Chapter 1

Positive Measures

1. Algebras of Sets

This section is intented to give the basic structures on sets, needed for the definition and properties of measures. We start with the following:

Preliminaries:

Let X be a set, and let $\mathcal{P}(X)$ be the power set of X. If I is any nonempty set, a function $f : I \longrightarrow \mathcal{P}(X)$ defines a family $\{A_i,\, i \in I\}$ of subsets of X, with $A_i = f(i) \in \mathcal{P}(X)$. For such family we perform the union and the intersection by:

$\underset{i}{\cup} A_i = \{x : \exists i \in I,\, x \in A_i\}$

$\underset{i}{\cap} A_i = \{x : \forall i \in I,\, x \in A_i\}$

Let us recall the frequently used **De Morgan's Laws:**

$\left(\underset{i}{\cup} A_i\right)^c = \underset{i}{\cap} A_i^c, \qquad \left(\underset{i}{\cap} A_i\right)^c = \underset{i}{\cup} A_i^c$

valid for any family $\{A_i,\, i \in I\}$, where A^c denotes the complement of the set A.

Definition 1.1.

Let $\mathcal{A}$ be a family of subsets of X.

We say that $\mathcal{A}$ is an algebra on X if:

(1) X, ϕ are in $\mathcal{A}$

(2) For every subset A in $\mathcal{A}$, the complement A^c of A is in $\mathcal{A}$

(3) For every subsets $A, B \in \mathcal{A}$, $A \cup B \in \mathcal{A}$

Example 1.2.

(a) For any X the power set $\mathcal{P}(X)$ is an algebra

(b) Let X be a set and let $\mathcal{A}$ be the family given by $\mathcal{A} = \{A \subset X : A \text{ or } A^c \text{ finite}\}$. It is not difficult to check that $\mathcal{A}$ is an algebra, using the De Morgan's Laws given in the Preliminaries

(c) If $\mathcal{A}$ is an algebra and if $A, B \in \mathcal{A}$ then $A \cap B \in \mathcal{A}$

(d) For any finite sequence $A_1, ..., A_n$ in $\mathcal{A}$ the union $\overset{n}{\underset{1}{\cup}} A_i$ and

the intersection $\overset{n}{\underset{1}{\cap}} A_i$ are in $\mathcal{A}$.

Definition 1.3.

Let $\mathcal{F}$ be a family of subsets of X.

We say that $\mathcal{F}$ is a $\sigma-$field or $\sigma-$algebra on X if:

(1) X, ϕ are in $\mathcal{F}$

(2) For every subset A in $\mathcal{F}$, the complement A^c of A is in $\mathcal{F}$

(3) For every sequence (A_n) of subsets $A_n \in \mathcal{F}$, $\underset{n}{\cup} A_n \in \mathcal{F}$

The pair $(X, \mathcal{F})$, where X is a set and $\mathcal{F}$ a $\sigma-$field on X is called a measurable space and sets A in $\mathcal{F}$ are called measurable sets.

Examples 1.4.
(a) For any X the power set $\mathcal{P}(X)$ is a $\sigma-$field on X.
(b) Let X be an infinite set and let $\mathcal{F}$ be the family given by $\mathcal{F} = \{A \subset X : A \text{ or } A^c \text{ countable}\}$.
Then it is not difficult to prove that $\mathcal{F}$ is a $\sigma-$field on X
(use the De Morgan's Laws given in the Preliminaries).
(c) Every $\sigma-$field on X is an algebra, but the converse is not true as is shown by the following:
take $X = \mathbb{Z}$, the integers and the algebra $\mathcal{A} = \{A \subset X : A \text{ or } A^c \text{ finite}\}$,
put $A_n = \{n\}, n \geq 0$; then $A_n \in \mathcal{A}, \forall n \geq 0$, but $\underset{n\geq 0}{\cup} A_n \notin \mathcal{A}$.

