COMPLEX MEASURES
Absolute Continuity and Representation Theorems

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Introduction

Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f : X \longrightarrow \mathbb{C}$ be a complex μ-integrable function. Let us consider the set function ν given by:

$$\nu(A) = \int_A f \, d\mu, \quad A \in \mathcal{F} \qquad (*)$$

Such set function has the following properties:

(1) ($\sigma-$additivity): For any sequence (A_n) of pairwise disjoint sets A_n in $\mathcal{F}$ we have $\nu\left(\underset{n}{\cup} A_n\right) = \sum\limits_n \nu(A_n)$

(2) (absolute continuity): Let $A \in \mathcal{F}$ with $\mu(A) = 0$ then $\nu(A) = 0$, because $f.I_A = 0 \ \mu - a.e$, we say that ν is absolutely continuous with respect to μ. This relation will be denoted by $\nu \ll \mu$

(3). If f is real valued let us write $f = f^+ - f^-$ then $\nu(A) = \int_A f \, d\mu = \int_A f^+ d\mu - \int_A f^- d\mu$ so we have $\nu(A) = \nu_1(A) - \nu_2(A)$, with $\nu_1(A) = \int_A f^+ d\mu$ and $\nu_2(A) = \int_A f^- d\mu$ positive and $\sigma-$additive.

In this chapter we consider complex valued $\sigma-$additive set functions $\lambda : \mathcal{F} \longrightarrow \mathbb{C}$ and we will show successively:

(a) λ is of bounded variation:

more precisely there is a positive finite measure $|\lambda|$ on $\mathcal{F}$ such that

$$|\lambda(E)| \leq |\lambda|(E), \ \forall E \in \mathcal{F}$$

$|\lambda|$ is called the total variation of λ.

(b) if λ is real valued then it can be written as $\lambda = \lambda^+ - \lambda^-$

where λ^+, λ^- are finite positive measures.

This is called the Jordan decomposition.

(c) λ has the integral form $(*)$ for some complex μ-integrable function f provided $\lambda \ll \mu$ for some $\sigma-$finite positive measure μ.

This is the Radon-Nicodym Theorem.

1. Complex Measures property

Definition 1.1

Let $(X, \mathcal{F})$ be a measurable space and $\lambda : \mathcal{F} \longrightarrow \mathbb{C}$ a complex set function.

We say that λ is a complex measure if for every sequence (A_n) of pairwise disjoint sets in $\mathcal{F}$ we have $\lambda\left(\underset{n}{\cup} A_n\right) = \sum\limits_n \lambda(A_n)$.

Remark (1) Let $\sum\limits_n z_n = M$ be a convergent series of real or complex numbers. We say that the series is unconditionally convergent to M if for any permutation (i.e bijection) $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$, the series $\sum\limits_n z_{\sigma(n)}$ converges to M. For real or complex numbers series unconditional convergence is equivalent to absolute convergence by **Riemann series theorem.**

(2) Let $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ be a permutation of $\mathbb{N}$ then $\underset{n}{\cup} A_n = \underset{n}{\cup} A_{\sigma(n)} = A$ and the sets $\left(A_{\sigma(n)}\right)$ are pairwise disjoint so $\lambda(A) = \sum\limits_n \lambda(A_n) = \sum\limits_n \lambda\left(A_{\sigma(n)}\right)$, since σ is arbitrary this implies that the series $\sum\limits_n \lambda(A_n)$ is unconditionally convergent and then absolutely convergent.
(3) If λ is a complex measure on $(X, \mathcal{F})$ then one can write $\lambda = \operatorname{Re}(\lambda) + i \operatorname{Im}(\lambda)$, where it is easy to see that $\operatorname{Re}(\lambda)$ and $\operatorname{Im}(\lambda)$ are real $\sigma-$additive set functions on $(X, \mathcal{F})$. This simple observation leads to the following definition:

Definition 1.2
Let $(X, \mathcal{F})$ be a measurable space and $\mu : \mathcal{F} \longrightarrow \mathbb{R}$ a real set function.
We say that μ is a real measure if for every sequence (A_n) of pairwise disjoint sets in $\mathcal{F}$ we have $\mu\left(\underset{n}{\cup} A_n\right) = \sum\limits_n \mu(A_n)$.
Let $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ be a permutation of $\mathbb{N}$ then $\underset{n}{\cup} A_n = \underset{n}{\cup} A_{\sigma(n)} = A$ and the sets $\left(A_{\sigma(n)}\right)$ are pairwise disjoint so $\mu(A) = \sum\limits_n \mu(A_n) = \sum\limits_n \mu\left(A_{\sigma(n)}\right)$, since σ is arbitrary this implies that the series $\sum\limits_n \mu(A_n)$ is unconditionally convergent and then absolutely convergent (see **Remark** (1))

