Chapter 3

Measurable Functions

1. Preliminaries

Definition.1.1.

Let X, Y be non empty sets.
To each function $f : X \longrightarrow Y$ it corresponds the preimage function $f^{-1} : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$ defined by: $B \in \mathcal{P}(Y)$, $f^{-1}(B) = \{x \in X : f(x) \in B\}$.
Also if $\Im$ is any subfamily of $\mathcal{P}(Y)$ put $f^{-1}(\Im) = \left\{f^{-1}(B), B \in \Im\right\}$.

Proposition.1.2.

The preimage function has the following properties:
(a) $f^{-1}\left(\underset{i}{\cup} B_i\right) = \underset{i}{\cup} f^{-1}(B_i)$ and $f^{-1}\left(\underset{i}{\cap} B_i\right) = \underset{i}{\cap} f^{-1}(B_i)$
for any family $(B_i) \subset \mathcal{P}(Y)$
(b) $f^{-1}(B^c) = \left(f^{-1}(B)\right)^c$, for any $B \in \mathcal{P}(Y)$
(c) $B \subset C \Longrightarrow f^{-1}(B) \subset f^{-1}(C)$ for any B, C in $\mathcal{P}(Y)$.

Proof. straightforward.■

Proposition.1.3.

Let $(X, \mathcal{F}), (Y, \mathcal{G})$ be measure spaces and $f : X \longrightarrow Y$ a function. Define the families:

$\Re_f = \left\{f^{-1}(G) : G \in \mathcal{G}\right\} = f^{-1}(\mathcal{G})$

$\mathcal{B}_f = \left\{B \subset Y : f^{-1}(B) \in \mathcal{F}\right\}$

Then $\Re_f$ is a $\sigma-$field on X and $\mathcal{B}_f$ a $\sigma-$field on Y
Moreover we have $f^{-1}(\mathcal{B}_f) \subset \mathcal{F}$.

Proof. We prove first that $\Re_f$ is a $\sigma-$field on X.
$X \in \Re_f$ since $X = f^{-1}(Y)$ and $Y \in \mathcal{G}$.
Let $A \in \Re_f$ with $A = f^{-1}(G)$ for some $G \in \mathcal{G}$, then $A^c = f^{-1}(G^c)$ since $G^c \in \mathcal{G}$, we deduce that $A^c \in \Re_f$.
Let (A_n) be a sequence in $\Re_f$ with $A_n = f^{-1}(G_n)$ for some $G_n \in \mathcal{G}$;
by Proposition. **1.2** (a) we have $\underset{n}{\cup} A_n = \underset{n}{\cup} f^{-1}(G_n) = f^{-1}\left(\underset{n}{\cup} G_n\right)$
since $\underset{n}{\cup} G_n \in \mathcal{G}$, we deduce that $\underset{n}{\cup} A_n \in \Re_f$. So $\Re_f$ is a $\sigma-$field on X.
The reader can do the remains by the same way.■

2. Measurable Functions Properties

Definition.2.1.

Let $(X, \mathcal{F}), (Y, \mathcal{G})$ be measure spaces and $f : X \longrightarrow Y$ a function. We say that f is measurable if $f^{-1}(\mathcal{G}) \subset \mathcal{F}$.This means that:
$f^{-1}(G) \in \mathcal{F}$ for every $G \in \mathcal{G}$.

Theorem.2.2.
Let $f : X \longrightarrow Y$ be a function and $\Im$ a family of subsets of Y.
Then we have $\sigma\left(f^{-1}(\Im)\right) = f^{-1}(\sigma(\Im))$.
This means that: the $\sigma-$field $\sigma\left(f^{-1}(\Im)\right)$ generated by $f^{-1}(\Im)$ coincides with the preimage of the $\sigma-$field $\sigma(\Im)$.

Proof. $\Im \subset \sigma(\Im) \Longrightarrow f^{-1}(\Im) \subset f^{-1}(\sigma(\Im))$ and $f^{-1}(\sigma(\Im))$ is a $\sigma-$field, since the preimage of a $\sigma-$field is a $\sigma-$field by Proposition.**1.3.**
So we deduce that $\sigma\left(f^{-1}(\Im)\right) \subset f^{-1}(\sigma(\Im))$. Now consider the $\sigma-$field
$\mathcal{B}_f = \left\{B \subset Y : f^{-1}(B) \in \sigma\left(f^{-1}(\Im)\right)\right\}$. If $B \in \mathcal{B}_f$, then $f^{-1}(B) \subset \sigma\left(f^{-1}(\Im)\right)$, so $f^{-1}(\mathcal{B}_f) \subset \sigma\left(f^{-1}(\Im)\right)$. But $\Im \subset \mathcal{B}_f$, and then $\sigma(\Im) \subset \mathcal{B}_f$,
so we get $f^{-1}(\sigma(\Im)) \subset f^{-1}(\mathcal{B}_f) \subset \sigma\left(f^{-1}(\Im)\right)$.■

Proposition.2.3.
Let $(X,\mathcal{F}),(Y,\mathcal{G})$ be measurable spaces and $f : X \longrightarrow Y$ a function. Suppose there is a family $\Im$ of subsets of Y with $\sigma(\Im) = \mathcal{G}$ and satsfying $f^{-1}(\Im) \subset \mathcal{F}$.Then f is measurable with respect to $(X,\mathcal{F}),(Y,\mathcal{G})$.

Proof. Since $f^{-1}(\Im) \subset \mathcal{F}$ we have $\sigma\left(f^{-1}(\Im)\right) \subset \mathcal{F}$.
By Theorem.**2.2** $\sigma\left(f^{-1}(\Im)\right) = f^{-1}(\sigma(\Im))$, but $\sigma(\Im) = \mathcal{G}$
and so $f^{-1}(\mathcal{G}) \subset \mathcal{F}$.■

Examples.2.4.
(a) Let $f : X \longrightarrow \mathbb{R}$ be a function from $(X,\mathcal{F})$ into $(\mathbb{R},\mathcal{B}_{\mathbb{R}})$.The Borel $\sigma-$field $\mathcal{B}_{\mathbb{R}}$ is defined in Proposition **3.6**, chap.**1.** For f to be measurable it is enough that $f^{-1}(]-\infty,t[) \in \mathcal{F}$ (the intervals $]-\infty,t[$ generates $\mathcal{B}_{\mathbb{R}}$)

(b) Let X be a topological space with a countable base (U_n), endowed with its Borel $\sigma-$field $\mathcal{B}_Y$. It is well known that $\mathcal{B}_Y$ is generated by the family (U_n) and any open set is the union of a subfamily of (U_n). So for a function from $(X,\mathcal{F})$ into $(Y,\mathcal{B}_Y)$ to be measurable it is enough that $f^{-1}(U_n) \in \mathcal{F}$ for every n.