Remark 1.5.
(a) If $\mathcal{F}$ is a $\sigma-$field on X, then for every sequence (A_n) in $\mathcal{F}$, $\underset{n}{\cap} A_n \in \mathcal{F}$.
(b) For every sequence (A_n) such that $A_i \cap A_j = \phi$, for $i \neq j$
we denote the set $\underset{n}{\cup} A_n$ by $\sum\limits_n A_n$.

2. Exercises

1. Prove that the family $\mathcal{F}$ is a $\sigma-$field on X, if and if the following conditions are satisfied:
(a) $\phi \in \mathcal{F}$
(b) For any finite sequence $A_1, ..., A_n$ in $\mathcal{F}$, $\overset{n}{\underset{1}{\cap}} A_i \in \mathcal{F}$
(c) For every sequence (A_n) such that $A_i \cap A_j = \phi$, for $i \neq j$. we have $\sum\limits_n A_n \in \mathcal{F}$

2. For every sequence (A_n), define the sequence (B_n) by the following recipe:
$B_1 = A_1,\ B_2 = A_2 \backslash A_1,\ B_3 = A_3 \backslash (A_1 \cup A_2), ...B_n \backslash \left(\underset{i<n}{\cup} A_i \right)$
Prove that $\underset{n}{\cup} A_n = \sum\limits_n B_n$.

3. Generations

Lemma 3.1.
Let $\mathcal{F}_i$, $i \in I$ be an arbitrary family of $\sigma-$fields
(resp. algebras). Then the family $\underset{i}{\cap} \mathcal{F}_i$ is a $\sigma-$field (resp. algebra).
Proof. Straightforward.$\blacksquare$
Corollary 3.2.
Let $\mathcal{H}$ be a family of subsets of a set X
Then there exist a smallest $\sigma-$field on X containing $\mathcal{H}$, denoted by $\sigma(\mathcal{H})$.
Smallest is taken in the sens of the inclusion ordering.
$\sigma(\mathcal{H})$.is called the $\sigma-$field generated by $\mathcal{H}$.
Proof. Let $\mathfrak{I} = \{\mathcal{F}\colon\ \mathcal{F}\ \sigma-\text{field on } X,\ \text{with } \mathcal{H} \subset \mathcal{F}\}$
then by **Lemma 3.1**, $\underset{\mathcal{F}\in\mathfrak{I}}{\cap} \mathcal{F}$ is a $\sigma-$field on X and it is clear that:
$\sigma(\mathcal{H}) = \underset{\mathcal{F}\in\mathfrak{I}}{\cap} \mathcal{F}$.$\blacksquare$

Example 3.3.
(a) Let $\mathcal{H}$ be a family given by one subset A, $\mathcal{H} = \{A\}$
then $\sigma(\mathcal{H}) = \{A, A^c, \phi, X\}$.
(b) If $\mathcal{I}$ is the family of one point sets given by $\mathcal{I} = \{\{x\} : x \in X\}$
then we have $\sigma(\mathcal{I}) = \{A \subset X : A$ or A^c countable$\}$(see **Example 1.4** (b))

Definition 3.4.(Product $\sigma-$field)

Let $(X_1, \mathcal{F}_1)$, $(X_2, \mathcal{F}_2)$ be measurable spaces. Consider on the product set $X_1 \times X_2$ the family $\mathcal{R} = \{A_1 \times A_2 : A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2\}$.
The product $\sigma-$field on $X_1 \times X_2$ is defined by $\mathcal{F}_1 \otimes \mathcal{F}_2 = \sigma(\mathcal{R})$.
The measurable space $(X_1 \times X_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ is called the product of $(X_1, \mathcal{F}_1)$, $(X_2, \mathcal{F}_2)$.

Definition 3.5. (Borel $\sigma-$field)

Let X be a topological space. The Borel $\sigma-$field of X is the $\sigma-$field generated by the family of all the open sets of X.
It is denoted by $\mathcal{B}_X$. Sets in $\mathcal{B}_X$ are called Borel sets of X. One can see that $\mathcal{B}_X$ is also generated by the closed sets of X.