2. Total variation of a complex measure

Let λ be a complex measure on $(X, \mathcal{F})$
Among all positive measures μ on $(X, \mathcal{F})$ satisfying
$|\lambda(E)| \leq \mu(E),\ \forall E \in \mathcal{F}$, there one and only one called the Total variation of λ and given by the following theorem:

Theorem 2.1 If λ is a complex measure on $(X, \mathcal{F})$
let us define the positive set function $|\lambda| : \mathcal{F} \longrightarrow [0, \infty]$ by:

$$E \in \mathcal{F},\ |\lambda|(E) = \sup\left\{\sum_n |\lambda(E_n)|, \quad (E_n) \quad \text{partition of } E \text{ in } \mathcal{F}\right\}$$

the supremum being taken over all partitions (E_n) of E in $\mathcal{F}$. Then:
$|\lambda|$ is a positive bounded measure on $(X, \mathcal{F})$ satisfying

$$|\lambda(E)| \leq |\lambda|(E),\ \forall E \in \mathcal{F}$$

moreover $|\lambda|$ is the smallest positive bounded measure with this property.
Proof. Let E be a set in $\mathcal{F}$
If (E_n) is a sequence of pairwise disjoint sets in $\mathcal{F}$ we have to prove that:

$$|\lambda|\left(\underset{n}{\cup} E_n\right) = \sum_n |\lambda|(E_n)$$

let us put $E = \underset{n}{\cup} E_n$ and take a partition (A_m) of E in $\mathcal{F}$ then we have:
$(A_m \cap E_n)_{n \geq 1}$ is a partition of A_m
$(A_m \cap E_n)_{m \geq 1}$ is a partition of E_n
so $|\lambda(A_m)| = \left|\sum\limits_n \lambda(A_m \cap E_n)\right| \leq \sum\limits_n |\lambda(A_m \cap E_n)|, \forall m \geq 1$ and then

$$\sum_m |\lambda(A_m)| \leq \sum_m \sum_n |\lambda(A_m \cap E_n)| = \sum_n \sum_m |\lambda(A_m \cap E_n)|$$

but we have from the definition of $|\lambda|$ $\sum\limits_m |\lambda(A_m \cap E_n)| \le |\lambda|(E_n) \quad \forall n \ge 1$ therefore we deduce that $\sum\limits_m |\lambda(A_m)| \le \sum\limits_n |\lambda|(E_n)$ inequality valid for every partition(A_m) of E and implies $|\lambda|(E) \le \sum\limits_n |\lambda|(E_n)$ by the definition of $|\lambda|(E)$.

It remains to prove that $\sum\limits_n |\lambda|(E_n) \le |\lambda|(E)$, to do this we use characteristic property of the supremum: for each $n \ge 1$ let $a_n > 0$ be any real number such that $a_n < |\lambda|(E_n)$, then from the definition of $|\lambda|(E_n)$ there is a partition $(A_{mn})_{m\ge1}$ of E_n with $a_n < \sum\limits_m |\lambda(A_{mn})|$, but we have $E = .\underset{n}{\cup}.E_n = \underset{mn}{\cup}.A_{mn}$ and $\sum\limits_n a_n < \sum\limits_n\sum\limits_m |\lambda(A_{mn})|$. Since $(A_{mn})_{m,n\ge1}$ is a partition of E we deduce that $\sum\limits_n\sum\limits_m |\lambda(A_{mn})| \le |\lambda|(E)$ and so $\sum\limits_n a_n < |\lambda|(E)$, but this is true for all $a_n > 0$ satisfying $\sum\limits_n a_n < \sum\limits_n |\lambda|(E_n)$ this implies that $\sum\limits_n |\lambda|(E_n) \le |\lambda|(E)$. The proof of the boundedness of $|\lambda|$ is left to the reader.■

Theorem 2.2 If λ is a complex measure on $(X, \mathcal{F})$
For any increasing or decreasing sequence (A_n) in $\mathcal{F}$ we have

$$\lambda\left(\lim_n A_n\right) = \lim_n \lambda(A_n)$$

where $\lim\limits_n A_n$ stands for $\underset{n}{\cup} A_n$ in the increasing case and for $\underset{n}{\cap} A_n$ in the decreasing one.