(c) Let X,Y be topological spaces endowed with their Borel $\sigma-$fields $\mathcal{B}_X,\mathcal{B}_Y$. A function $f : X \longrightarrow Y$ is measurable with respect to $\mathcal{B}_X,\mathcal{B}_Y$ iff $f^{-1}(G) \in \mathcal{B}_X$ for every open set $G \subset Y$. In particular any continuous function is measurable.

(d) Let $I_A : X \longrightarrow \mathbb{R}$ be the indicator function of the set A, i.e $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ if $x \notin A$. We have $I_A^{-1}(\mathcal{B}_{\mathbb{R}}) = \{A, A^c, X, \phi\}$, then I_A is measurable from $(X,\mathcal{F})$ into $(\mathbb{R},\mathcal{B}_{\mathbb{R}})$ iff $A \in \mathcal{F}$.

Now we state some important properties of measurable functions.

Proposition.2.5.
Let $(X,\mathcal{F}),(Y,\mathcal{G}),(Z,\mathcal{H})$ be measurable spaces and
$f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ measurable functions. Then the composition function $g \circ f : X \longrightarrow Z$ is measurable from $(X,\mathcal{F})$ into $(Z,\mathcal{H})$.

Proof. We have $(g \circ f)^{-1}(\mathcal{H}) = \left(f^{-1} \circ g^{-1}\right)(\mathcal{H}) = f^{-1}\left(g^{-1}(\mathcal{H})\right)$
Since g is measurable $g^{-1}(\mathcal{H}) \subset \mathcal{G}$, so $f^{-1}\left(g^{-1}(\mathcal{H})\right) \subset f^{-1}(\mathcal{G})$. But f is measurable then $f^{-1}(\mathcal{G}) \subset \mathcal{F}$. We deduce that $(g \circ f)^{-1}(\mathcal{H}) \subset \mathcal{F}$ and $g \circ f$ is measurable.■

Proposition.2.6.

Let $(X \times Y, \mathcal{F} \otimes \mathcal{G})$ be the product of the measurable spaces $(X, \mathcal{F}), (Y, \mathcal{G})$ (see Definition **3.4.** Chap.**1**). Then the projection $\pi_1(x, y) = x$ is measurable from $(X \times Y, \mathcal{F} \otimes \mathcal{G})$ into $(X, \mathcal{F})$. Similarly the projection $\pi_2(x, y) = y$ is measurable from $(X \times Y, \mathcal{F} \otimes \mathcal{G})$ into $(Y, \mathcal{G})$.

Proof. By Definition **3.4** Chap.**1** the $\sigma-$field $\mathcal{F} \otimes \mathcal{G}$ contains the family $\{A \times B : \ A \in \mathcal{F}, \ \ B \in \mathcal{G}\}$. We get $\pi_1^{-1}(A) = A \times Y \in \mathcal{F} \otimes \mathcal{G}$ for every $A \in \mathcal{F}$ and $\pi_2^{-1}(B) = X \times B \in \mathcal{F} \otimes \mathcal{G}$ for every $B \in \mathcal{G}$. So π_1 and π_2 are measurable.$\blacksquare$

Proposition.2.7.

Let $(Z, \mathcal{H})$ be a measurable space and let $f : Z \longrightarrow X \times Y$ be a function with $f_1 = \pi_1 \circ f : Z \longrightarrow X$ and $f_2 = \pi_2 \circ f : Z \longrightarrow Y$. Then f is measurable from $(Z, \mathcal{H})$ into $(X \times Y, \mathcal{F} \otimes \mathcal{G})$ if and only if f_1 is measurable from $(Z, \mathcal{H})$ into $(X, \mathcal{F})$ and f_2 is measurable from $(Z, \mathcal{H})$ into $(Y, \mathcal{G})$.

Proof. The <if> part comes from the measurability of π_1 and π_2 (Proposition **2.6**) and the measurability of the composition function (Proposition **2.5**).
We prove the <only if> part:. Since the family $\{A \times B : \ A \in \mathcal{F}, \ \ B \in \mathcal{G}\}$ generates the product $\sigma-$field $\mathcal{F} \otimes \mathcal{G}$ it is enough to prove that $f^{-1}(A \times B) \in \mathcal{H}$ (Proposition **2.3**). Since f_1 and f_2 are measurable we have

$f_1^{-1}(A) = (\pi_1 \circ f)^{-1}(A) = f^{-1}(A \times Y) \in \mathcal{H}$

and $f_2^{-1}(A) = (\pi_2 \circ f)^{-1}(B) = f^{-1}(X \times B) \in \mathcal{H}$

$f^{-1}(A \times Y) \cap f^{-1}(X \times B) = f^{-1}((A \times Y) \cap (X \times B)) = f^{-1}(A \times B) \in \mathcal{H}$.$\blacksquare$

Remark. 2.8.

Let Let X be a topological space. Let us recall that the Borel $\sigma-$field of X is the $\sigma-$field generated by the family of all the open sets of X.
It is denoted by $\mathcal{B}_X$. Sets in $\mathcal{B}_X$ are called Borel sets of X. If X, Y are topological spaces whose product $X \times Y$ is endowed with the product topology then on the space $X \times Y$ one may put two $\sigma-$fields that are $\mathcal{B}_X \otimes \mathcal{B}_Y$ and $\mathcal{B}_{X \otimes Y}$. An interesting question is when do we have $\mathcal{B}_{X \otimes Y} = \mathcal{B}_X \otimes \mathcal{B}_Y$. It is known that if X and Y are separable metric spaces then $\mathcal{B}_{X \otimes Y} = \mathcal{B}_X \otimes \mathcal{B}_Y$. This result is of particular importance when $X = Y = \mathbb{R}$:

Theorem.2.9.

The space $\mathbb{R}$ is separable, since the countable set $\mathbb{Q}$ of rational numbers is dense. So the set $\mathbb{R}^2$ with the product topology is separable and we have $\mathcal{B}_{\mathbb{R}^2} = \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$.

As a consequence of this Theorem we have:

Proposition. 2.10.

Let $f, g : X \longrightarrow \mathbb{R}$ be measurable functions from $(X, \mathcal{F})$ into $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Then the following functions $f + g$, $f.g$, $\sup(f, g)$, $\inf(f, g)$ are measurable.

Proof. Since the functions f, g are measurable, the function $\varphi : X \longrightarrow \mathbb{R}^2$ defined by $\varphi(x) = (f(x), g(x))$ is measurable with respect to $\mathcal{F}$ and $\mathcal{B}_{\mathbb{R}^2}$(Proposition**.2.7**). On the other hand the functions $S, P, M, m : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by: $S(u, v) = u+v$, $P(u, v) = uv$, $M(u, v) = \sup(u, v)$, $m(u, v) = \inf(u, v)$ are continuous and so measurable with respect to $\mathcal{B}_{\mathbb{R}^2}$ and $\mathcal{B}_{\mathbb{R}}$. Now we have $S \circ \varphi = f + g$, $P \circ \varphi = fg$, $M \circ \varphi = \sup(f, g)$, $m \circ \varphi = \inf(f, g)$; the conclusion comes from Proposition**.2.5.**$\blacksquare$

Corollary. The family $\mathcal{M}(X,\mathbb{R})$ of measurable functions from $(X,\mathcal{F})$ into $(\mathbb{R},\mathcal{B}_{\mathbb{R}})$ is a vector space on the field $\mathbb{R}$ and even an algebra of functions.

