Solutions of Exercises (MEA-INT)

2

For every sequence (A_n), define the sequence (B_n) by the following recipe:

$B_1 = A_1$, $B_2 = A_2 \backslash A_1$, $B_3 = A_3 \backslash (A_1 \cup A_2)$, ..., $B_n = A_n \backslash \left(\underset{i<n}{\cup} A_i \right)$, ...

Prove that $\underset{n}{\cup} A_n = \sum_n B_n$. It is clear that the B_n are pairwise disjoint and that $B_n \subset A_n, \forall n$. Then $\sum_n B_n \subset \underset{n}{\cup} A_n$.

Now we show that $\underset{n}{\cup} A_n \subset \sum_n B_n$. Let $x \in \underset{n}{\cup} A_n$, so there is $n \geq 1$ such that $x \in A_n$. Put $n_1 = \inf \{n \geq 1, \quad x \in A_n\}$

this means that $x \in A_{n_1}$ and $x \notin \underset{i<n_1}{\cup} A_i$, that is $x \in B_{n_1}$. Therefore for each $x \in \underset{n}{\cup} A_n$ there is n_1 such that $x \in B_{n_1}$ which means that $\underset{n}{\cup} A_n \subset \sum_n B_n$.

1

First we must add the missing condition: $A \in \mathcal{F} \Longrightarrow A^c \in \mathcal{F}$.

We have to prove that for every sequence $(A_n) \subset \mathcal{F}$ we have $\underset{n}{\cup}.A_n \in \mathcal{F}$.

By the solution above we have $\underset{n}{\cup} A_n = \sum_n B_n$ and by definition

$B_n = A_n \backslash \left(\underset{i<n}{\cup} A_i \right) = A_n \cap A_1^c \cap A_2^c \cap ... \cap A_{n-1}^c$

so we have each $A_i^c \in \mathcal{F}$ and by condition (b) $B_n \in \mathcal{F}$,$\forall n$

moreover the sets B_n are paiwise disjoint and condition (c) implies $\sum_n B_n \in \mathcal{F}$ and since $\underset{n}{\cup} A_n = \sum_n B_n$ we deduce that $\underset{n}{\cup}.A_n \in \mathcal{F}$.

3

Let $\mathcal{A}$ be a family of subsets of a set X. If E is any subset in X, we define the trace of $\mathcal{A}$ on E by the family $\mathcal{A} \cap E = \{A \cap E, A \in \mathcal{A}\}$.

Prove that $\sigma(\mathcal{A} \cap E) = \sigma(\mathcal{A}) \cap E$.

solution.

First we prove that $\sigma(\mathcal{A}) \cap E$ is a $\sigma-$algebra on E:

$\sigma(\mathcal{A}) \cap E = \{F \subset X : \exists K \in \sigma(\mathcal{A}) \text{ with } F = K \cap E\}$

(1) $E \in \sigma(\mathcal{A}) \cap E$ since $E = X \cap E$ and $X \in \sigma(\mathcal{A})$

(2) let $H \in \sigma(\mathcal{A}) \cap E$ with $H = K \cap E$ and $K \in \sigma(\mathcal{A})$

we prove that the complement of H in E that is $E \backslash H$ is in $\sigma(\mathcal{A}) \cap E$

we have $E \backslash H = E \cap H^c = E \cap (K \cap E)^c = E \cap K^c$ and $K^c \in \sigma(\mathcal{A})$

so $E \backslash H \in \sigma(\mathcal{A}) \cap E$

(3) Let (H_n), be a sequence in $\sigma(\mathcal{A}) \cap E$ with $H_n = K_n \cap E$, $K_n \in \sigma(\mathcal{A})$

then $\underset{n}{\cup} H_n = \left(\underset{n}{\cup} K_n \right) \cap E$, and $\underset{n}{\cup} K_n \in \sigma(\mathcal{A})$, since $\sigma(\mathcal{A})$ is a $\sigma-$algebra.$\blacksquare$