Proof.
use the $\sigma-$additivity of λ and the fact that $|\lambda(E)| < \infty, \ \forall E \in \mathcal{F}$.■

Theorem 2.3
Let $M(X, \mathcal{F})$ be the family of all complex measures on $(X, \mathcal{F})$
let λ, ν be in $M(X, \mathcal{F})$ and let $\alpha \in \mathbb{C}$, then define:
$\lambda + \nu, \quad \alpha.\nu, \quad \|\lambda\|$, by the following recipe

$$(\lambda + \nu)(E) = \lambda(E) + \nu(E), \ (\alpha.\nu)(E) = \alpha.\nu(E) \ , \ \ E \in \mathcal{F}$$

$$\|\lambda\| = |\lambda|(X)$$

Then $M(X, \mathcal{F})$ is a vector space on $\mathbb{C}$, and $\|\lambda\|$ is a norm on $M(X, \mathcal{F})$
Moreover $M(X, \mathcal{F})$, endowed with the norm $\|\lambda\|$, is a Banach space.

Proof. see any standard book on measure theory for a classical proof.e.g.$[R-F]$ ■

3. Hahn-Jordan Decomposition of a Real Measure

Theorem 3.1 Jordan Decomposition of a Real Measure
Let $(X, \mathcal{F})$ be a measurable space and $\mu : \mathcal{F} \longrightarrow \mathbb{R}$ a real measure (Definition **1.2**)
Let us define the set functions μ^+, μ^- as follows: for each $E \in \mathcal{F}$

$\mu^+(E) = \sup\{\mu(F) : F \in \mathcal{F}, \quad F \subset E\}$

$\mu^-(E) = -\inf\{\mu(F) : F \in \mathcal{F}, \quad F \subset E\}$

Then μ^+, μ^- are positive bounded measures on $(X, \mathcal{F})$ satisfying:

$(i)\ \mu^+ = \frac{1}{2}(|\mu| + \mu) \qquad (ii)\ \mu^- = \frac{1}{2}(|\mu| - \mu)$

$(iii)\ \mu = \mu^+ - \mu^-$ **(Jordan Decomposition)**

$(iv)\ |\mu| = \mu^+ + \mu^-$

Proof
First it is enough to prove that μ^+ is a positive bounded measure on $(X, \mathcal{F})$ because $\mu^- = (-\mu)^+$.
From the definition of the total variation of μ we can see that for $E \in \mathcal{F}$

$\forall F \in \mathcal{F},\ F \subset E.....\mu(F) \leq |\mu(F)| \leq |\mu|(F) \leq |\mu|(E) \leq |\mu|(X) < \infty$

so we deduce that μ^+ is positive bounded;
it remains to prove that μ^+ is $\sigma-$additive
Let (E_n) be pairwise disjoint sets in $\mathcal{F}$, we have to prove:

$\mu^+\left(\underset{n}{\cup} E_n\right) = \sum_n \mu^+(E_n)$

Since $\forall n \geq 1 \quad \mu^+(E_n) < \infty$, we have from the definition of μ^+ :
let $\epsilon > 0$ then for each $n \geq 1$, $\exists F_n \subset E_n$ such that $\mu^+(E_n) - \frac{\epsilon}{2^n} < \mu(F_n)$
summing over n we get $\sum_n \mu^+(E_n) - \epsilon < \sum_n \mu(F_n) = \mu\left(\underset{n}{\cup} F_n\right)$

but $\underset{n}{\cup} F_n \subset \underset{n}{\cup} E_n \Longrightarrow \mu\left(\underset{n}{\cup} F_n\right) \leq \mu^+\left(\underset{n}{\cup} E_n\right)$, therefore $\sum_n \mu^+(E_n) - \epsilon < \mu^+\left(\underset{n}{\cup} E_n\right)$

since $\epsilon > 0$ is arbitrary, we obtain $\sum_n \mu^+(E_n) \leq \mu^+\left(\underset{n}{\cup} E_n\right)$ after making $\epsilon \longrightarrow 0$.

On the other hand let $F \subset \underset{n}{\cup} E_n ... F \in \mathcal{F}$, then $F = \underset{n}{\cup}(E_n \cap F)$ and

$\mu(F) = \sum_n \mu(E_n \cap F) \leq \sum_n \mu^+(E_n)$ because $E_n \cap F \subset E_n$

since $F \subset \underset{n}{\cup} E_n$..is arbitrary in $\mathcal{F}$, we deduce that $\mu^+\left(\underset{n}{\cup} E_n\right) \leq \sum_n \mu^+(E_n)$

so μ^+ is a positive measure on $(X, \mathcal{F})$.
We have $(i) = (ii) - (iii)$ and $(iv) = (ii) + (iii)$
it is enough to prove (i) because (ii) follows with $\mu^- = (-\mu)^+$
then we get (iii) with $(ii) - (i)$ and (iv) with $(ii) + (iii)$.