Definition.2.11.

Let $\{f_i, i\in I\}$ be a family of functions defined on a set X such that each $f_i : X \longrightarrow E_i$ sends X into the measurable space $(E_i,\mathcal{F}_i)$. The $\sigma-$field generated by the family $\{f_i, i\in I\}$ is defined as the smallest $\sigma-$field $\mathcal{F}$ on X making each function f_i measurable from $(X,\mathcal{F})$ into the space $(E_i,\mathcal{F}_i)$. We denote this $\sigma-$field $\mathcal{F}$ by $\sigma\{f_i, i\in I\}$; in other words $\sigma\{f_i, i\in I\}$ is the smallest $\sigma-$field $\mathcal{F}$ on X containing all the families $f_i^{-1}(\mathcal{F}_i), i\in I$.

Examples.2.12.

(a) Let X be a set and take $\{f_i, i\in I\} = \{I_A, A\in\mathcal{P}(X)\}$ where I_A is the indicator function, then $\sigma\{I_A, A\in\mathcal{P}(X)\} = \mathcal{P}(X)$.

(b) Let X be a topological space. The Baire $\sigma-$field on X is defined as the $\sigma-$field $\mathcal{B}_0(X)$ generated by all continuous functions $f_i : X \longrightarrow \mathbb{R}$, that is the smallest $\sigma-$field on X making each continuous function $f_i : X \longrightarrow \mathbb{R}$ measurable with respect to $\mathcal{B}_0(X)$ and $\mathcal{B}_{\mathbb{R}}$.

(c) If in Example (b) the space X is a metric space whose topology is defined by the distance d then $\mathcal{B}_0(X)$ coincides with the Borel $\sigma-$field $\mathcal{B}_X$ on X.

Indeed we have $\mathcal{B}_0(X)\subset\mathcal{B}_X$ since $\mathcal{B}_X$ makes each continuous function measurable as easily may be seen. On the other hand let F be a closed set in X and consider the continuous function $f : X \longrightarrow \mathbb{R}$ given by $f(x) = d(x,F)$. Then we have $F = \{x\in X : f(x) = 0\} = f^{-1}(0)\in\mathcal{B}_0(X)$; so $\mathcal{B}_0(X)$ contains all the closed sets of X and then $\mathcal{B}_X\subset\mathcal{B}_0(X)$ since $\mathcal{B}_X$ is generated by the family of closed sets in X (see Definition **3.5** Chap.**1**).

(d) Let $(X\times Y,\mathcal{F}\otimes\mathcal{G})$ be the product of the measurable spaces $(X,\mathcal{F}),(Y,\mathcal{G})$. Then the projection $\pi_1(x,y) = x$ and the projection $\pi_2(x,y) = y$ are measurable on $(X\times Y,\mathcal{F}\otimes\mathcal{G})$ (Proposition.**2.6**). Then $\pi_1^{-1}(A) = A\times Y\in\mathcal{F}\otimes\mathcal{G}$ for every $A\in\mathcal{F}$ and $\pi_2^{-1}(B) = X\times B\in\mathcal{F}\otimes\mathcal{G}$ for every $B\in\mathcal{G}$.

We deduce that $\sigma\{\pi_1,\pi_2\}\subset\mathcal{F}\otimes\mathcal{G}$. On the other hand we have:

$\pi_1^{-1}(A)\cap\pi_2^{-1}(B) = (A\times Y)\cap(X\times B) = A\times B\in\mathcal{F}\otimes\mathcal{G}$. So every set of the form $A\times B$ with $A\in\mathcal{F}$ and $B\in\mathcal{G}$ is in $\sigma\{\pi_1,\pi_2\}$. But $\sigma\{A\times B : A\in\mathcal{F}, B\in\mathcal{G}\} = \mathcal{F}\otimes\mathcal{G}$, finally $\mathcal{F}\otimes\mathcal{G}\subset\sigma\{\pi_1,\pi_2\}$. Then $\mathcal{F}\otimes\mathcal{G} = \sigma\{\pi_1,\pi_2\}$.

3. Exercises

20. Let X be a non empty set. Determine the $\sigma-$field $\mathcal{F}$ generated by the constant functions $f : X \longrightarrow \mathbb{R}$. Let $\Im$ be the family of measurable functions from $(X, \mathcal{F})$ into $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, prove that $\Im$ is isomorphic to $\mathbb{R}$.
21. Let f be a measurable function from $(X, \mathcal{F})$ into $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, prove that $|f|$ is measurable. Let E be a set not Lebesgue measurable (see section 5 for the definition of Lebesgue measurable sets). Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = xI_{E^c} - xI_E$, prove that f is not Lebesgue measurable but $|f|$ is measurable.
22. Let $\{(X_i, \mathcal{F}_i), 1 \leq i \leq n\}$ be a finite family of measurable spaces and form the product set $X = \prod_1^n X_i = X_1 \times X_2 \times \cdots \times X_n$. We denote by $p_i : X \longrightarrow X_i$ the projection from X onto X_i given by $p_i(x_1, x_2, \cdots, x_n) = x_i$. Consider the $\sigma-$field $\sigma\{p_i, 1 \leq i \leq n\}$ generated by the functions $\{p_i, 1 \leq i \leq n\}$ and denoted by $\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \cdots \otimes \mathcal{F}_n = \overset{n}{\underset{1}{\otimes}} \mathcal{F}_i$. The space $\left(X, \overset{n}{\underset{1}{\otimes}} \mathcal{F}_i\right)$ is called the product of the spaces $(X_i, \mathcal{F}_i), 1 \leq i \leq n$.
(a) Prove that $\overset{n}{\underset{1}{\otimes}} \mathcal{F}_i$ is generated by the subsets of X of the form
$A = A_1 \times A_2 \times \cdots \times A_n,\ A_i \in \mathcal{F}_i\ 1 \leq i \leq n.$
(b) Let $(Y, \mathcal{G})$ be a measurable space and let $g : Y \longrightarrow \prod_1^n X_i$ be a function, prove that g is measurable with respect to $(Y, \mathcal{G})$ and $\left(X, \overset{n}{\underset{1}{\otimes}} \mathcal{F}_i\right)$ if and only if $p_i \circ g$ is measurable from $(Y, \mathcal{G})$ into $(X_i, \mathcal{F}_i)$ for each $1 \leq i \leq n$.
23. Let X be a non empty set and let $\{f_i, i \in I\}$ be a family of functions defined on X such that each $f_i : X \longrightarrow E_i$ sends X into the measurable space $(E_i, \mathcal{B}_i)$. Suppose that X is endowed with the $\sigma-$field $\sigma\{f_i, 1 \leq i \leq n\}$ generated by the functions $\{f_i, 1 \leq i \leq n\}$ (see Definition **2.11**). Let $(Y, \mathcal{G})$ be a measurable space and let $g : Y \longrightarrow X$, prove that g is measurable with respect to $(Y, \mathcal{G})$ and $(X, \sigma\{f_i, 1 \leq i \leq n\})$ if and only if $f_i \circ g$ is measurable from $(Y, \mathcal{G})$ into $(E_i, \mathcal{B}_i)$ for each $1 \leq i \leq n$.