4. Let $\mathcal{S}$ be a family of subsets of a set X. We say that $\mathcal{S}$ is a semialgebra if it satisfies:
(a) ϕ, X are in $\mathcal{S}$
(b) If A, B are in $\mathcal{S}$ then $A \cap B$ is in $\mathcal{S}$
(c) If A is in $\mathcal{S}$ then $A^c = \sum\limits_1^n A_k$, where the sets A_k are pairwise disjoint in $\mathcal{S}$.
Prove that the algebra $\mathcal{A}(\mathcal{S})$ generated by the semialgebra $\mathcal{S}$ is the family

$$\mathcal{A} = \left\{ A : A = \sum_1^n S_k, \text{ where the } S_k \text{ are pairwise disjoint in } \mathcal{S}. \right\}$$

solution.

$\mathcal{S} \subset \mathcal{A}$ since any $S \in \mathcal{S}$ can be written as $S = S + \phi$ and S, ϕ are in $\mathcal{S}$.

now we prove that $\mathcal{A}$ is an algebra:

Let $A \in \mathcal{A}$ with $A = \sum\limits_1^n S_k$, then $A^c = \bigcap\limits_1^n . S_k^c$, we apply condition (c) to each S_k^c and obtain:

$S_k^c = \sum\limits_1^{n_k} A_{i_k}$, so $A^c = \bigcap\limits_{k=1}^n . \sum\limits_1^{n_k} A_{i_k} = \left(\sum\limits_1^{n_1} A_{i_k} \right) \cap \left(\sum\limits_1^{n_2} A_{i_k} \right) \cap ... \cap \left(\sum\limits_1^{n_n} A_{i_k} \right) = \sum\limits_{i_1=1}^{n_1} \sum\limits_{i_2=1}^{n_2} .... \sum\limits_{i_n=1}^{n_n} . A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_n}$, and since each $A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_n}$ is in $\mathcal{S}$ we deduce that A^c is in $\mathcal{A}$.

Let $A = \sum\limits_1^n S_k$ and $B = \sum\limits_1^m T_j$ both in $\mathcal{A}$, then $A \cap B = \sum\limits_k \sum\limits_j . S_k \cap T_j \in \mathcal{A}$, because $S_k \cap T_j \in \mathcal{S}$ $\forall k, j$. So $\mathcal{A}$ is an algebra, and since it contains $\mathcal{S}$ it also contains the algebra $\mathcal{A}(\mathcal{S})$ generated by $\mathcal{S}$, that is $\mathcal{A}(\mathcal{S}) \subset \mathcal{A}$. On the other hand let $A = \sum\limits_1^n S_k \in \mathcal{A}$, since each S_k is in $\mathcal{S}$, we have $A = \sum\limits_1^n S_k \in \mathcal{A}(\mathcal{S})$ so $\mathcal{A} \subset \mathcal{A}(\mathcal{S})$.■

5. Let $\mathbb{R}$ the set of real numbers equiped with the usual topology, prove that the family of all intervals is a semialgebra.

solution. Straightforward and left to the reader.■

8. Prove that for any sequence (A_n) in $\mathcal{P}(X)$ we have:

1. $\liminf\limits_n A_n \subset \limsup\limits_n A_n$
2. $\left(\liminf\limits_n A_n \right)^c = \limsup\limits_n A_n^c$
3. $\left(\limsup\limits_n A_n \right)^c = \liminf\limits_n A_n^c$

solution,

Recall the frequently used **De Morgan's Laws:**

$\left(\bigcup\limits_i A_i \right)^c = \bigcap\limits_i A_i^c, \quad \left(\bigcap\limits_i A_i \right)^c = \bigcup\limits_i A_i^c$

valid for any family $\{A_i,\, i \in I\}$, where A^c denotes the complement of the set A.