Proposition 3.6.

The Borel $\sigma-$field $\mathcal{B}_{\mathbb{R}}$of $\mathbb{R}$ is generated by the open intervals of $\mathbb{R}$.
In fact $\mathcal{B}_{\mathbb{R}}$ is generated by the family $\{]-\infty, t[, t \in \mathbb{R}\}$.

Proof. Every open set of $\mathbb{R}$ is the union of a sequence of open intervals.■

Definition 3.7. (Monotone family)

Let $\mathcal{M}$ be a family of subsets of a set X. $\mathcal{M}$ is said to be monotone if:
(i) For any sequence (A_n) with $A_1 \subset A_2 \subset ... \subset A_n \subset ...$, we have $\underset{n}{\cup} A_n \in \mathcal{M}$
(ii) For any sequence (A_n) with $A_1 \supset A_2 \supset ... \supset A_n \supset ...$, we have $\underset{n}{\cap} A_n \in \mathcal{M}$

Example 3.8.

(a) Any $\sigma-$field is a monotone family
(b) Let $\mathcal{A}$ be an algebra, then $\mathcal{A}$ is a $\sigma-$field iff $\mathcal{A}$ is a monotone family.

Lemma 3.9.
Let $\mathcal{M}_i$, $i \in I$ be an arbitrary class of monotone families
Then the family $\underset{i}{\cap} \mathcal{M}_i$ is a monotone family.

Proof. Straightforward.■

Corollary 3.10.

Let $\mathcal{H}$ be a family of subsets of a set X
Then there exist a smallest monotone family on X containing $\mathcal{H}$, denoted by $\mathcal{M}(\mathcal{H})$. Smallest is taken in the sens of the inclusion ordering.

$\mathcal{M}(\mathcal{H})$.is called the monotone family generated by $\mathcal{H}$.

Proof. Let $\mathfrak{I} = \{\mathcal{M}$: $\mathcal{M}$ monotone family on X, with $\mathcal{H} \subset \mathcal{M}\}$
then by **Lemma 3.9**, $\underset{\mathcal{M} \in \mathfrak{I}}{\cap} \mathcal{M}$ is a monotone family on X and it is clear that:

$\mathcal{M}(\mathcal{H}) = \underset{\mathcal{M} \in \mathfrak{I}}{\cap} \mathcal{M}$.■

Theorem 3.11.
Let $\mathcal{A}$ be an algebra on the set X. Then the $\sigma-$field generated by $\mathcal{A}$ is identical to the monotone family generated by $\mathcal{A}$.

Proof. Put $\mathcal{M} = \mathcal{M}(\mathcal{A})$, $\mathcal{B} = \sigma(\mathcal{A})$. Then $\mathcal{M} \subset \mathcal{B}$ (**Example 3.8.** (a)).
To show that $\mathcal{B} \subset \mathcal{M}$ it is enough to prove that $\mathcal{M}$ is an algebra
(see **Example 3.8.** (b))
First we prove that $B \in \mathcal{M} \Longrightarrow B^c \in \mathcal{M}$. To this end let $\mathcal{M}^{'} = \{B \in \mathcal{M} : \quad B^c \in \mathcal{M}\}$
Then we have $\mathcal{A} \subset \mathcal{M}^{'} \subset \mathcal{M}$. Moreover $\mathcal{M}^{'}$ is monotone and so $\mathcal{M}^{'} = \mathcal{M}$.
It remains to prove that $\mathcal{M}$ is stable by intersection. For each $A \in \mathcal{M}$, consider the family $\mathcal{M}_A = \{B \in \mathcal{M} : \quad A \cap B \in \mathcal{M}\}$, then $\mathcal{M}_A$ is a monotone family with $\mathcal{M}_A \subset \mathcal{M}$. Moreover if $A \in \mathcal{A}$, we have $\mathcal{A} \subset \mathcal{M}_A$, so we deduce that $\mathcal{M}_A = \mathcal{M}$. On the other hand it is clear that $A \in \mathcal{M}_B$ iff $B \in \mathcal{M}_A$, therefore $A \in \mathcal{M}_B$ for every $A \in \mathcal{A}$ and $B \in \mathcal{M}$. Finally $\mathcal{M}_B = \mathcal{M}$, for all $B \in \mathcal{M}$. This proves that $\mathcal{M}$ is an algebra.■