4. Measurable Functions with values in $\mathbb{R}, \overline{\mathbb{R}}, \mathbb{C}$

Definition.4.1
(a) The set $\mathbb{R}$ is the real numbers system endowed with the Borel $\sigma-$field $\mathcal{B}_{\mathbb{R}}$.
(b) The set $\overline{\mathbb{R}}$ is defined as $\{\mathbb{R}, -\infty, +\infty\}$. The $\sigma-$field we need on $\overline{\mathbb{R}}$ is given by $\sigma\{\mathcal{B}_{\mathbb{R}}, -\infty, \infty\}$ and denoted by $\mathcal{B}_{\overline{\mathbb{R}}}$.
(c) It is well known that the set $\mathbb{C}$ of complex numbers can be identified with the product space $\mathbb{R} \times \mathbb{R}$; so we can identify the Borel $\sigma-$field $\mathcal{B}_{\mathbb{C}}$with $\mathcal{B}_{\mathbb{R}\times\mathbb{R}}$, which is $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$ by Theorem.**2.9.**

Notations. 4.2.

Let $(X, \mathcal{F})$ be a measurable space. In the sequel.we will use the following notations:

$\mathcal{M}(X,\mathbb{R})$ is the family of measurable functions f from $(X,\mathcal{F})$ into $(\mathbb{R},\mathcal{B}_{\mathbb{R}})$.
$\mathcal{M}(X,\mathbb{C})$ is the family of measurable functions f from $(X,\mathcal{F})$ into $(\mathbb{C},\mathcal{B}_{\mathbb{C}})$

We already have seen that $\mathcal{M}(X,\mathbb{R})$ is a vector space on the field $\mathbb{R}$ (see the Corollary of Proposition**2.10).**
It is not difficult to prove the same for $\mathcal{M}(X,\mathbb{C})$

Arithmetic in $\overline{\mathbb{R}}$. 4.3.

We will agree with the following conventions in $\overline{\mathbb{R}} = \{\mathbb{R}, -\infty, +\infty\}$:
$0 \cdot (\pm\infty) = (\pm\infty) \cdot 0 = 0$
$(+\infty) + (+\infty) = +\infty$
$(-\infty) + (-\infty) = -\infty$
$a \pm (\pm\infty) = \pm\infty, \forall a \in \mathbb{R}$
$(-1) \cdot (\pm\infty) = (\mp\infty)$

Definition. 4.4.

Let $(X,\mathcal{F})$ be a measurable space.
A function $f : X \longrightarrow \overline{\mathbb{R}}$ is measurable from $(X,\mathcal{F})$ into $\left(\overline{\mathbb{R}},\mathcal{B}_{\overline{\mathbb{R}}}\right)$ if:
$f^{-1}(B) \in \mathcal{F}, \forall B \in \mathcal{B}_{\mathbb{R}}$, and $f^{-1}(+\infty) \in \mathcal{F}, f^{-1}(-\infty) \in \mathcal{F}$
this comes from the fact that $\mathcal{B}_{\overline{\mathbb{R}}} = \sigma\{\mathcal{B}_{\mathbb{R}}, -\infty, \infty\}$ and Proposition **2.3.**
We denote by $\mathcal{M}\left(X,\overline{\mathbb{R}}\right)$ the the family of measurable functions f from $(X,\mathcal{F})$ into $\left(\overline{\mathbb{R}},\mathcal{B}_{\overline{\mathbb{R}}}\right)$.

Proposition. 4.5.

The $\sigma-$field $\mathcal{B}_{\overline{\mathbb{R}}}$ is generated by all the intervals of the form $[-\infty, t\ [$.

Proof. Use the fact that $\mathcal{B}_{\mathbb{R}}$ is generated by all the open intervals by Proposition **3.6.**Chap.**1**■

Corollary.

A function $f : X \longrightarrow \overline{\mathbb{R}}$ is measurable from $(X,\mathcal{F})$ into $\left(\overline{\mathbb{R}},\mathcal{B}_{\overline{\mathbb{R}}}\right)$ if:
$f^{-1}([-\infty, t\ [) \in \mathcal{F}, \forall t \in \mathbb{R}$.

Definition. 4.6.

Let $(X,\mathcal{F}),(Y,\mathcal{G})$ be measurable spaces and $E\subset X$ a subset of X.
If $f:X\longrightarrow Y$ is a function. We say that f is measurable on E if the restriction of f to E considered as a function from $(E,E\cap\mathcal{F})$ into $(Y,\mathcal{G})$ is measurable.

Example. 4.7.
If f,g are in $\mathcal{M}\left(X,\overline{\mathbb{R}}\right)$, then the function $f+g$ is measurable on the set E with:
$E^c=(\{f=\infty\}\cap\{g=-\infty\})\cup(\{f=-\infty\}\cap\{g=\infty\})$
Let φ be the restriction of $f+g$ to E then we have
φ is well defined on E and $\{\varphi<t\}=\{f+g<t\}\cap E\in E\cap\mathcal{F}$.

5. Sequences of Measurable Functions

Definition. 5.1. (simple function)

Let $f:X\longrightarrow\mathbb{R}$ be a function from X into $\mathbb{R}$. The function f is simple if it takes a finite number of values, that is, f is simple if the set $f(X)$ is a finite subset of $\mathbb{R}$. So if $f(X)=\{a_1,a_2,...,a_n\}$ and $A_i=\{x:f(x)=a_i\},i=1,2,...,n$, then $\{A_1,A_2,...,A_n\}$ is a partition of X and the function f can be written as $f(\cdot)=\sum\limits_1^n a_i.I_{A_i}(\cdot)$, where I_{A_i} is the indicateur function of the set $A_i, i=1,2,...,n$.

Proposition. 5.2

A simple function $f(\cdot)=\sum\limits_{i=1}^n a_i.I_{A_i}(\cdot)$ is measurable from $(X,\mathcal{F})$ into $(\mathbb{R},\mathcal{B}_{\mathbb{R}})$ iff $A_i\in\mathcal{F},i=1,2,...,n$.