So let us prove (i) that is $\mu^+ = \frac{1}{2}(|\mu| + \mu)$:
fix $\epsilon > 0$ then from the definition of the total variation $|\mu|$
there is a partition (E_n) of E in $\mathcal{F}$ such that $(*)\ |\mu|(E) - \epsilon < \sum_n |\mu(E_n)|$.

Let us put $L = \{l : \mu(E_l) \geq 0\}$ and $K = \{k : \mu(E_k) < 0\}$, we get:

$\sum_n |\mu(E_n)| = \sum_L \mu(E_l) - \sum_K \mu(E_k)$

now define $F = \underset{L}{\cup} E_l$ and $G = \underset{K}{\cup} E_k$, so by the $\sigma-$additivity of μ we deduce

$\mu(F) = \sum_L \mu(E_l)$ and $\mu(G) = \sum_K \mu(E_k)$, then $\sum_n |\mu(E_n)| = \mu(F) - \mu(G)$

therefore we obtain from the inequality $(*)$ that

$(2*)$ $|\mu|(E) - \epsilon < \sum_n |\mu(E_n)| = \mu(F) - \mu(G)$

but since $E = F \cup G$ we have

$(3*)$ $\mu(E) = \mu(F) + \mu(G)$

adding $(2*) + (3*)$ we get:

$|\mu|(E) + \mu(E) - \epsilon < 2\mu(F)$

the definition of μ^+ implies $\mu(F) \leq \mu^+(E)$, because $F \subset E$

finally $\frac{1}{2}(|\mu|(E) + \mu(E) - \epsilon) < \mu^+(E)$ with nothing depending on ϵ apart ϵ,

making $\epsilon \longrightarrow 0$ we get $\frac{1}{2}(|\mu|(E) + \mu(E)) \leq \mu^+(E)$.

This inequality cannot be strict:

indeed suppose that we have $\frac{1}{2}(|\mu|(E) + \mu(E)) < \mu^+(E)$, this would imply the existence of an F in $\mathcal{F}$ with $F \subset E$ and $\frac{1}{2}(|\mu|(E) + \mu(E)) < \mu(F) < \mu^+(E)$

but the set function $\frac{1}{2}(|\mu|(E) + \mu(E))$ is a positive measure on $(X, \mathcal{F})$,

since $F \subset E$ we should have $\frac{1}{2}(|\mu|(F) + \mu(F)) \leq \frac{1}{2}(|\mu|(E) + \mu(E))$

since $\mu(F) \leq |\mu|(F)$, we have $\mu(F) = \frac{1}{2}(\mu(F) + \mu(F)) \leq \frac{1}{2}(|\mu|(F) + \mu(F)) \leq \frac{1}{2}(|\mu|(E) + \mu(E)) < \mu(F)$ which is absurd

so we deduce that $\frac{1}{2}(|\mu|(E) + \mu(E)) = \mu^+(E)$.■

Remark.

The Jordan decomposition of a real measure μ as difference of two positive bounded measures $\mu = \mu^+ - \mu^-$ is not unique, since for any positive bounded measure ν one can write $\mu = (\mu^+ + \nu) - (\mu^- + \nu)$.

However such decomposition is minimal in the sense that if $\mu = \lambda - \nu$ with λ, ν positive bounded measures then $\mu^+ \leq \lambda$ and $\mu^- \leq \nu$; to see this use the facts $\mu \leq \lambda$ and $(-\mu) \leq \nu$ in the definition of μ^+, μ^-.

Theorem 3.2 The Hahn Decomposition

Let $\mu : \mathcal{F} \longrightarrow \mathbb{R}$ be a real measure on the measurable space $(X, \mathcal{F})$.

There exists a partition of X in two sets A, B in $\mathcal{F}$ such that:

(1) $\mu(F) \geq 0$ for every $F \subset A$ and $\mu^-(A) = 0$

(2) $\mu(F) \leq 0$ for every $F \subset B$ and $\mu^+(B) = 0$

The partition $X = A \cup B$ satisfying $(1), (2)$ is unique in the following sense: if C, D is another partition of X satisfying $(1), (2)$ then $|\mu|(A \Delta C) = 0$ and $|\mu|(B \Delta D) = 0$, (see $[R - F]$ for the Proof).

