8. Complete Measures

Definition 8.1.

Let $(X, \mathcal{F}, \mu)$ be a measure space and let N be a subset of X, we say that N is a null set if there is $A \in \mathcal{F}$, with $\mu(A) = 0$ such that $N \subset A$. Let $\mathcal{N}$ be the family of null subsets of X. The space $(X, \mathcal{F}, \mu)$ is said to be complete if $\mathcal{N} \subset \mathcal{F}$ i.e every null set is mesurable.

Examples 8.2.

(a) The counting measure on any set X is complete since in this case ϕ is the only null set.

(b) If μ_s is the Dirac measure at s on $(X, \mathcal{F})$ (**Example 7.2.**(a)), every subset N not containing s is a null set

Lemma 8.3.

The family $\mathcal{N}$ is closed by countable union.

Proof. Let (N_k) be a sequence in $\mathcal{N}$, then for each k there is $A_k \in \mathcal{F}$, with $\mu(A_k) = 0$ such that $N_k \subset A_k$. So $N = \underset{k}{\cup} N_k \subset \underset{k}{\cup} A_k$; by the sub $\sigma-$ additivity of μ we have $\mu\left(\underset{n}{\cup} A_n\right) \leq \sum\limits_n \mu(A_n) = 0.$■

It is possible to complete any measure space $(X, \mathcal{F}, \mu)$ according to the following:

Theorem 8.4.

Let $(X, \mathcal{F}, \mu)$ be a measure space and let $\mathcal{N}$ be the family of null subsets of X. Let us put:

$\mathcal{F}_0 = \{E \subset X: \ E = F \cup N, \ F \in \mathcal{F}, \, N \in \mathcal{N}\}$

$\mu_0(E) = \mu_0(F \cup N) = \mu(F)$, if $E = F \cup N$, $F \in \mathcal{F}$, $N \in \mathcal{N}$

Then: $\mathcal{F}_0$ is a $\sigma-$field on X containing $\mathcal{F}$, and $\mathcal{N}$

μ_0 is a well defined measure on $\mathcal{F}_0$ that coincides with μ on $\mathcal{F}$.

The measure space $(X, \mathcal{F}_0, \mu_0)$ is complete.

Proof. First $\mathcal{F}_0$ is a $\sigma-$field

it is clear that ϕ and X are in $\mathcal{F}_0$

let $E \in \mathcal{F}_0$ with $E = F \cup N$, $F \in \mathcal{F}$, $N \in \mathcal{N}$ and let $A \in \mathcal{F}$, such that $\mu(A) = 0, N \subset A$; then we have $E^c = F^c \cap N^c = (F^c \cap N^c \cap A) + (F^c \cap N^c \cap A^c) = (F^c \cap N^c \cap A) + (F^c \cap A^c)$; since $F^c \cap N^c \cap A \in \mathcal{N}$ and $F^c \cap A^c \in \mathcal{F}$

we have $E^c \in \mathcal{F}_0$. Finally $\mathcal{F}_0$ is closed by countable union and this comes from the same property for the family $\mathcal{N}$ (**Lemma 8.3**).

To finish the proof, we consider the set function μ_0. First it is well defined, indeed suppose the set $E \in \mathcal{F}_0$ can be written as $E = F_1 \cup N_1 = F_2 \cup N_2$, then $F_1 \cap F_2^c \subset N_1 \cup N_2$ and $F_2 \cap F_1^c \subset N_1 \cup N_2$ which gives $\mu(F_1 \cap F_2^c) = \mu(F_2 \cap F_1^c) = 0$, so $\mu(F_1) = \mu(F_2)$ and $\mu_0(E) = \mu_0(F \cup N) = \mu(F)$ is well defined.

To prove the $\sigma-$additivity of μ_0, let (E_n) be a pairwise disjoint sequence in $\mathcal{F}_0$, and write $E_k = F_k \cup N_k$, $k \geq 1$, with $F_k \in \mathcal{F}$, $N_k \in \mathcal{N}$.