For any sequence (A_n) of sets, we defined the sets $\limsup_n A_n$ and $\liminf_n A_n$ by:

$\limsup_n A_n = \underset{n\geq 1}{\cap} \underset{k\geq n}{\cup} A_k$

$\liminf_n A_n = \underset{n\geq 1}{\cup} \underset{k\geq n}{\cap} A_k$

solution of 1. we have $\forall n \geq 1, \underset{k\geq n}{\cap} A_k \subset \underset{k\geq n}{\cup} A_k \Longrightarrow$

whence $\underset{n\geq 1}{\cup} \underset{k\geq n}{\cap} A_k \subset \underset{n\geq 1}{\cap} \underset{k\geq n}{\cup} A_k$

we apply **De Morgan's Laws**

solution of 2. $\left(\liminf_n A_n\right)^c = \left(\underset{n\geq 1}{\cup} \underset{k\geq n}{\cap} A_k\right)^c = \underset{n\geq 1}{\cap} \underset{k\geq n}{\cup} A_k^c = \limsup_n A_n^c$

solution of 3. $\left(\limsup_n A_n\right)^c = \left(\underset{n\geq 1}{\cap} \underset{k\geq n}{\cup} A_k\right)^c = \underset{n\geq 1}{\cup} \underset{k\geq n}{\cap} A_k^c = \liminf_n A_n^c$

9. Let I_A be the indicator function of the set A, i.e $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ if $x \notin A$.
Prove that for any sequence (A_n) in $\mathcal{P}(X)$ we have::

$I_{\limsup_n A_n} = \limsup_n I_{A_n}$ and $I_{\liminf_n A_n} = \liminf_n I_{A_n}$

solution

we have to show that $I_{\limsup_n A_n}(x) = \limsup_n I_{A_n}(x), \forall x \in X$

$I_{\limsup_n A_n}(x) = 1 \Longleftrightarrow x \in \underset{n\geq 1}{\cap} \underset{k\geq n}{\cup} A_k \Longleftrightarrow \forall n \geq 1, \exists k \geq n : x \in A_k$

so $I_{A_k}(x) = 1$ which is equivalent to $\limsup_n I_{A_n}(x) = 1$

10. A family σ of subsets of X is $\sigma-$additive if:

(1) ϕ and X are in σ

(2) If (A_n) is an increasing sequence in σ then $\underset{n}{\cup} A_n \in \sigma$

(3) For any A, B in σ we have:

$A \subset B \Longrightarrow B \cap A^c \in \sigma$

$A \cap B = \phi \Longrightarrow A + B \in \sigma$

(a) prove that any $\sigma-$algebra is a $\sigma-$additive family

(b) let μ, λ be two measures on the same measurable space $(X, \mathcal{F})$ such that $\mu(X) = \lambda(X) < \infty$.

Prove that the family $\sigma = \{A \in \mathcal{F}: \ \mu(A) = \lambda(A)\}$ is $\sigma-$additive.

(c) Let C be a family of subsets of X then prove that there exists a smallest $\sigma-$additive family on X containing C called the $\sigma-$additive family generated by C.

solution

(a) any $\sigma-$algebra satisfies conditions $(1), (2), (3)$, of a $\sigma-$additive family

(b) let $\sigma = \{A \in \mathcal{F}: \ \mu(A) = \lambda(A)\}$:

(1) is satisfied since $\mu(X) = \lambda(X)$ and $\mu(\phi) = \lambda(\phi)$ imply X, ϕ in σ

(2) is satisfied by the sequential continuity of measures

(3) is satisfied because μ, λ are finite: $A \subset B \Longrightarrow \mu(B \cap A^c) = \mu(B) - \mu(A)$ and $\lambda(B \cap A^c) = \lambda(B) - \lambda(A)$ so $B \cap A^c \in \sigma$

(c) It is not difficult to prove that the intersection $\bigcap\limits_i .\sigma_i$ of a family $\{\sigma_i, \quad i \in I\}$ of $\sigma-$additive families is a $\sigma-$additive family. Now we take the intersection $\sigma(C)$ of all $\sigma-$additive families σ containing C, then it is clear that $\sigma(C)$ is the smallest $\sigma-$additive family on X containing C