C_0 semigroup of contractions

In all what follows X will be a Banach space on the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

1.Definition A one parameter family S_t, $t \geq 0$ of linear bounded operators from X into X is a semigroup if:

(i) $S_0 = I$, the identity operator of X

(ii) $S_{t+s} = S_t S_s$, for every $t, s \geq 0$

the linear operator A defined by $A(x) = \lim\limits_{t \to 0} \dfrac{S_t x - x}{t}$, with domain $D(A) = \left\{ x \in X : \lim\limits_{t \to 0} \dfrac{S_t(x) - x}{t}, \text{ exists} \right\}$ is the infinitesimal generator of the semigroup S_t.

2.Definition A C_0 semigroup of linear bounded operators on X is a semigroup satisfying:

$\lim\limits_{t \to 0} S_t x = x$ for every $x \in X$, that is $\lim\limits_{t \to 0} \|S_t x - x\| = 0$.

3.Definition A C_0 semigroup S_t, $t \geq 0$ on X satisfying $\|S_t\| \leq 1, \forall t \geq 0$ is called a C_0 semigroup of contractions.

The following Theorem gives some useful properties of a C_0 semigroup of contractions:

4.Theorem Let S_t, $t \geq 0$ be a C_0 semigroup of linear bounded operators on X then we have:

(1). For each $x \in X$ the function $t \longrightarrow S_t x$ from $[0, \infty[$ into X is continuous on $[0, \infty[$.

(2). For all $x \in X$ and all $t \geq 0$, $\lim\limits_{h \longrightarrow 0} \dfrac{1}{h} \displaystyle\int_t^{t+h} S_s x.ds = S_t x$

(3). For all $x \in X$ and all $t \geq 0$, $\displaystyle\int_0^t S_s x.ds \in D(A)$ and

$$A\left(\int_0^t S_s x.ds \right) = S_t x - x$$

(4). For all $x \in X$ and all $t > 0$, $S_t x \in D(A)$ and the function $t \longrightarrow S_t x$ is differentiable from $]0, \infty[$ into X and $\dfrac{d}{dt} S_t x = A(S_t x) = S_t A(x), \forall t > 0$

(5). For all $x \in X$ and all $t > s \geq 0$, we have:

$$S_t x - S_s x = \int_s^t A(S_u x).du = \int_s^t S_u A(x).du$$

Proof:

(1). By definition 2 it is clear that the function $t \longrightarrow S_t x$ is continuous at $t = 0$.

Now take any $t > 0$ so that for $h \geq 0$ we can write $S_{t+h}x - S_t x = S_t.(S_h x - x)$ which implies $\|S_{t+h}x - S_t x\| \leq \|S_t\| . \|S_h x - x\| \leq \|S_h x - x\|$ since $\|S_t\| \leq 1$ by the contraction condition. But $\lim\limits_{h \to 0} \|S_h x - x\| = 0$ by definition 2. If $h < 0$ and $t + h > 0$ write $S_{t+h}x - S_t x = S_{t+h}.(x - S_{-h}x)$, then $\|S_{t+h}x - S_t x\| \leq \|S_{t+h}\| . \|x - S_{-h}x\| \leq \|x - S_{-h}x\|$, because $\|S_{t+h}\| \leq 1$. Finally use the fact $\lim\limits_{h \to 0} \|x - S_{-h}x\| = 0$ to get $\lim\limits_{h \to 0} \|S_{t+h}x - S_t x\| = 0$.
So for both cases $h \geq 0$ and $h < 0$, $S_{t+h}x - S_t x$ goes to 0 as $h \longrightarrow 0$.
(2). comes from the continuity of the function $t \longrightarrow S_t x$, proved in (1), and usual properties of Riemann integral for Banach space valued functions.

(3). Fix $x \in X$ and $h > 0$, then we have $\frac{S_h - I}{h} \int\limits_0^t S_s x.ds = \frac{1}{h} \int\limits_0^t (S_{s+h}x - S_s x).ds$ because the operator $S_h - I : X \longrightarrow X$ is continuous. So we write:
$\frac{1}{h} \int\limits_0^t (S_{s+h}x - S_s x).ds = \frac{1}{h} \int\limits_0^t S_{s+h}x.ds - \frac{1}{h} \int\limits_0^t S_s x.ds$ and evaluate each integral as follows: making variable change $s + h = u$ we get
$\frac{1}{h} \int\limits_0^t S_{s+h}x.ds = \frac{1}{h} \int\limits_h^{t+h} S_u x.du = \frac{1}{h} \int\limits_h^t S_u x.du + \frac{1}{h} \int\limits_t^{t+h} S_u x.du$ and
$\frac{1}{h} \int\limits_0^t (S_{s+h}x - S_s x).ds = \frac{1}{h} \int\limits_h^t S_u x.du + \frac{1}{h} \int\limits_t^{t+h} S_u x.du - \frac{1}{h} \int\limits_0^t S_s x.ds$
$= \frac{1}{h} \int\limits_t^{t+h} S_u x.du - \frac{1}{h} \int\limits_0^h S_u x.du$, letting h goes to 0 we get $\frac{1}{h} \int\limits_t^{t+h} S_u x.du \longrightarrow S_t x$ and $\frac{1}{h} \int\limits_0^h S_u x.du \longrightarrow x$ then $\frac{S_h - I}{h} \int\limits_0^t S_s x.ds \longrightarrow S_t x - x$ so we deduce that $\int\limits_0^t S_s x.ds \in D(A)$, and $A\left(\int\limits_0^t S_s x.ds\right) = S_t x - x$.
(4). Let $x \in D(A)$ and $t, h > 0$, then by the semigroup property:

$$\frac{S_h - I}{h} S_t x = S_t \frac{S_h - I}{h} x = \frac{S_{t+h}x - S_t x}{h}$$