4. Exercises

3. Let $\mathcal{A}$ be a family of subsets of a set X. If E is any subset in X, we define the trace of $\mathcal{A}$ on E by the family $\mathcal{A} \cap E = \{A \cap E, A \in \mathcal{A}\}$.
Prove that $\sigma(\mathcal{A} \cap E) = \sigma(\mathcal{A}) \cap E$.
4. Let $\mathcal{S}$ be a family of subsets of a set X. We say that $\mathcal{S}$ is a semialgebra if it satisfies:
(a) ϕ, X are in $\mathcal{S}$
(b) If A, B are in $\mathcal{S}$ then $A \cap B$ is in $\mathcal{S}$
(c) If A is in $\mathcal{S}$ then $A^c = \sum_1^n A_k$, where the sets A_k are pairwise disjoint in $\mathcal{S}$.
Prove that the algebra generated by the semialgebra $\mathcal{S}$ is the family

$$\mathcal{A} = \left\{ A : A = \sum_1^n S_k, \text{ where the } S_k \text{ are pairwise disjoint in } \mathcal{S}. \right\}$$

5. Let $\mathbb{R}$ the set of real numbers equiped with the usual topology, prove that the family of all intervals is a semialgebra.
6. Let $\mathcal{S}_1$,$\mathcal{S}_2$ be semialgebras on the set X and consider the family $\mathcal{S} = \{S_1 \cap S_2, \quad S_1 \in \mathcal{S}_1, S_2 \in \mathcal{S}_2\}$.
Prove that $\mathcal{S}$ is a semialgebra and that the algebra generated by $\mathcal{S}$ is identical to the algebra generated by $\mathcal{S}_1$ and $\mathcal{S}_2$.
7. Let $(X_1, \mathcal{F}_1)$, $(X_2, \mathcal{F}_2)$ be measurable spaces. Prove that the family $\{A_1 \times A_2 : \quad A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2\}$ is a semialgebra.on $X_1 \times X_2$, (see exercise **4.**).

5. Limsup and Liminf

Let X be a set, and let $\mathcal{P}(X)$ be the power set of X. We assume that $\mathcal{P}(X)$ is endowed with the inclusion ordering $\subset$. then:

Definition 5.1.
For any sequence (A_n) in $\mathcal{P}(X)$, we define the sets $\limsup_n A_n$ and $\liminf_n A_n$ by:

$\limsup_n A_n = \bigcap_{n\geq 1} \bigcup_{k\geq n} A_k$

$\liminf_n A_n = \bigcup_{n\geq 1} \bigcap_{k\geq n} A_k$

Similarly let $\mathbb{R}, \leq$ be the ordered real number system and:

Definition 5.2.
For any sequence (a_n) in $\mathbb{R}$, we define the numbers $\limsup_n a_n$ and $\liminf_n a_n$ in $\overline{\mathbb{R}} = [-\infty, \infty]$ by:

$\limsup_n a_n = \inf_{n\geq 1} \sup_{k\geq n} a_k$

$\liminf_n a_n = \sup_{n\geq 1} \inf_{k\geq n} a_k$

Definition 5.3.
If $f_n : X \longrightarrow \mathbb{R}$ us a sequence of functions from a set X into $\mathbb{R}$, we define the functions $\limsup_n f_n$ and $\liminf_n f_n$ from X into $\overline{\mathbb{R}}$,by:

