0$. To demonstrate $ii)$, suppose that there exists $\varepsilon > 0$, such that $\forall x \in E$, $x \leq M_0 - \varepsilon$, that is $M_0 - \varepsilon$ is an upper bound of E less than M_0, contradiction with the definition of $supE$. Reciprocally $i)$ implies that M_0 is an upper bound of E.To demonstrate that M_0 is the least upper bound of E, suppose that there exists $M_0' < M_0$, such that $M_0' = supE$. According to $i)$ and $ii)$ $\forall \varepsilon > 0$, there exists $x_\varepsilon \in E$, such that $M_0 - \varepsilon < x_\varepsilon \leq M_0' < M_0$, so $M_0 < M_0' + \epsilon$, using the remark 2.2, we get $M_0' = M_0$. Property 2) is demonstrated in the same way.

Example 2.1.

$a)$ If, $E = \{-1, 0, 1\}$ then, $\inf E = \min E = -1$ and $supE = maxE = 1$.

$b)$ If $E = [0, 1]$ then, $\inf E = \min E = 0$ and $supE = maxE = 1$.

$c)$ If $E = [0, 1[$ then, $\inf E = \min E = 0$ and $supE = 1$.

$d)$ If $E =]0, 1]$ then, $\inf E = 0$ and $supE = 1$.

$e)$ If $E =]0, 1[$ then, $\inf E = 0$ and $supE = 1$.

Let us demonstrate, for example that in $e)$ $supE = 1$. Using property $a)$ in Proposition 2.2, it is clear that $i)$ $\forall x \in E$, $x < 1$. To demonstrate $ii)$, let $\varepsilon > 0$, if $\varepsilon \leq 1$ then $0 \leq 1 - \epsilon < 1$, as $\mathbb{R}$ is an interval, there exists $x_\varepsilon \in \mathbb{R}$ such that $1 - \epsilon < x_\varepsilon < 1$, so $x_\varepsilon \in E$. If, $1 < \varepsilon$, then $1 - \varepsilon < 0 < x$, $\forall x \in E$.

Example 2.2. Let $E = \{r \in \mathbb{Q}_+,\ r^2 < 2\}$ be a part of $\mathbb{Q}$, then $E \neq \phi$ and $\forall r \in E$, $0 \leq r < \sqrt{2} < 2$, hence E is bounded in $\mathbb{Q}$, and $\min E = 0 \in \mathbb{Q}_+$. But $supE$ it isn't in $\mathbb{Q}$, which shows that the axiom a_{16} of the upper bound is not true in $\mathbb{Q}$. Hence, once again the need to introduce $\mathbb{R}$.

Let us prof that $supE \notin \mathbb{Q}$. Suppose that, there exist $p \in \mathbb{Z}$, $q \in \mathbb{N}^*$ with $p \wedge q = 1$, such that $supE = \dfrac{p}{q} = r$. In the case when $0 < 2 - r^2$, we have $s = \dfrac{2 - r^2}{5} \in \mathbb{Q}_+^*$, so $s < 1$ and $(r + s)^2 = r^2 + 2rs + s^2 < r^2 + 5s = 2$, witch implies that $r + s \in E$, therefore $s \leq 0$ contradiction. In the case when $0 < r^2 - 2$, we have $s = \dfrac{r^2 - 2}{5} \in \mathbb{Q}_+^*$, so $s < 1$ and $(r - s)^2 > r^2 - 2rs > r^2 - 4s = \dfrac{r^2 + 8}{5} > 2$, it follows that $r - s \in \mathbb{Q}_+^*$ and $r - s$ is an upper bound of E, witch is less than r, a contradiction.

1.4 Archimedes' axiom, density of $\mathbb{Q}$ in $\mathbb{R}$

In the sequel, S^C denotes the complement of any set $S \subset \mathbb{R}$.

Proposition 2.2 (Archimedes' axiom). For every $x, y \in \mathbb{R}_+^*$ satisfying $x < y$ there exists $n \in \mathbb{N}^*$, such that $y \leq nx$, that is $\mathbb{R}$ an Archimedian.

Proof. Suppose that, there exists x_0 and y_0 in $\mathbb{R}$, $x_0 < y_0$ and for all $n \in \mathbb{N}^*$, $nx_0 < y_0$. Since a non empty part $E = \{nx_0;\ n \in \mathbb{N}^*\}$ is bounded above by y_0. For $M_0 = \sup E$ and $\epsilon = \dfrac{M_0}{2} > 0$, there exists $n_0 \in \mathbb{N}^*$ such that, $M_0 - \dfrac{M_0}{2} < n_0 x_0$, hence $M_0 < (2n_0)\, x_0$, as $2n_0 \in E$, contradiction.