Proof. We have $f^{-1}\{a_i\}=A_i\in\mathcal{F},i=1,2,...,n$; so if $B\in\mathcal{B}_{\mathbb{R}}$ and
$n_B=\{i:a_i\in B\}$, we deduce that $f^{-1}(B)=\underset{i\in n_B}{\cup}A_i\in\mathcal{F}$.■

Notation. 5.3. We denote by $\mathcal{E}$ the family of measurable simple functions from $(X,\mathcal{F})$ into $(\mathbb{R},\mathcal{B}_{\mathbb{R}})$

Proposition. 5.4.

Let s,t be in $\mathcal{E}$ and $\lambda\in\mathbb{R}$, then:
the functions $s+t$, $s\cdot t$, $\lambda\cdot s$, $\sup(s,t)$, $\inf(s,t)$ are in $\mathcal{E}$.

Proof. Write $s(\cdot)=\sum\limits_1^n a_i.I_{A_i}(\cdot)$, $t(\cdot)=\sum\limits_1^m b_j.I_{Bj}(\cdot)$, then we have:

$$s+t=\sum_{i=1}^n\sum_{j=1}^m(a_i+b_j).I_{A_i\cap B_j}$$

$$s\cdot t=\sum_{i=1}^n\sum_{j=1}^m(a_ib_j).I_{A_i\cap B_j},\ \lambda\cdot s=\sum_1^n(\lambda a_i).I_{A_i}$$

(so the family $\mathcal{E}$ is an algebra on $\mathbb{R}$.)

$$\sup(s,t)=\sum_{i=1}^n\sum_{j=1}^m\sup(a_i,b_j).I_{A_i\cap B_j},\ \inf(s,t)=\sum_{i=1}^n\sum_{j=1}^m\inf(a_i,b_j).I_{A_i\cap B_j}$$

Since $\{A_i\cap B_j,\ \ 1\le i\le n,\ 1\le j\le m\}$ is a partition of X we get the result.■

Proposition. 5.5.

Let (f_n) be a sequence of functions in $\mathcal{M}(X, \mathbb{R})$ or either in $\mathcal{M}\left(X, \overline{\mathbb{R}}\right)$ then: the functions $\sup\limits_n f_n$ and $\inf\limits_n f_n$ are in $\mathcal{M}\left(X, \overline{\mathbb{R}}\right)$.

Proof. For any $t \in \mathbb{R}$ we have $\left\{\sup\limits_n f_n \leq t\right\} = \bigcap\limits_n \{f_n \leq t\}$ whence the mesurability of $\sup\limits_n f_n$. Since $\inf\limits_n f_n = -\sup\limits_n - f_n$ we deduce the mesurability of $\inf\limits_n f_n$.■

Corollary. 1.

Let (f_n) be a sequence of functions in $\mathcal{M}(X, \mathbb{R})$ or either in $\mathcal{M}\left(X, \overline{\mathbb{R}}\right)$ then: the functions $\lim\limits_n \sup f_n$ and $\lim\limits_n \inf f_n$ are measurable

Proof. Comes directly from the proposition above since $\lim\limits_n \sup f_n = \inf\limits_{n \geq 1} \sup\limits_{k \geq n} f_k$ and $\lim\limits_n \inf f_n = \sup\limits_{n \geq 1} \inf\limits_{k \geq n} f_k$.■

Corollary. 2.

Let (f_n) be a sequence of functions in $\mathcal{M}(X, \mathbb{R})$ or either in $\mathcal{M}\left(X, \overline{\mathbb{R}}\right)$ then: The set $C = \left\{x : \lim\limits_n \sup f_n(x) = \lim\limits_n \inf f_n(x)\right\}$ belongs to $\mathcal{F}$.

Proof. Observe that C is the convergence set of the sequence (f_n). Put :

$$C_1 = \left(\left\{x : \lim\limits_n \sup f_n(x) = \infty\right\} \cap \left\{x : \lim\limits_n \inf f_n(x) = \infty\right\}\right)$$

$$C_2 = \left(\left\{x : \lim\limits_n \sup f_n(x) = -\infty\right\} \cap \left\{x : \lim\limits_n \inf f_n(x) = -\infty\right\}\right)$$

$$C_3 = \left\{x : \lim\limits_n \sup f_n(x) \in \mathbb{R}\right\} \cap \left\{x : \lim\limits_n \sup f_n(x) = \lim\limits_n \inf f_n(x)\right\}$$

Then C_1 and C_2 and C_3 are in $\mathcal{F}$ and $C = C_1 \cup C_2 \cup C_3$.■.

Corollary. 3.

Let (f_n) be a sequence of functions in $\mathcal{M}(X, \mathbb{R})$ or either in $\mathcal{M}\left(X, \overline{\mathbb{R}}\right)$
Suppose that: $\lim\limits_n f_n(x) = f(x) \in \overline{\mathbb{R}}$ exists for each $x \in X$. Then $f \in \mathcal{M}\left(X, \overline{\mathbb{R}}\right)$.

Proof. The convergence set $C = \left\{x : \lim\limits_n \sup f_n(x) = \lim\limits_n \inf f_n(x)\right\}$ given in Corollary 2 is equal to X here.
So the function $f(x)$ is equal to $\lim\limits_n \sup f_n(x) = \lim\limits_n \inf f_n(x), \forall x \in X$. Then f is measurable by Corollary 1.■

The following theorem is fundamental and will be used in the construction of the integral of a measurable function.

Theorem. 5.6.

Let $f \in \mathcal{M}\left(X, \overline{\mathbb{R}}\right)$ be such that $f(x) \in [0, \infty]$, $\forall x \in X$.Then:
there exists a sequence $(s_n)$of positive measurable simple functions
from $(X, \mathcal{F})$ into $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ with:

(i) $0 \leq s_n \leq s_{n+1}$

(ii) $\lim\limits_n s_n(x) = f(x)$, $\forall x \in X$.

Proof. For each $n \geq 1$ and each $x \in X$, define s_n by:

$s_n(x) = \dfrac{i-1}{2^n}$ if $\dfrac{i-1}{2^n} \leq f(x) < \dfrac{i}{2^n}$, $i = 1, 2, ..., n2^n$

$s_n(x) = n$ if $f(x) \geq n$

we can use a consolidated form for s_n:

$$s_n(x) = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} I_{\left\{\frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n}\right\}} + n\, I_{\{f(x) \geq n\}}$$

recall that I_A is the function defined by $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ if $x \notin A$.

Then (s_n) is an increasing sequence of positive simple functions (check it!).

Let us prove that $\lim_n s_n(x) = f(x)$, $\forall x \in X$:

if $f(x) < \infty$ then for every $n > f(x)$ we have $0 < f(x) - s_n(x) < \dfrac{1}{2^n}$, so $\lim_n s_n(x) = f(x)$

if $f(x) = \infty$ then $f(x) \geq n$ for every n and so we have $s_n(x) = n$ for all n whence $\lim_n s_n(x) = \infty$.■

Definition. 5.7.