$\left(\limsup_n f_n\right)(x) = \limsup_n \left(f_n(x)\right)$

$\left(\liminf_n f_n\right)(x) = \liminf_n \left(f_n(x)\right)$

6. Exercises

8. Prove that for any sequence (A_n) in $\mathcal{P}(X)$ we have:

$\liminf_n A_n \subset \limsup_n A_n$

$\left(\liminf_n A_n\right)^c = \limsup_n A_n^c$

$\left(\limsup_n A_n\right)^c = \liminf_n A_n^c$

9. Let I_A be the indicator function of the set A, i.e $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ if $x \notin A$.
Prove that for any sequence (A_n) in $\mathcal{P}(X)$ we have::

$I_{\limsup_n A_n} = \limsup_n I_{A_n}$ and $I_{\liminf_n A_n} = \liminf_n I_{A_n}$

7. Positive Measures

Let $(X, \mathcal{F})$ be a measurable space.

Definition 7.1.
A positive measure μ on $\mathcal{F}$ is a set function
$\mu : \mathcal{F} \longrightarrow [0\ \infty]$ such that:

(i) $\mu(\phi) = 0$
(ii) For every pairwise disjoint sequence (A_n) in $\mathcal{F}$:
$\mu\left(\sum\limits_n A_n\right) = \sum\limits_n \mu(A_n)$ ($\sigma-$additivity of μ).

The triple $(X, \mathcal{F}, \mu)$ is called measure space.
Let us observe that for a finite pairwise disjoint sequence
$A_k, 1 \leq k \leq n$ in $\mathcal{F}$, we have: $\mu\left(\sum\limits_1^n A_k\right) = \sum\limits_1^n \mu(A_k)$.

Example 7.2.

(a) Let X be a set and fix $x_0 \in X$. Define μ on $\mathcal{P}(X)$ by:
$A \in \mathcal{P}(X)$, $\mu(A) = I_A(x_0)$ (see exercise **9** defining the function I_A). $I_{(\cdot)}(x_0)$ is called Dirac measure at x_0.
To prove the $\sigma-$additivity of μ, observe that $I_{\sum\limits_n A_n} = \sum\limits_n I_{A_n}$ for pairwise disjoint sequences (A_n).
(b) For $A \subset X$ put $\mu(A) = \infty$ if A is an infinite set and $\mu(A) = n$ if A is a finite set with n elements. This measure is called the cardinality measure on $\mathcal{P}(X)$.

Proposition 7.3.
Let $(X, \mathcal{F}, \mu)$ be a measure space and let A, B be in $\mathcal{F}$, then:
(a) $A \subset B \Longrightarrow \mu(A) \leq \mu(B)$.
(b) $A \subset B$ and $\mu(A) < \infty \Longrightarrow \mu(B \backslash A) = \mu(B) - \mu(A)$.
($B \backslash A$ is the difference set $B \cap A^c$)

Proof. If $A \subset B$, then $B = (B - A) \cup A$ and $\mu(B) = \mu(B \backslash A) + \mu(A)$, by additivity; so $\mu(B) \geq \mu(A)$ .If moreover $\mu(A) < \infty$ we deduce that:
$\mu(B \backslash A) = \mu(B) - \mu(A)$.■

Proposition 7.4. Let $(X, \mathcal{F}, \mu)$ be a measure space. Then for any sequence (A_n) in $\mathcal{F}$ we have:
$\mu\left(\bigcup\limits_n A_n\right) \leq \sum\limits_n \mu(A_n)$ (sub $\sigma-$additivity of μ).