Then we have $\sum\limits_k E_k = \sum\limits_k F_k \cup \sum\limits_k N_k$, with $\sum\limits_k N_k \in \mathcal{N}$ (**Lemma 8.3**).

and $\mu_0\left(\sum\limits_k E_k\right) = \mu\left(\sum\limits_k F_k\right) = \sum\limits_k \mu(F_k) = \sum\limits_k \mu_0(E_k)$, since μ is $\sigma-$additive.

Finally we prove that $(X, \mathcal{F}_0, \mu_0)$ is complete. Let M_0 be a μ_0 null set in X, so there is $E_0 \in \mathcal{F}_0$ with $\mu_0(E_0) = 0$ and $M_0 \subset E_0$; write $E_0 = F \cup N$, $F \in \mathcal{F}$, $N \in \mathcal{N}$ with $\mu_0(E_0) = \mu(F) = 0$ and $N \subset A \in \mathcal{F}, \mu(A) = 0$, so $M_0 \subset F \cup A$, with $\mu(F \cup A) = 0$; this proves that $M_0 \in \mathcal{N} \subset \mathcal{F}_0$ and M_0 is $\mathcal{F}_0$ measurable.$\blacksquare$

9. Exercises

10. A family σ of subsets of X is $\sigma-$additive if:

(1) ϕ and X are in σ

(2) If (A_n) is an increasing sequence in σ then $\bigcup\limits_n A_n \in \sigma$

(3) For any A, B in σ we have:

$A \subset B \Longrightarrow B \cap A^c \in \sigma$

$A \cap B = \phi \Longrightarrow A + B \in \sigma$

(a) prove that any $\sigma-$field is a $\sigma-$additive family

(b) let μ, λ be two measures on the same measurable space $(X, \mathcal{F})$ such that $\mu(X) = \lambda(X) < \infty$.

Prove that the family $\sigma = \{A \in \mathcal{F}: \ \mu(A) = \lambda(A)\}$ is $\sigma-$additive.

Let $\boldsymbol{C}$ be a family of subsets of X then there exists a smallest $\sigma-$additive family on X containing $\boldsymbol{C}$ called the $\sigma-$additive family generated by $\boldsymbol{C}$.

11. Let $\Im$ be a family of subsets of X closed by finite intersection. Prove that the $\sigma-$field generated by $\Im$ coincides with the $\sigma-$additive family generated by $\Im$.

Chapter 2

Outer measures Extension of measures
1. Outer measures

Definition 1.1.

An outer measure on a set X is a set function
$\lambda : \mathcal{P}(X) \longrightarrow [0\ \infty]$ such that:
(1) $\lambda(\phi) = 0$
(2) if $A \subset B$ then $\lambda(A) \leq \lambda(B)$
(3) if (E_n) is any sequence in $\mathcal{P}(X)$ then $\lambda\left(\underset{n}{\cup} E_n\right) \leq \sum_n \lambda(E_n)$

Remark.1.2.
It is not difficult to see that if λ is additive then λ is a positive measure on $\mathcal{P}(X)$.

Example.1.3.

(a) Any positive measure on $\mathcal{P}(X)$ is an outer measure.

(b) Define λ on $\mathcal{P}(X)$ by $\lambda(\phi) = 0$ and $\lambda(E) = 1$ if $E \neq \phi$; if X has more than one point then λ is an outer measure but not a measure.
We can say that the notion of outer measure is a natural generalization of that of positive measure. We will see below that an outer measure acts as a true measure on a some specific family of subsets of X. Let us start with the following:

Definition 1.4.

Let λ be an outer measure on X. A subset $E \subset X$ is said to be outer measurable or $\lambda-$measurable if we have:

$$\text{for every } A \subset X, \quad \lambda(A) = \lambda(A \cap E) + \lambda(A \cap E^c)$$

Example.1.5.
(a) A subset $E \subset X$ with $\lambda(E) = 0$ is $\lambda-$measurable.
(b) X, ϕ are $\lambda-$measurable for every outer measure λ.
(c) .Let λ be defined on X by $\lambda(\phi) = 0$, $\lambda(X) = 2$, $\lambda(E) = 1$ for $E \neq \phi, X$.
Then λ is an outer measure and ϕ, X are the only $\lambda-$measurable sets.