Let $f \in \mathcal{M}(X, \overline{\mathbb{R}})$. Define the positive measurable functions f^+, f^- by:

$f^+ = \sup(f, 0)$, $f^- = -\inf(f, 0)$

Remark. 5.8.

It is easy to check that:

$f = f^+ - f^-$

$|f| = f^+ + f^-$

Proposition. 5.9.

Let $f \in \mathcal{M}(X, \overline{\mathbb{R}})$. Then there exists a sequence (s_n) of measurable simple functions from $(X, \mathcal{F})$ into $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ with $\lim_n s_n(x) = f(x)$, $\forall x \in X$.

Proof. We have $f = f^+ - f^-$ where f^+, f^- are simple positives.

By Theorem **5.6**.there exist simple positive functions s'_n, s''_n such that: $\lim_n s'_n(x) = f^+(x)$, $\forall x \in X$ and $\lim_n s''_n(x) = f^-(x)$, $\forall x \in X$. Then $s_n = s'_n - s''_n$ is measurable simple and $\lim_n s_n(x) = f^+(x) - f^-(x) = f(x)$, $\forall x \in X$.■

Corollary.

Let $f \in \mathcal{M}(X, \mathbb{R})$ and suppose f bounded. Then there is a sequence (s_n) of measurable simple functions converging uniformly to f on X.

Proof. By the Proposition above it is enough to consider the case f positive. Since f is bounded there is n such that $n > f(x)$ for every $x \in X$. So there exists a sequence (s_n) of positive measurable simple functions

with $0 \leq f(x) - s_m(x) < \dfrac{1}{2^m}, \forall x \in X, \forall m > n$, from which we deduce the uniform convergence of s_n to f on X.■

6. Convergence of Measurable Functions

Let us recall that if $(X, \mathcal{F}, \mu)$ is a measure space, a subset N of X is a null set if there is $A \in \mathcal{F}$, with $\mu(A) = 0$ such that $N \subset A$.
In this section we describe different type of convergence of measurable functions and the relations between them.

Definition. 6.1.

Let $\mathcal{P}$ be a property depending on a variable $x \in X$, that is $\mathcal{P}$ may be true or false according to x. We say that $\mathcal{P}$ is true almost every where if there is a null subset N of X such that $\mathcal{P}$ is true for any x outside N.

Examples. 6.2.

(a) A function $f : X \longrightarrow \overline{\mathbb{R}}$ is said to be finite almost every where if there is a null subset N of X such that $f(x) \in \mathbb{R}\ \forall x \in X \backslash N$. If moreover $f \in \mathcal{M}\left(X, \overline{\mathbb{R}}\right)$ then $\{f = \pm\infty\} \in \mathcal{F}$ and the condition of finiteness almost every where may be written simply as $\mu\{f = \pm\infty\} = 0$.
(b).A function $f : X \longrightarrow \mathbb{R}$ is said to be bounded almost every where if there is a constant $M > 0$ and a null subset N such that $|f(x)| \leq M, \forall x \in X \backslash N$. If moreover $f \in \mathcal{M}(X, \mathbb{R})$ then $\{|f| > M\} \in \mathcal{F}$ and the condition of boundedness almost every where may be written simply as $\mu\{|f| > M\} = 0$.

(c). Let $f, g : X \longrightarrow \overline{\mathbb{R}}$ be functions. We say that $f = g$ almost every where if there is a null subset N such that $f(x) = g(x), \forall x \in X \backslash N$. If moreover $f \in \mathcal{M}\left(X, \overline{\mathbb{R}}\right)$, the condition may be written as $\mu\{f \neq g\} = 0$.

Abbreviation. almost every where with respect to μ is abbreviated to: $\mu - a.e$

Definition. 6.3.

Let $f_n : X \longrightarrow \mathbb{R}$ be a sequence of functions. We say that f_n converges $\mu - a.e$ if the set $N = \left\{\lim_n \sup f_n \neq \lim_n \inf f_n\right\}$ is a null set. In other words f_n converges $\mu - a.e$ if for each $x \in X \backslash N$ the real sequence $f_n(x)$ converge to the real number $f(x)$, that is: $\forall \epsilon > 0, \exists m(\epsilon, x) \geq 1$ such that $\forall n \geq m(\epsilon, x), |f_n(x) - f(x)| < \epsilon$.

Definition. 6.4.

Let $f_n : X \longrightarrow \mathbb{R}$ be a sequence of functions. We say that f_n is a Cauchy sequence $\mu - a.e$ if there is a null subset N such that for each $x \in X \backslash N$ the real sequence $f_n(x)$ is a Cauchy sequence in $\mathbb{R}$, that is satisfies the following condition:

$$\forall \epsilon > 0, \exists M(\epsilon, x) \geq 1 \text{ such that } \forall n, m \geq M(\epsilon, x), |f_n(x) - f_m(x)| < \epsilon$$

Proposition. 6.5.

Let $f_n : X \longrightarrow \mathbb{R}$ be a sequence of functions. The following conditions are equivalent:
(a) The sequence f_n converges to $\mu - a.e$ to a function $f : X \longrightarrow \mathbb{R}$
(b) f_n is a Cauchy sequence $\mu - a.e$

Proof. For each x outside of a null set $f_n(x)$ is a Cauchy sequence in $\mathbb{R}$, so the Proposition results from the validity of the same properties in $\mathbb{R}$.■

Now let us come to the convergence of measurable functions.

Proposition. 6.6.

Let f_n be a sequence of functions in $\mathcal{M}\left(X,\overline{\mathbb{R}}\right)$ converging $\mu - a.e$ on X. Then there is $f \in \mathcal{M}\left(X,\overline{\mathbb{R}}\right)$ such that f_n converges $\mu - a.e$ to f.
Conversely if there is $f : X \longrightarrow \overline{\mathbb{R}}$ such that f_n converges $\mu - a.e$ to f, then f is measurable on a set E with $\mu\left(E^c\right) = 0$.

Proof. Take $E = \left\{x : \limsup_n f_n(x) = \liminf_n f_n(x)\right\}$ and take f defined by:

$f(x) = \liminf_n f_n(x)$ for $x \in E$ and $f(x) = 0$ for $x \in E^c$

(see Definition **4.6** for the measurability of f on E).■

Definition. 6.7. (uniform convergence $\mu - a.e$)

Let $f_n : X \longrightarrow \mathbb{R}$ be a sequence of functions. We say that f_n converges uniformly $\mu - a.e$ to the function $f : X \longrightarrow \mathbb{R}$ if there is a null set N such that f_n converges uniformly to f on $X \backslash N$, that is:

$\forall \epsilon > 0, \exists M(\epsilon) \geq 1$ such that $\forall n \geq M(\epsilon), |f_n(x) - f(x)| < \epsilon, \forall x \in X \backslash N$

We say that f_n is a Cauchy sequence for the uniform convergence $\mu - a.e$ if there is a null set N such that:

$\forall \epsilon > 0, \exists M(\epsilon) \geq 1$ such that $\forall n, m \geq M(\epsilon), |f_n(x) - f_m(x)| < \epsilon, \forall x \in X \backslash N$

let us observe that the integer $M(\epsilon)$ does not depend on x.