5. Absolute Continuity of Measures
Radon-Nikodym Theorem

Definition 5.1

Let λ be a complex measure on $(X, \mathcal{F})$ and μ a positive measure.
We say that λ is absolutely continuous with respect to μ if:
for $E \in \mathcal{F}$ satisfying $\mu(E) = 0 \Longrightarrow \lambda(E) = 0$
notation: $\lambda \ll \mu$

Example 5.2

Let $f \in L_1(\mu)$, then the complex measure λ on $(X, \mathcal{F})$ given by:

$$E \in \mathcal{F},\ \lambda(E) = \int_E f.d\mu \qquad (*)$$

is absolutely continuous with respect to μ, indeed suppose $\mu(E) = 0$
then $.f.I_E = 0\ \ \mu{-}a.e$ and $\int_E f.d\mu = \int_X f.I_E.d\mu = 0 \Longrightarrow \lambda$ absolutely continuous with respect to μ.(by the property of the integral)
This example is fundamental in the following sense:
If μ is $\sigma-$finite then the complex measure in the integral form $(*)$ is the only one which is absolutely continuous with respect to μ
this is due to the **Radon-Nikodym Theorem.** First let us look at some properties of the absolute continuity.

Theorem 5.3

Let λ be a complex measure on $(X, \mathcal{F})$ and μ a positive measure.
The following properties are equivalent:
$(a)\quad \lambda \ll \mu$
(b) For any $\epsilon > 0$ there is $\delta = \delta_\epsilon > 0$ such that for $A \in \mathcal{F}$
$\mu(A) < \delta \implies |\lambda|(A) < \epsilon$

Proof.

$(b) \Longrightarrow (a)$
if $\mu(A) = 0$ then $\mu(A) < \delta\ \forall \delta > 0$, so $|\lambda|(A) < \epsilon\ \forall \epsilon > 0$
that is $|\lambda|(A) = 0$

$(a) \Longrightarrow (b)$
we prove that *not* $(b) \Longrightarrow$ *not* (a)
suppose *not* (b) then there is $\epsilon > 0$ such that for each $n \geq 1$
there exists $E_n \in \mathcal{F}$ with $\mu(E_n) < \dfrac{1}{2^n}$ and $|\lambda|(E_n) \geq \epsilon$
put $E = \limsup_n E_n = \bigcap_n . \bigcup_{k \geq n} .E_k$, then since $|\lambda|$ is bounded we get:

$$|\lambda|(E) = \lim_n |\lambda| \left(\bigcup_{k \geq n} .E_k \right) \geq \lim_n |\lambda|(E_n) \geq \epsilon > 0.$$

On the other hand since we have $\sum_n \mu(E_n) < \sum_n \dfrac{1}{2^n} < \infty$, we can apply the Borel-Cantelli Lemma to the sequence E_n to get $\mu(E) = \mu\left(\limsup_n E_n\right) = 0$
so *not* (a) is satisfied and *not* $(b) \Longrightarrow$ *not* (a) is proved.■

Definition 5.4 (Singular Measures)
Let μ, ν be positive measures on $(X, \mathcal{F})$.
We say that μ, ν are singular if there is a partition $\{A, B\}$ of X such that
$\mu(A) = \nu(B) = 0$
notation: $\mu \perp \nu$
If μ, ν are complex measures then they are singular if $|\mu| \perp |\nu|$

Remark.
Suppose μ, ν positive or complex measures and $\mu \perp \nu$
if $\{A, B\}$ is the partition of X defining the singularity then we have
for $E \in \mathcal{F}$ $\mu(E) = \mu(E \cap B)$ and $\nu(E) = \nu(E \cap A)$

Proposition 5.5
Let $\mu, \lambda_1, \lambda_2$, be measures on $(X, \mathcal{F})$ with μ positive and λ_1, λ_2, complex then
(a) $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu \Longrightarrow \lambda_1 + \lambda_2 \perp \mu$
(b) $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu \Longrightarrow \lambda_1 + \lambda_2 \ll \mu$
(c) $\lambda_1 \ll \mu \Longrightarrow |\lambda_1| \ll \mu$
$(d).\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu \Longrightarrow \lambda_1 \perp \lambda_2$
$(e).\lambda_1 \ll \mu$ and $\lambda_1 \perp \mu \Longrightarrow \lambda_1 = 0$