Now we go to the important assertion:

Theorem.1.6.
Let λ be an outer measure on X
and let $\mathcal{F}$ be the family of the $\lambda-$measurable sets.
Then $\mathcal{F}$ is a $\sigma-$field and the restriction of λ to $\mathcal{F}$ is a positive measure.
Proof. see [7].

2. Exercises

12. Let λ be an outer measure on X and let H be a $\lambda-$measurable set. Let λ_0 be the restriction of λ to $\mathcal{P}(H)$, prove that:

(a) λ_0 is an outer measure on $\mathcal{P}(H)$.

(b) $A \subset H$ is λ_0-measurable iff A is $\lambda-$measurable.

13. Let λ be an outer measure on X and let A be a $\lambda-$measurable set. If $B \subset X$ is a subset with $\lambda(B) < \infty$, prove that:

$\lambda(A \cup B) = \lambda(A) + \lambda(B) - \lambda(A \cap B)$

3. Extension of Measures

We start this section with the construction of an outer measure from a measure defined on an algebra of sets.

Definition 3.1.

Let $\mathcal{A}$ be an algebra on X. A positive measure on $\mathcal{A}$ is a set function $\mu : \mathcal{A} \longrightarrow [0\ \infty]$ such that:

(i) $\mu(\phi) = 0$

(ii) For every pairwise disjoint sequence (A_n) in $\mathcal{A}$ with $\underset{n}{\cup} A_n \in \mathcal{A}$:

$$\mu\left(\sum_n A_n\right) = \sum_n \mu(A_n) \quad (\sigma-\text{additivity of } \mu).$$

Any measure on an algebra $\mathcal{A}$ gives rise toan outer measure according to:

Theorem.3.2.

Let μ be a measure on an algebra $\mathcal{A}$.

For each subset $E \subset X$ define $\lambda(E)$ by the recipe:

$$\lambda(E) = \inf\left\{\sum_n \mu(A_n) : E \subset \underset{n}{\cup} A_n, \quad (A_n) \subset \mathcal{A}\right\}$$

the lower bound being taken over all sequences $(A_n) \subset \mathcal{A}$.

Then λ is an outer measure whose restriction to $\mathcal{A}$ coincides with μ.

Moreover the sets of $\mathcal{A}$ are $\lambda-$measurable.

Proof. see [7].

Definition 3.3. ($\sigma-$finite measures)

Let $(X, \mathcal{F}, \mu)$ be a measure space. We say that the measure μ is $\sigma-$finite if there is a sequence (A_n) in $\mathcal{F}$, such that $\underset{n}{\cup} A_n = X$ and $\mu(A_n) < \infty$, $\forall n$.

A measure μ on an algebra $\mathcal{A}$ is $\sigma-$finite if there is a sequence (A_n) in $\mathcal{A}$ such that $\underset{n}{\cup} A_n = X$ and $\mu(A_n) < \infty$, $\forall n$.

Example 3.4.

(a) Any finite measure μ, i.e $\mu(X) < \infty$, is $\sigma-$finite

(b) The counting measure on $\mathbb{N}$ or on any infinite countable set is $\sigma-$finite but not finite.

(c) we will see later that the Lebesgue measure on $\mathbb{R}$ is a non trivial $\sigma-$finite measure.

Now we give the main extension theorem:

Theorem 3.5.

Let μ be a measure on an algebra $\mathcal{A}$ of subsets of X.
Then μ can be extended to a measure $\overline{\mu}$ on the $\sigma-$field $\sigma(\mathcal{A})$ generated by $\mathcal{A}$. Moreover if μ is $\sigma-$finite on $\mathcal{A}$ the extension $\overline{\mu}$ is unique.

Proof.