Remark. 6.8.

In most of our discussion, especially in integration theory, we frequently use a complete measure space $(X, \mathcal{F}, \mu)$ as our basic space.
So in this case every null set is in $\mathcal{F}$ and this avoids some cumbersome measurability character of functions.

The following Theorem localizes the points of the space X where the convergence of a sequence fails to be uniform. Let us start with an example:

Example. 6.9.

Consider the space $X = [0, 1]$ endowed with the Lebesgue measure μ and let $f_n : X \longrightarrow \mathbb{R}$ be the sequence of functions given by $f_n(x) = x^n, x \in [0, 1]$. The sequence converges pointwise to the function f given by $f(x) = 0$ for $0 \leq x < 1$, and $f(x) = 1$ for $x = 1$, but the convergence is not uniform (why?). However for $\epsilon > 0$, we see that the sequence f_n converges uniformly on the interval $\left[0, 1 - \frac{\epsilon}{2}\right]$; intuitevely the points where the uniform convergence fails are localized in the set $B = \left[1 - \frac{\epsilon}{2}, 1\right]$ and $\mu(B) < \epsilon$.

Theorem. 6.10. (Egorov)

Let $(X, \mathcal{F}, \mu)$ be a measure space, with $\mu(X) < \infty$. Let $f_n, f \in \mathcal{M}\left(X,\overline{\mathbb{R}}\right)$ be functions finite $\mu - a.e.$
Suppose that the sequence f_n converges $\mu - a.e$ to f on X. Then we have:

For every $\epsilon > 0$ there is $B \in \mathcal{F}$ such that $\mu(B) < \epsilon$
and f_n converges uniformly to f on $X \backslash B$.

Proof. Without losing general hypothesis, we can assume that:
f_n, f take values in $\mathbb{R}$ and f_n converges everywhere to f on X.

Let $E_n^m = \underset{j \geq n}{\cap} \left\{ |f_j - f| < \frac{1}{m} \right\}$, since f_n, f are measurable we get $E_n^m \in \mathcal{F}, \forall n, m$. Moreover it is clear that $E_n^m \subset E_{n+1}^m \subset .... \subset \underset{n \geq 1}{\cup} E_n^m$. Since f_n converges everywhere to f on X, we have $\underset{n \geq 1}{\cup} E_n^m = X, \forall m \geq 1$.
So $X \backslash E_n^m \supset X \backslash E_{n+1}^m \supset .... \supset \underset{n \geq 1}{\cap} (X \backslash E_n^m) = \emptyset$ for each $m \geq 1$. Since $\mu(X) < \infty$ we deduce that $\lim_n \mu(X \backslash E_n^m) = 0$; so for each $m \geq 1$ there is $n(m) \geq 1$ such that $\mu\left(X \backslash E_{n(m)}^m\right) < \frac{\epsilon}{2^m}$. Now put $B = \underset{m \geq 1}{\cup} X \backslash E_{n(m)}^m$; then we have:
$\mu(B) \leq \sum_{m \geq 1} \mu\left(X \backslash E_{n(m)}^m\right) < \sum_{m \geq 1} \frac{\epsilon}{2^m} = \epsilon$. So $\mu(B) < \epsilon$ and $X \backslash B = \underset{m \geq 1}{\cap} E_{n(m)}^m$, therefore $|f_n(x) - f(x)| < \frac{1}{m}, \forall x \in X \backslash B, \forall n > n(m)$ and then the uniform convergence of f_n to f on $X \backslash B$.$\blacksquare$

Remark. 6.11.

Egorov'Theorem is not valid in the case μ infinite as is shown by the following:
Take for $(X, \mathcal{F}, \mu)$ the space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ with μ the counting measure;
if $f_n = I_{\{1,2,...,n\}}$ then $f_n(k)$ converges to 1 for each $k \in \mathbb{N}$; nevertheless there is no $F \subset \mathbb{N}$ such that $\mu(F) < \epsilon$ and f_n converges uniformly to 1 on $X \backslash F$ (indeed take $0 < \epsilon < 1$).

Remark. 6.12.

It is not difficult to prove the equivalence of the following assertions:
(a) f_n converges almost uniformly
(b) f_n is a Cauchy sequence for the almost uniform convergence.

Definition. 6.13.

Let $(X, \mathcal{F}, \mu)$ be a measure space, and let $f_n, f \in \mathcal{M}\left(X, \overline{\mathbb{R}}\right)$ be functions finite $\mu - a.e.$
(a) the sequence f_n converges almost uniformly if:
$\forall \epsilon > 0\ \exists B \in \mathcal{F}$ such that $\mu(B) < \epsilon$ and f_n converges uniformly to f on $X \backslash B$.
(b) the sequence f_n is a Cauchy sequence for the almost uniform convergence if:
$\forall \epsilon > 0\ \exists B \in \mathcal{F}$ such that $\mu(B) < \epsilon$ and f_n is a Cauchy sequence for the uniform convergence on $X \backslash B$.

Here is a specific type of convergence of measurable functions:

Definition. 6.14.

Let $f_n, f \in \mathcal{M}\left(X, \overline{\mathbb{R}}\right)$ be functions finite $\mu - a.e.$.
We say that the sequence (f_n) converges in measure to f if:
$\forall \epsilon > 0$, $\lim_n \mu\{x : |f_n(x) - f(x)| > \epsilon\} = 0$

Notation: $f_n \xrightarrow{\mu} f$

Proposition. 6.15.

The almost uniform convergence implies:
(a) The convergence $\mu - a.e$
(b) The convergence in measure

Proof. By almost uniform convergence we have:

$\forall k \geq 1, \exists F_k \in \mathcal{F}$, with $\mu(F_k) < \frac{1}{k}$, and f_n converges uniformly on $X \backslash F_k$.
Take $F = \bigcap_k F_k$ then $F \in \mathcal{F}$, $\mu(F) = 0$. If $x \in X \backslash F$, there is k such that $x \in X \backslash F_k$, so $\lim_n f_n(x) = f(x)$ and proves (a).
By almost uniform convergence we have:
$\forall \delta > 0, \exists F_\delta \in \mathcal{F}$, with $\mu(F_\delta) < \delta$, and f_n converges uniformly on $X \backslash F_\delta$.
Put $E_n(\epsilon) = \{x : |f_n(x) - f(x)| > \epsilon\}$, then $E_n(\epsilon) = E_n(\epsilon) \cap F_\delta + E_n(\epsilon) \cap X \backslash F_\delta$; we deduce that $\mu(E_n(\epsilon)) < \delta + \mu(E_n(\epsilon) \cap X \backslash F_\delta)$. Now since f_n converges uniformly on $X \backslash F_\delta$ there is $N(\epsilon, \delta) \geq 1$ such that for $n \geq N(\epsilon, \delta)$, $\mu(E_n(\epsilon) \cap X \backslash F_\delta) = 0$. This proves that $\forall \epsilon > 0, \lim_n \mu(E_n(\epsilon)) = 0$ whence $f_n \xrightarrow{\mu} f$.■

Proposition. 6.16.