Proof
(b) and (c) are deduced from definition, (e) is a consequence of (d)
so let us prove (a) and (d).
To see (a) let $\{A_1, B_1\}, \{A_2, B_2\}$ be two partitions of X with
$|\lambda_1|(A_1) = \mu(B_1) = 0$ and $|\lambda_2|(A_2) = \mu(B_2) = 0$
put $A = A_1 \cap A_2$ and $B = B_1 \cup B_2$, then $\{A, B\}$ is a partition of X and we have
$|\lambda_1 + \lambda_2|(A) \leq (|\lambda_1| + |\lambda_2|)(A) = 0$ and $\mu(B) \leq \mu(B_1) + \mu(B_2) = 0$
therefore $\lambda_1 + \lambda_2 \perp \mu$ this proves (a).
To prove (d): since $\lambda_2 \perp \mu$ let $\{A, B\}$ be a partition of X with $|\lambda_2|(A) = \mu(B) = 0$
but we have also $\lambda_1 \ll \mu$ so we will have $|\lambda_1|(B) = 0$ by (c), finally we get $|\lambda_2|(A) = |\lambda_1|(B) = 0$
that is $\lambda_1 \perp \lambda_2$, then we deduce (d).■

Theorem 5.6 Radon-Nikodym-Lebesgue Theorem.
Let μ, λ, be measures on $(X, \mathcal{F})$ with μ positive $\sigma-$finite and λ complex then:
(1) There is a unique pair of complex measures λ_a, λ_s such that
$\lambda_a \ll \mu$, $\lambda_s \perp \mu$ and $\lambda_a \perp \lambda_s$
moreover $\lambda = \lambda_a + \lambda_s$ **Lebesgue Decomposition**
(2) There is a unique function $h \in L_1(\mu)$ such that

$\lambda_a(E) = \int_E h.d\mu$ for every $E \in \mathcal{F}$.**Radon-Nikodym Theorem**

Proof
Let us point out:
(i) uniqueness in (1) of λ_a, λ_s: if $\lambda_a^{'}, \lambda_s^{'}$ is an other pair of complex measures with $\lambda_a^{'} \ll \mu$, $\lambda_s^{'} \perp \mu$ and $\lambda = \lambda_a^{'} + \lambda_s^{'} = \lambda_a + \lambda_s$ then we have
$\lambda_a^{'} - \lambda_a = \lambda_s - \lambda_s^{'}$; we deduce from Proposition **5.5** (a) that $\lambda_s - \lambda_s^{'} \perp \mu$

and from Proposition **5.5** (b) that $\lambda_a^{'} - \lambda_a \ll \mu$
so part (e) of the same Proposition implies $\lambda_a^{'} - \lambda_a = \lambda_s - \lambda_s^{'} = 0$, whence the uniqueness of the decomposition.
(ii) uniqueness in (2) of the function h: if $h^{'} \in L_1(\mu)$ is an other function with $\lambda_a(E) = \int\limits_E h^{'}.d\mu$ for every $E \in \mathcal{F}$ then $\int\limits_E h.d\mu = \int\limits_E h^{'}.d\mu$, $\forall E \in \mathcal{F}$, and so by the property of the integral we get $h = h^{'}$ $\mu - a.e.$
We prove the theorem when μ, λ are positive bounded measures
the proof in this case is due to John Von Neuman.
(for the general case see reference $[R - F]$).
Let μ, λ be positive bounded measures on $(X, \mathcal{F})$ and put $m = \mu + \lambda$
then m is a positive bounded measure on $(X, \mathcal{F})$ and we have

$$\int\limits_X f.dm = \int\limits_X f.d\mu + \int\limits_X f.d\lambda$$

for any measurable positive function f on X
this can be checked easily for simple positive functions
by using **Beppo-Levy** convergence theorem one can prove it
for measurable positive functions.
Observe that $L_1(m) = L_1(\mu) \cap L_1(\lambda)$ because $\int\limits_X |f|.dm = \int\limits_X |f|.d\mu + \int\limits_X |f|.d\lambda$
Now take f in the Hilbert space $L_2(m)$ then we have by the Schwarz inequality

$$\left|\int\limits_X f.d\lambda\right| \leq \int\limits_X |f|.d\lambda \leq \int\limits_X |f|.dm \leq \left(\int\limits_X |f|^2.dm\right)^{\frac{1}{2}}.(m(X))^{\frac{1}{2}}$$

consequently the linear functional $f \longrightarrow \int\limits_X f.d\lambda$ is continuous on the Hilbert Space $L_2(m)$. Therefore there is a unique function g in $L_2(m)$ such that :
$(*)$ $\int\limits_X f.d\lambda = \int\limits_X f.g.dm$, by the isomorphism between $L_2(m)$ and its strong dual
take $f = I_E$ in equation $(*)$ to get
$(**)$ $\lambda(E) = \int\limits_E g.dm$, then since $0 \leq \lambda(E) \leq m(E)$ $\forall E \in \mathcal{F}$
we obtain $0 \leq g \leq 1$ $m - a.e$, but $m = \mu + \lambda$, from which $(*)$ gives
$(***)$ $\int\limits_E f.(1-g).d\lambda = \int\limits_E f.g.d\mu.$
Now put $A = \{0 \leq g < 1\}$, $B = \{g = 1\}$ and define measures λ_a, λ_s as follows