Let μ^* be the outer measure given by Theorem**.** **3.2** and let $\mathcal{F}$ be the $\sigma-$field of μ^*-measurable sets. By the same theorem we have $\mathcal{A} \subset \mathcal{F}$ and μ^* coincides with μ on $\mathcal{A}$. So we have $\sigma(\mathcal{A}) \subset \mathcal{F}$. By Theorem**.** **1.6** μ^* acts as a true measure on $\mathcal{F}$. Then it is enough to take $\overline{\mu}$ as the restriction of μ^* to $\sigma(\mathcal{A})$. We prove the uniqueness in the case μ finite. Suppose the existence of two extensions μ_1, μ_2 for μ and consider the family $\mathcal{M} = \{A \in \sigma(\mathcal{A}) : \ \mu_1(A) = \mu_2(A)\}$. It is not difficult to prove that $\mathcal{M}$ is a monotone class which contains $\mathcal{A}$ (use the finiteness of the measures) So we have $\mathcal{A} \subset \mathcal{M} \subset \sigma(\mathcal{A})$ and since $\mathcal{A}$ is an algebra the monotone class generated by $\mathcal{A}$ is idendical to the $\sigma-$field generated by $\mathcal{A}$ (Theorem **3.10,** Chap. **1**) We deduce that $\mathcal{M} = \sigma(\mathcal{A})$. We leave the $\sigma-$finiteness case to the reader.■

Theorem 3.6.

Let μ be a $\sigma-$finite measure on an algebra $\mathcal{A}$ of subsets of X.
Let $\overline{\mu}$ be the unique extension of μ to the $\sigma-$field $\sigma(\mathcal{A})$ generated by $\mathcal{A}$. If $B \in \sigma(\mathcal{A})$ with $\overline{\mu}(B) < \infty$, then:

$\forall \epsilon > 0$ there is $A_\epsilon \in \mathcal{A}$ such that $\overline{\mu}(B \triangle A_\epsilon) < \epsilon$

where $B \triangle A_\epsilon$ is the symmetric difference $(B \cap A_\epsilon^c) \cup (A_\epsilon \cap B^c)$.

Proof. By Theorems **3.2** and **3.5** the unique extension $\overline{\mu}$ has the form:

$$\overline{\mu}(B) = \inf \left\{ \sum_n \mu(A_n) : B \subset \bigcup_n A_n, \quad (A_n) \subset \mathcal{A} \right\}$$

If $B \in \sigma(\mathcal{A})$ with $\overline{\mu}(B) < \infty$, $\forall \epsilon > 0$ $\exists (A_n) \subset \mathcal{A}$ such that $B \subset \bigcup_n A_n$ and $\sum_n \mu(A_n) < \overline{\mu}(B) + \frac{\epsilon}{2}$.then use the fact that $\bigcup_n A_n = \lim_N \bigcup_1^N A_n$ and $\bigcup_1^N A_n \in \mathcal{A}$; put $B_N = \bigcup_1^N A_n$ then $\overline{\mu}\left(\bigcup_n A_n\right) = \lim_N \overline{\mu}(B_N) = \lim_N \mu(B_N)$.

So for some N_0 we have $\overline{\mu}\left(\bigcup_n A_n\right) < \overline{\mu}(B_{N_0}) + \frac{\epsilon}{2}$, then the set $A_\epsilon = B_{N_0}$ is in $\mathcal{A}$ and works.■

4. Exercises

14. An outer measure μ^* on X is regular if for any $A \subset X$ there is a μ^*-measurable set E such that $A \subset E$ and $\mu^*(A) = \mu^*(E)$.

(a) If μ^* is regular then for any sequence (A_n) of subsets of X we have

$$\mu^*\left(\liminf_n A_n\right) \leq \liminf_n \mu^*(A_n).$$

(b) If moreover the sequence (A_n) is increasing then $\mu^*\left(\lim_n A_n\right) = \lim_n \mu^*(A_n)$.