Remark 2.3.

a) The set $\mathbb{N}$ of natural numbers is unbounded above. That is for every $y \in \mathbb{R}_+^*$, there exists $n \in \mathbb{N}^*$, such that $y \leq n$. It suffices to take $x = 1$ in the proposition 2.2.

b) The set $\mathbb{Z}$ of relative numbers is both unbounded above and below, since $-\mathbb{N}$ is unbounded below.

Definition 2.4 (dense part in $\mathbb{R}$). A non-empty part in $\mathbb{R}$, is said to be dense in $\mathbb{R}$ if, for all x, y in $\mathbb{R}$, $x < y$ there exists $z \in E$, such that $x < z < y$.

Proposition 2.3. $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. Let x, y be in $\mathbb{R}$ with $x < y$. Let us prove that there exists r in $\mathbb{Q}$ such that: $x < r < y$. Since $z = \dfrac{1}{y-x} > 0$, there exists $n \in \mathbb{N}^*$ such that $z = \dfrac{1}{y-x} < n$, or $nx + 1 < ny$ $(*)$, likewise for $nx \in \mathbb{R}$, there exists $k \in \mathbb{N}^*$ such that $nx < k$. Let $E = \{k \in \mathbb{N}^*;\ nx < k\}$ and $F = \{nx \in \mathbb{R};\ z < n\}$, E and F are non-empty, and F is bounded above by the elements of E. Let $p = \sup F$, then $p \in E$ and for $\varepsilon = 1$, there exists $n \in \mathbb{N}^*$ such that $p - 1 < nx < p$, witch implies that $nx < p < nx + 1$, using $(*)$ we obtain $nx < p < ny$ or $x < \dfrac{p}{n} < y$, $(r = \dfrac{p}{n} \in \mathbb{Q})$.

In the sequel, S^C denotes the complement of any set $S \subset \mathbb{R}$.

Applications:

a) $\sqrt{2}$ is the supremum of $E = \{r \in \mathbb{Q}_+,\ r^2 < 2\}$, indeed *i*) $\forall r \in E$, $r < \sqrt{2}$, *ii*) For $0 < \epsilon \leq \sqrt{2}$, we have $0 \leq \sqrt{2} - \epsilon < \sqrt{2}$, since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $r_\epsilon \in \mathbb{Q}$ such that, $0 \leq \sqrt{2} - \epsilon < r_\epsilon < \sqrt{2}$ $(r_\epsilon \in E)$. If, $\sqrt{2} < \epsilon$, then $\sqrt{2} - \epsilon < 0 \leq r$, $\forall r \in E$.

b) The set $\mathbb{Q}^C$, of the irrational numbers is dense in $\mathbb{R}$. Note that, for every $\alpha, \beta \in \mathbb{Q}$ $(\beta \neq 0)$, $\alpha + \beta\sqrt{2} \in \mathbb{Q}^C$. Then if, $x, y \in \mathbb{R}$ $x < y$, there exists $r \in \mathbb{Q}$ such that $x < r < y$. Since $\dfrac{\sqrt{2}}{y-r} \in \mathbb{R}_+^*$, there exists $n \in \mathbb{N}^*$, such that $\dfrac{\sqrt{2}}{y-r} \leq n$, then $x < r + \dfrac{1}{n}\sqrt{2} < y$ $(r + \dfrac{1}{n}\sqrt{2} \in \mathbb{Q}^C)$.

Exercise series n^{o}1

Exercise 1. Prof that

a) If $r \in \mathbb{Q}$ and $x \notin \mathbb{Q}$, then $r + x \notin \mathbb{Q}$ and if $r \neq 0$, then $rx \notin \mathbb{Q}$.

b) If x and y are irrational, then xy is not always an irrational.

c) $\forall x \in \mathbb{R}_+^*$, $-x < 0$ and $0 < \dfrac{1}{x}$.

d) $\forall x, y \in \mathbb{R}_+$, such that $x \leq y$, then: $-y \leq -x$ and if $x, y \in \mathbb{R}^*$, then $\dfrac{1}{y} \leq \dfrac{1}{x}$.

Exercise 2.

a) Prof that $\forall x, y \in \mathbb{R}$;

1) $|x| = 0 \Leftrightarrow x = 0$;

2) $|x| \leq \alpha \Leftrightarrow -\alpha \leq x \leq \alpha$ where $\alpha \in \mathbb{R}_+$.