Let $(X, \mathcal{F}, \mu)$ be a measure space, with $\mu(X) < \infty$. Then:
The convergence $\mu - a.e$ implies the convergence in measure.

Proof. By Egorov Theorem (**6.10**) convergence $\mu - a.e$ implies almost uniform convergence from which the convergence in measure comes by Proposition**. 6.15.**■

Proposition. 6.17.

If $f_n \xrightarrow{\mu} f$ then f_n is a Cauchy sequence for the convergence in measure that is:

$$\forall \epsilon > 0, \lim_{n,m} \mu\{x : |f_n(x) - f_m(x)| > \epsilon\} = 0$$

Moreover if also $f_n \xrightarrow{\mu} g$ then $f = g \ \mu - a.e.$

Proof. Since $|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)|$, we deduce that:

$$\{x : |f_n(x) - f_m(x)| > \epsilon\} \subset \left\{x : |f_n(x) - f(x)| > \tfrac{\epsilon}{2}\right\} \cup \left\{x : |f_m(x) - f(x)| > \tfrac{\epsilon}{2}\right\}$$

and we have:

$$\mu\{x : |f_n(x) - f_m(x)| > \epsilon\} \leq \mu\left\{x : |f_n(x) - f(x)| > \tfrac{\epsilon}{2}\right\} + \mu\left\{x : |f_m(x) - f(x)| > \tfrac{\epsilon}{2}\right\}$$

so

$$\lim_{n,m} \mu\{x : |f_n(x) - f_m(x)| > \epsilon\} \leq \lim_n \mu\left\{x : |f_n(x) - f(x)| > \tfrac{\epsilon}{2}\right\} + \lim_m \mu\left\{x : |f_m(x) - f(x)| > \tfrac{\epsilon}{2}\right\} = 0$$

now suppose $f_n \xrightarrow{\mu} g$; it is clear that

$$\{x : |f(x) - g(x)| > 0\} = \bigcup_n \left\{x : |f(x) - g(x)| > \tfrac{1}{n}\right\}$$

and

$$\left\{x : |f(x) - g(x)| > \tfrac{1}{n}\right\} \subset \left\{x : |f(x) - f_k(x)| > \tfrac{1}{2n}\right\} \cup \left\{x : |f_k(x) - g(x)| > \tfrac{1}{2n}\right\}, \forall k, n;$$

then

$$\mu\left\{x : |f(x) - g(x)| > \tfrac{1}{n}\right\} \leq \mu\left\{x : |f(x) - f_k(x)| > \tfrac{1}{2n}\right\} + \mu\left\{x : |f_k(x) - g(x)| > \tfrac{1}{2n}\right\}$$

the right side goes to 0 as $k \longrightarrow \infty$, for each n since $f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{\mu} g$, so $\mu\left\{x : |f(x) - g(x)| > \frac{1}{n}\right\} = 0$ for all n and then $\mu\{x : |f(x) - g(x)| > 0\} = 0$ whence $f = g \ \mu - a.e.$■

Lemma. 6.18.

Every Cauchy sequence in measure f_n contains a subsequence f_{n_k} satisfying Cauchy condition for the almost uniform convergence (Definition **6.13**(b)).

Proof. Left to the reader.■

Theorem. 6.19.

Every Cauchy sequence in measure f_n converges in measure to a measurable function f

Proof. By Lemma **6.18**, f_n contains a subsequence f_{n_k} satisfying the Cauchy condition for the almost uniform convergence. So from Remark**.6.12** the subsequence f_{n_k} converges almost uniformly to some measurable function f and then f_{n_k} converges in measure to f by Proposition**. 6.15** (b)**.** But f_n itself converges in measure to f, indeed we have:
$\{x : |f_n(x) - f(x)| > \epsilon\} \subset \left\{x : |f_n(x) - f_{n_k}(x)| > \frac{\epsilon}{2}\right\} \cup \left\{x : |f(x) - f_{n_k}(x)| > \frac{\epsilon}{2}\right\}$
and $\mu\{x : |f_n(x) - f(x)| > \epsilon\} \leq$
$\mu\left\{x : |f_n(x) - f_{n_k}(x)| > \frac{\epsilon}{2}\right\} + \mu\left\{x : |f(x) - f_{n_k}(x)| > \frac{\epsilon}{2}\right\}$
so if $n, k \longrightarrow \infty, \mu\left\{x : |f_n(x) - f_{n_k}(x)| > \frac{\epsilon}{2}\right\} \longrightarrow 0$, since f_n is Cauchy sequence in measure and $\mu\left\{x : |f(x) - f_{n_k}(x)| > \frac{\epsilon}{2}\right\} \longrightarrow 0$ because f_{n_k} converges in measure to f .■

7. Exercises

24. (a) Prove that in any measure space the uniform convergence implies the convergence in measure.
(b) In the counting measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ the uniform convergence is equivalent to the convergence in measure.
25. In the space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ consider the sequence of indicator functions $f_n = I_{\{1,2,\ldots,n\}}$; prove that f_n converges $\mu - a.e$ but does not converge in measure.What do we deduce about Proposition**. 4.3.16.**
26. Let $f_n, f \in \mathcal{M}\left(X, \overline{\mathbb{R}}\right)$ be functions finite $\mu - a.e.$. Suppose f_n converges pointwise to f and there is a positive measurable function g satisfying $\lim_n \mu\{g > \epsilon_n\} = 0$ for some sequence of positive numbers ϵ_n with $\lim_n \epsilon_n = 0$. Then if $|f_n| \leq g, \forall n$, prove that f_n converges in measure to f.
27. Let $f : X \longrightarrow \mathbb{R}$ be measurable in the space $(X, \mathcal{F}, \mu)$ and put:
$M(f) = \inf\{\alpha \geq 0 : \ \mu\{|f| > \alpha\} = 0\}$,Prove that $|f| \leq M(f) \ \ \mu - a.e.$
Prove that $\lim_n M(f_n - f) = 0$ iff $\lim_n f_n = f$ uniformly $\mu - a.e.$
28 Let $f_n, f : X \longrightarrow \mathbb{R}$ be measurable functions in the space $(X, \mathcal{F}, \mu)$ and suppose that f_n converges in measure to f ; if $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a uniformly continuous function prove that the sequence $g \circ f_n$ converges in measure to $g \circ f$
.