$\lambda_a(E) = \lambda(A \cap E)$ and $\lambda_s(E) = \lambda(E \cap B)$ $\quad \forall E \in \mathcal{F}$
putting $E = X$, $f = I_B$ in the relation $(***)$ gives

$$\int_B (1-g).d\lambda = \int_B g.d\mu = \mu(B)$$

since $g = 1$ on the set B we have $\int_B (1-g).d\lambda = 0$, so $\mu(B) = 0$

but $\lambda_s(E) = 0$ for every $E \subset A$, therefore $\lambda_s \perp \mu$.

Again consider $(***)$ but with $f = \left[1 + g + g^2 + ... + g^n\right].I_E$, we get:

$$\int_E \left(1-g^{n+1}\right).d\lambda = \int_{E\cap A} \left(1-g^{n+1}\right).d\lambda + \int_{E\cap B} \left(1-g^{n+1}\right).d\lambda = \int_{E\cap A} \left(1-g^{n+1}\right).d\lambda$$

since $g = 1$ on the set B.

On the other hand we have $g^{n+1}(x) \downarrow 0 \ \forall x \in E \cap A$, since $A = \{0 \le g < 1\}$ so $1 - g^{n+1}(x) \uparrow 1 \forall x \in E \cap A$ and by **Beppo-Levy** convergence theorem we get:

$$(4*)\ \lim_n \int_{E\cap A} \left(1-g^{n+1}\right).d\lambda = \lambda(E\cap A) = \lambda_a(E)$$

but we have $g.\left(1 + g + g^2 + ... + g^n\right) \uparrow h = \left\{ \begin{array}{cc} \dfrac{g}{1-g} & \text{on } A \\ \infty & \text{on } B \end{array} \right\}$

since $\mu(B) = 0$, we deduce that

$$(5*)\ \int_E g.\left(1 + g + g^2 + ... + g^n\right).d\mu \uparrow \int_E h.d\mu$$

now properties $(4*)$ and $(5*)$ jointly imply

$$\lambda_a(E) = \int_E h.d\mu \text{ and } h \in L_1(\mu)$$

which ends the proof of the Theorem. the function h is called the Radon-Nikodym density of λ with respect to μ and is denoted by $h = \dfrac{d\lambda}{d\mu}$.■

The following theorem is a version of the preceding one in the case λ, μ positive $\sigma-$finite measures, the proof can be found in $[R-F]$.

Theorem 5.7

Let μ, λ, be positive $\sigma-$finite measures on $(X, \mathcal{F})$

(1) There is a unique pair of positive measures λ_a, λ_s such that

$\lambda_a \ll \mu$, $\lambda_s \perp \mu$ and $\lambda_a \perp \lambda_s$

moreover $\lambda = \lambda_a + \lambda_s$ **Lebesgue Decomposition**

(2) There is a unique positive function h locally $\mu-$integrable such that

$\lambda_a(E) = \int_E h.d\mu$ for every $E \in \mathcal{F}$.**Radon-Nikodym Theorem**

h locally $\mu-$integrable means that there is a partition (X_n) of X in $\mathcal{F}$ such that

$$\int_{X_n} h.d\mu < \infty \quad \forall n.$$

6. Applications

We recall that the structures given in Theorem **5.6** are valid on $(X, \mathcal{F})$ with μ positive $\sigma-$finite and λ complex although its proof has been given for μ, λ positive bounded. So hereafter we consider the general context of complex measures. Let us start by the relation of a complex mesure and its total variation.

Proposition 6.1
Let λ be a complex measure on $(X, \mathcal{F})$ then
there exists a measurable function $h : X \longrightarrow \mathbb{C}$ such that

$$|h(x)| = 1 \text{ for every } x \in X \text{ and } \lambda(E) = \int_E h.d|\lambda| \quad \forall E \in \mathcal{F}$$

where $|\lambda|$ is the total variation of λ (see Theorem.**2.1** for total variation of λ)
Proof
Observe that $\lambda \ll |\lambda|$ and use the Radon-Nikodym Theorem.■

Proposition 6.2
Let $(X, \mathcal{F}, \mu)$ be a measure space and let f be in $L_1(\mu)$.