3) Suppose that $x_1, x_2, ..., x_n$ are real numbers. Prove that $|x_1+x_2+...+x_n \leq |x_1|+|x_2|+...+|x_n|$

4) $|x+y| \leq |x+y|+|x-y|$.

5) $max(|x|,|y|) \leq \sqrt{x^2+y^2} \leq \sqrt{2}max(|x|,|y|)$

6) $\dfrac{|x+y|}{1+|x+y|} \leq \dfrac{|x|}{1+|x|+|y|}+\dfrac{|y|}{1+|x|+|y|}$.

7) Prove that, $|x-y| < \epsilon$ for every $\epsilon > 0$ iff $x = y$.

$b)$ Solve in $\mathbb{R}$; $|x-2| \leq 1$ and $\left|x^2-1\right| \leq 2$.

Exercise 3. Let E be a non-empty part of $\mathbb{R}$ and $m \in \mathbb{R}$.

1) If E is bounded, then $\inf E$ and $\sup E$ are unique.

2) Prof that, E is bounded iff there exists $k \in \mathbb{R}_+$ such that, $\forall x \in E$; $|x| \leq k$.

3) Suppose that E is bounded below. Prof that:

4) $m = \inf E \Leftrightarrow \begin{cases} i)\ \forall x \in E;\ m \leq x; \\ ii)\ \forall \varepsilon > 0;\ \exists x_\varepsilon \in E \text{ such that, } x_\varepsilon < m+\epsilon. \end{cases}$

5) $-E = \{-x;\ x \in E\}$ is bounded above and $\sup(-E) = -\inf E$.

6) Prof that if F is a non-empty part of $\mathbb{R}$ included in a bounded part E of $\mathbb{R}$, then F is bounded; and: $\inf E \leq \inf F$; $\sup F \leq \sup E$.

7) Noting that $\forall x, y \in \mathbb{R}_+; x+y \geq 2\sqrt{xy}$. Prof that if $E = \left\{x+\dfrac{1}{x};\ x \in \mathbb{R}_+^*\right\}$; then E is bounded below and $\min E = 2$.

Exercise 4. Let E, F be two non-empty and bounded parts in $\mathbb{R}$. We define the sum and the product of E and F, by: $E+F = \{x+y;\ x \in E,\ y \in F\}$ and $EF = \{xy;\ x \in E,\ y \in F\}$. Prof that:

1) $E+F$ is bounded; $\inf(E+F) = \inf E + \inf F$ and $\sup(E+F) = \sup E + \sup F$.

2) In the case where E and F are positive terms, EF is bounded, $\inf(EF) = \inf E \inf F$ and $\sup(EF) = \sup E \sup F$.

3) Calculate the min. and max of $E+F$ and EF in the following two cases:

$i)$ $E = \{-1, 0\}$ and $F = \{-2, -1\}$.

$ii)$ $E = \{-1, 0\}$ and $F = \{0, 1\}$.

What can we deduce?

4) Let $E = \left\{1-\dfrac{1}{n}; n \in \mathbb{N}^*\right\}$. Determine $minE$ and $supE$, justify.

Exercise 5. Let E, F be two non-empty and bounded parts in $\mathbb{R}$.

1) Prof that: $E \cup F$ is bounded and that: $\sup(E \cup F) = \max(\sup E, \sup F)$ and $\inf(E \cup F) = \min(\inf E, \inf F)$.

2) Suppose that $E \cap F$ is non-empty. Prof that: $E \cap F$ is bounded, $\sup(E \cap F) \leq \min(\sup E, \sup F)$ and $\max(\inf E, \inf F) \leq \inf(E \cap F)$.

Exercise 6. The integer part of $x \in \mathbb{R}$, is the largest element of $\mathbb{Z}$, noted $[x]$ such that: $[x] \leq x$. Prof that: $\forall x, y \in \mathbb{R}$

1) $[x]$ exists and it is unique.

2) $[x] \leq x < [x]+1$ and $x-[x] \in [0, 1[$.

3) $\forall z \in \mathbb{Z}$, $[z] = z$ and $[z+x] = z+[x]$.

4) $[x]+[y] \leq [x+y] < [x]+[y]+1$, if $x \leq y$ then $[x] \leq [y]$.

5) $\forall n \in \mathbb{N}$, $n[x] \leq [nx]$.

Exercise 7. Let $E = \left\{r \in \mathbb{Q}_-;\ r^2 < 2\right\}$..

1) Prof that E is bounded and determine $infE$ and $maxE$.
2) Prof that $E = \left\{r^3;\ r \in \mathbb{Q}\right\}$ is dense in $\mathbb{R}$.