By the definition of $D(A)$ and the continuity of the semigroup we get:

$$\lim_{h \longrightarrow 0_+} \frac{S_h - I}{h} S_t x = S_t \left(\lim_{h \longrightarrow 0_+} \frac{S_h - I}{h} x \right) = S_t A(x)$$

This shows that $S_t x \in D(A)$ and $A(S_t x) = S_t A(x) = \frac{d^+}{dt} S_t x$
where $\frac{d^+}{dt} S_t x$ is the right derivative of $S_t x$ at t.
For the left derivative take $0 < h < t$ and write $t = h + t - h$

so $\dfrac{S_t x - S_{t-h} x}{h} - S_t A(x) = S_{t-h}\left(\dfrac{S_h x - x}{h} - Ax\right) + S_{t-h}Ax - S_t A(x)$

since $\left\| S_{t-h}\left(\dfrac{S_h x - x}{h} - Ax\right)\right\| \leq \left\| \dfrac{S_h x - x}{h} - Ax\right\|$ because $\|S_{t-h}\| \leq 1$

thus making $h \longrightarrow 0$ gives $\lim\limits_{h\longrightarrow 0_+} S_{t-h}\left(\dfrac{S_h x - x}{h} - Ax\right) + S_{t-h}Ax - S_t A(x) = 0$

finally $\lim\limits_{h\longrightarrow 0_+} \dfrac{S_t x - S_{t-h} x}{h} - S_t A(x) = 0$.so the function $t \longrightarrow S_t x$ is differentiable from $]0, \infty[$ into X and $\dfrac{d}{dt} S_t x = A(S_t x) = S_t A(x), \forall t > 0$.

(5).For all $x \in X$ and all $t > s \geq 0$, we have to prove that:

$$S_t x - S_s x = \int_s^t A(S_u x).du = \int_s^t S_u A(x).du$$

From point (4) we have $\dfrac{d}{dt} S_t x = A(S_t x) = S_t A(x), \forall t > 0$ then
we get point (5) by integration from s to t :

$$\int_s^t \frac{d}{dt} S_t x.dt = S_t x - S_s x = \int_s^t A(S_u x).du = \int_s^t S_u A(x).du. \blacksquare$$

Corollary

If A is the generator of a C_0 semigroup of contractions S_t, $t \geq 0$ then
A is a closed operator with a dense domain $D(A)$.

Proof:

Let us start with some facts about closed operators.
Let X,Y be normed spaces and $T : X \longrightarrow Y$ a linear operator. The graph of T is the subspace Γ of $X \times Y$ defined by $\Gamma = \{(x, T(x)) : x \in X\}$.
We say that T is closed if its graph Γ is closed in the product space $X \times Y$ endowed with the product topology.

Remark: Let (x_n) be a sequence in X and consider the conditions:
(i) $x_n \longrightarrow x, n \longrightarrow \infty$
(ii) $T(x_n) \longrightarrow y$
(iii) $y = T(x)$
then it is easy to see that:
T is closed $\Longleftrightarrow$ (i) and (ii) $\Longrightarrow$ (iii).
T is continuous $\Longleftrightarrow$ (i) $\Longrightarrow$ (ii) and (iii).

It is known that if $T : X \longrightarrow Y$ is linear continuous then T is closed.
But the converse is not true in general
(see any standard book on functional analysis).

going back to the proof of the corollary, let $x \in X$ and $t > 0$ then
put $x_t = \dfrac{1}{t}\displaystyle\int_0^t S_s x.ds$; by point (3) of Theorem 4, $x_t \in D(A)$ and
$\lim\limits_{t\longrightarrow 0_+} x_t = S_0 x = x$ this shows that $D(A)$ is indeed dense in X.

Now we prove that A is closed: let $x_n \in D(A)$ be such that

$x_n \longrightarrow x$ and $A(x_n) \longrightarrow y$ when $n \longrightarrow \infty$

we have to show that $x \in D(A)$ and $A(x) = y$.

By point (5) of Theorem 4, for any $t > 0$ we have

$$S_t x_n - x_n = \int_0^t S_u A(x_n) .du$$

but $S_u A(x_n) \longrightarrow S_u y$ for each u because S_u is a bounded linear operator.

Let $t > 0$ and $\epsilon > 0$ then $\exists N_{\epsilon,t} \geq 1$ such that $n \geq \exists N_{\epsilon,t} \implies \|A(x_n) - y\| < \frac{\epsilon}{t}$

but S_u is a contraction so

$$n \geq \exists N_{\epsilon,t} \implies \|S_u A(x_n) - S_u y\| \leq \|S_u\| \|A(x_n) - y\| < \frac{\epsilon}{t}$$

then for $n \geq \exists N_{\epsilon,t}$ we have

$$\left\| \int_0^t S_u A(x_n) .du - \int_0^t S_u y.du \right\| \leq \int_0^t \|S_u A(x_n) - S_u y\| .du < t.\frac{\epsilon}{t} = \epsilon$$

so for each $t > 0$ $\int_0^t S_u A(x_n) .du - \int_0^t S_u y.du \longrightarrow 0, n \longrightarrow \infty$ and we get:

for each $t > 0$ $S_t x_n - x_n \longrightarrow S_t x - x = \int_0^t S_u y.du$

whence $\frac{1}{t}(S_t x - x) = \frac{1}{t}\int_0^t S_u y.du$ for each $t > 0$

letting $t \longrightarrow 0$ gives $\frac{1}{t}(S_t x - x) \longrightarrow Ax$ and by point (2) of Theorem 4, $\frac{1}{t}\int_0^t S_u y.du \longrightarrow y$ so $Ax = y$ and A is closed.■