Consider the complex measure λ on $(X, \mathcal{F})$ given by $\lambda(E) = \int_E f.d\mu$

then we have $|\lambda|(E) = \int_E |f|.d\mu \quad \forall E \in \mathcal{F}$.

Proof

Let us consider the set measure $\nu(E) = \int_E |f|.d\mu$ on $(X, \mathcal{F})$ then we have:

$$|\lambda(E)| = \left|\int_E f.d\mu\right| \leq \int_E |f|.d\mu = \nu(E) \Longrightarrow |\lambda(E)| \leq \nu(E)$$

but we now that the total variation $|\lambda|$ is the smallest positive measure satisfying $|\lambda(E)| \leq |\lambda|$ by theorem **2.1**, so we deduce that $|\lambda| \leq \nu$ and therefore $|\lambda| \ll \nu$. Since ν is bounded by the Radon-Nikodym theorem there is $\varphi \in L_1(\nu)$ with φ positive and $|\lambda|(E) = \int_E \varphi.d\nu$. The integral form of ν allows to write $|\lambda|(E) = \int_E \varphi.|f|.d\mu$. By Proposition **6.1** there exists a measurable function $h : X \longrightarrow \mathbb{C}$ such that $|h(x)| = 1$ for every $x \in X$ and $\lambda(E) = \int_E h.d|\lambda| \quad \forall E \in \mathcal{F}$.

We deduce from the integration process that $\lambda(E) = \int_E h.d|\lambda| = \int_E h.\varphi.|f|.d\mu$.

By hypothesis $\lambda(E) = \int_E f.d\mu$ so we get $\int_E h.\varphi.|f|.d\mu = \int_E f.d\mu, \forall E \in \mathcal{F}$ and then $h.\varphi.|f| = f \quad \mu - a.e$. But $|h(x)| = 1$ for every $x \in X$ so $\varphi = 1 \quad \mu - a.e$ because $\varphi \geq 0$, finally $|\lambda|(E) = \int\limits_E \varphi.d\nu = \nu(E) = \int\limits_E |f|.d\mu.$■

Proposition 6.3
Let μ be a $\sigma-$finite measure on $(X, \mathcal{F})$ and let us denote by $\mathcal{A}_\mu$ the family of all complex measures absolutely continuous with respect to μ Then $\mathcal{A}_\mu$ is a closed subspace of the Banach space $M(X, \mathcal{F})$. Moreover there is a linear isometry from $L_1(\mu)$ onto $\mathcal{A}_\mu$.

Proof
Recall the Banach space $M(X, \mathcal{F})$ of all complex measures on $(X, \mathcal{F})$ with the norm $\|\lambda\| = |\lambda|(X)$ defined in Theorem **2.3.**
It is easy to check that $\mathcal{A}_\mu$ is a subspace of $M(X, \mathcal{F})$. We prove that it is closed: let (λ_n) be a sequence in $\mathcal{A}_\mu$ converging to λ that is $\lim\limits_n \|\lambda_n - \lambda\| = \lim\limits_n |\lambda_n - \lambda|(X) = 0$. This implies that $(\lambda_n(E))$ converges to $\lambda(E)$ even uniformly with respect to E.
If we have $\mu(E) = 0$ then $\lambda_n(E) = 0 \ \forall n$, so $\lambda(E) = 0$ that is $\lambda \in \mathcal{A}_\mu$ this shows that $\mathcal{A}_\mu$ is closed.
Now we define the linear isometry Ψ from $L_1(\mu)$ onto $\mathcal{A}_\mu$ as follows: for $f \in L_1(\mu)$ put $\Psi(f) = \lambda$, where λ is the complex measure on $(X, \mathcal{F})$ given by

$\lambda(E) = \int\limits_E f.d\mu$. Then it is clear that Ψ is linear, moreover it is invertible, indeed
if $\lambda \in \mathcal{A}_\mu$ then $\lambda \ll \mu$ and since μ is $\sigma-$finite there is a unique $f \in L_1(\mu)$ such that

$\lambda(E) = \int\limits_E f.d\mu = \Psi(f)$, by the Radon-Nikodym Theorem. On the other hand Ψ is an isometry
since we have $\|\Psi(f)\| = \|\lambda\| = |\lambda|(X) = \int\limits_X |f|.d\mu = \|f\|_{L_1(\mu)}$.■

References

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