15. Let $(X,\mathcal{F},\mu)$ be a measure space. Define μ^* on $\mathcal{P}(X)$ by the recipe:
$\mu^*(E)=\inf\{\mu(A): A\in\mathcal{F}\quad E\subset A\}$
(a) Prove that μ^* is an outer measure.
(b) Prove that $\forall E\subset X\quad \exists A\in\mathcal{F}$ such that $E\subset A$ and $\mu^*(E)=\mu(A)$.
(c) Let us define μ^* on $\mathcal{P}(X)$ by the recipe:
$\mu_*(E)=\sup\{\mu(A): A\in\mathcal{F}\quad E\subset A\}$
Prove that $\forall E\subset X$, in either case $\mu_*(E)<\infty$ or $\mu_*(E)=\infty$, there is$A\in\mathcal{F}$ such that $E\subset A$ and $\mu_*(E)=\mu(A)$.
(d) Prove that $\mu_*(E)\leq\mu^*(E), \forall E\subset X$ and if E is μ^*-measurable then $\mu_*(E)=\mu^*(E)$ .If $\mu_*(E)=\mu^*(E)<\infty$ then E is μ^*-measurable.

5. Lebesgue Measure on $\mathbb{R}$

Measure on the Algebra generated by the semialgebra of intervals
Let us recall that a family $\mathcal{S}$ of subsets of a set X is a semialgebra if it satisfies:
(a) ϕ, X are in $\mathcal{S}$
(b) If A,B are in $\mathcal{S}$ then $A\cap B$ is in $\mathcal{S}$
(c) If A is in $\mathcal{S}$ then $A^c=\sum_1^n A_k$, where the sets A_k are pairwise disjoint in $\mathcal{S}$
(see Chapter 1 exercise 4)
We recall also that the algebra generated by the semialgebra $\mathcal{S}$ is the family
$$\left\{A : A=\sum_1^n S_k,\ \text{where the } S_k \text{ are pairwise disjoint in } \mathcal{S}.\right\}$$
It is easy to prove that the family $\mathcal{I}$ of all intervals of $\mathbb{R}$ is a semialgebra. Let $\mathcal{A}$ be the algebra generated by $\mathcal{I}$. It is well known that the borel $\sigma-$field $\mathcal{B}_{\mathbb{R}}$ of $\mathbb{R}$ is generated by $\mathcal{A}$ or simply by $\mathcal{I}$. Now if $A\in\mathcal{A}$ has the form $A=\sum_1^n I_k$, where the I_k are pairwise disjoint in $\mathcal{I}$, put $\mu(A)=\sum_1^n\lambda(I_k)$, where $\lambda(I)$ is the lengh of the interval I. Then μ is unambiguously defined on $\mathcal{A}$. Moreover μ is a $\sigma-$finite measure on the algebra $\mathcal{A}$. By Theorems **3.2** and **3.5** the unique extension $\overline{\mu}$ of μ to the $\sigma-$field $\sigma(\mathcal{A})=\mathcal{B}_{\mathbb{R}}$ generated by $\mathcal{A}$ has the form:
$$\overline{\mu}(B)=\inf\left\{\sum_n\mu(A_n): B\subset\bigcup_n A_n,\quad (A_n)\subset\mathcal{A}\right\}$$
The completion of the measure space $(\mathbb{R},\mathcal{B}_{\mathbb{R}},\overline{\mu})$ is the Lebesgue space $(\mathbb{R},\mathcal{L}_{\mathbb{R}},\overline{\mu}^*)$ (see Theorem **8.4,** Chap.**1**). In fact each set $E\in\mathcal{L}_{\mathbb{R}}$ has the form $E=B\cup N$, where $B\in\mathcal{B}_{\mathbb{R}}$ and N is a $\overline{\mu}-$null set. Let us note the following approximation result:

Theorem 5.1.
Let $E\in\mathcal{L}_{\mathbb{R}}$, then we have:
$\forall\epsilon>0$ there is a closed set F and an open set G such that:
$F\subset E\subset G$ and $\overline{\mu}^*(G\diagdown F)<\epsilon$