Proof. Define the sequence (B_n) by the following recipe: $B_1 = A_1$, $B_2 = A_2 \backslash A_1$, $B_3 = A_3 \backslash (A_1 \cup A_2)$, ...$B_n \backslash \left(\bigcup\limits_{i<n} A_i\right)$, then $\bigcup\limits_n A_n = \sum\limits_n B_n$ and $B_n \subset A_n$, $\forall n$. So $\mu\left(\bigcup\limits_n A_n\right) = \mu\left(\sum\limits_n B_n\right) = \sum\limits_n \mu(B_n)$;by Proposition **7.3**$(a)$ $\mu(B_n) \leq \mu(A_n)$,$\forall n$.■

Proposition 7.5. (sequential continuity of a measure)
Let $(X, \mathcal{F}, \mu)$ be a measure space. If (A_n) is a sequence in $\mathcal{F}$, then we have
(a) if $A_1 \subset A_2 \subset ... \subset A_n \subset .... \subset A = \underset{n}{\cup} A_n$ then $\mu(A) = \underset{n}{Lim} \mu(A_n)$
(b) if $A_1 \supset A_2 \supset ... \supset A_n \supset .... \supset A = \underset{n}{\cap} A_n$ and if $\mu(A_{n_0}) < \infty$ for some n_0 then $\mu(A) = \underset{n}{Lim} \mu(A_n)$

Proof. (a) Define the sequence (B_n) by:
$B_1 = A_1$, $B_2 = A_2 \backslash A_1$, $B_3 = A_3 \backslash A_2, ..., B_n = A_n \backslash A_{n-1}$, so we have $A = \sum_n B_n$ and $\mu(A) = \sum_n \mu(B_n) = \sum_n \mu(A_n \backslash A_{n-1}) = \underset{n}{Lim} \sum_{k=1}^{n} \mu(A_k \backslash A_{k-1}) = \underset{n}{Lim} \mu\left(\sum_{k=1}^{n} A_k \backslash A_{k-1}\right)$; but $\sum_1^n A_k \backslash A_{k-1} = A_n$ by construction and we deduce that $\mu(A) = \underset{n}{Lim} \mu(A_n)$.

(b) We can assume $n_0 = 1$, so $\mu(A_n) < \infty$ for all n. On the other hand we have $A_1 \backslash A_1 \subset A_1 \backslash A_2 \subset ... \subset A_1 \backslash A_n \subset ... \underset{n}{\cup} A_1 \backslash A_n = A_1 \backslash A$. By (a) we deduce $\mu(A_1 \backslash A) = \underset{n}{Lim} \mu(A_1 \backslash A_n)$. Since $\mu(A_n) < \infty$ for all n we get, by Proposition **7.3**(b), $\mu(A_1 \backslash A) = \mu(A_1) - \mu(A)$ and $\mu(A_1 \backslash A_n) = \mu(A_1) - \mu(A_n)$, whence $\mu(A) = \underset{n}{Lim} \mu(A_n)$.■

Example 7.6. The condition (b) above is essential as is shown by taking μ the counting measure on $\mathbb{N}$ and taking $A_n = \{p : p \geq n\}$; indeed we have $\underset{n}{\cap} A_n = \phi$, so $\mu(\phi) = 0$ but $\mu(A_n) = \infty$, for all n, and then $\underset{n}{Lim} \mu(A_n) = \infty$.■

Proposition 7.7. (Borel-Cantelli Lemma)
Let $(X, \mathcal{F}, \mu)$ be a measure space. Let (A_n) be a sequence in $\mathcal{F}$ such that: $\sum_n \mu(A_n) < \infty$, then: $\mu\left(\limsup_n A_n\right) = 0$

Proof. Put $B_n = \underset{k \geq n}{\cup} A_k$, then B_n is decreasing and $\limsup_n A_n = \underset{n \geq 1}{\cap} B_n$. Since $\mu(B_n) = \mu\left(\underset{k \geq n}{\cup} A_k\right) \leq \sum_{k \geq n} \mu(A_n) \leq \sum_n \mu(A_n) < \infty$ for all n, we deduce, from Proposition **7.5** (b), that $\mu\left(\limsup_n A_n\right) = \underset{n}{Lim} \mu(B_n) \leq \underset{n}{Lim} \sum_{k \geq n} \mu(A_n) = 0$, because $\sum_{k \geq n} \mu(A_n)$ is the remainder of a convergent series.■