Chapter 4

INTEGRATION

1. Preliminaries

Introduction.

Let $(X, \mathcal{F}, \mu)$ be a measure space. This chapter concerns the Lebesgue integration process $\int\limits_X f.d\mu$ of numerical measurable functions on X with respect to the measure μ. Such classes of functions have been introduced with their convergence properties in sections **1-3** of chapter **3**.

If X is the closed interval $[a, b]$ in the real system $\mathbb{R}$, it is also possible to define the Riemann integral $\int\limits_a^b f.dx$ of some function $f : [a, b] \longrightarrow \mathbb{R}$ (e.g continuous function).

If the Lebesgue integration process is applied to a sequence of Riemann integrable functions, it leads to a kind of convergence properties less restrictive and easier in applications than those needed in the Riemann process framework. Let us recall:

Classes of functions.1.1. (see sections 1-3 of chapter 3.)

$\mathcal{E} = \{s : X \longrightarrow \mathbb{R},\ s \text{ simple measurable}\}$

$\mathcal{E}_+ = \{s \in \mathcal{E} :\ s \text{ positive}\}$

$\mathcal{M}_+ = \{f : X \longrightarrow [0, \infty],\ \ f \ \text{ measurable}\}$

$\mathcal{M}(\mathbb{R}) = \{f : X \longrightarrow \mathbb{R},\ \ f \ \text{ measurable}\}$

$\mathcal{M}(\mathbb{C}) = \{f : X \longrightarrow \mathbb{C},\ \ f \ \text{ measurable}\}$

Let us recall that if $f \in \mathcal{M}_+$, there is an increasing sequence s_n in $\mathcal{E}_+$ with: $\lim\limits_n s_n(x) = f(x), \forall x \in X$.

2. Integration in $\mathcal{E}_+$

Definition.2.1.

Let $s \in \mathcal{E}_+$ with $s(\cdot) = \sum\limits_1^n a_i.I_{A_i}(\cdot)$, where I_A is the Dirac function of the set A, and the sets $A_i, 1 \leq i \leq n$ form a partition of X in $\mathcal{F}$.

The integral of s with respect to μ is defined by:

$$\int\limits_X s.d\mu = \sum\limits_1^n a_i.\mu(A_i)$$

with the convention $0 \cdot \infty = 0$.

Remark.2.2.

Suppose $s \in \mathcal{E}_+$ with $s(\cdot) = \sum\limits_{i=1}^n a_i.I_{A_i}(\cdot) = \sum\limits_{j=1}^m b_j.I_{B_j}(\cdot)$, where $\{A_i, 1 \leq i \leq n\}$ and $\{B_j, 1 \leq j \leq m\}$ are partitions of X. Then we have:

$A_i = \{x \in X : \ s(x) = a_i\}$ and $B_j = \{x \in X : \ s(x) = b_j\}$
so $a_i.I_{A_i \cap B_j}(\cdot) = b_j.I_{A_i \cap B_j}(\cdot)$ for $1 \le i \le n, 1 \le j \le m$.
$a_i.I_{A_i}(\cdot) = \sum_{j=1}^{m} a_i.I_{A_i \cap B_j}(\cdot)$ and $\sum_{i=1}^{n} a_i.I_{A_i}(\cdot) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i.I_{A_i \cap B_j}(\cdot)$
likewise $\sum_{j=1}^{m} b_j.I_{B_j}(\cdot) = \sum_{i=1}^{n}\sum_{j=1}^{m} b_j.I_{A_i \cap B_j}(\cdot)$ and the terms in the two double sums are equivalent so $\sum_{i=1}^{n} a_i.\mu(A_i) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i.\mu(A_i \cap B_j)$
and $\sum_{j=1}^{m} b_j.\mu(B_j) = \sum_{j=1}^{m}\sum_{i=1}^{n} b_j.\mu(A_i \cap B_j)$ then $\sum_{i=1}^{n} a_i.\mu(A_i) = \sum_{j=1}^{m} b_j.\mu(B_j)$
we deduce that the integral $\int_X s.d\mu = \sum_{1}^{n} a_i.\mu(A_i)$ is well defined.

Proposition.2.3.

Let s, t be in $\mathcal{E}_+$ and $c \ge 0$ then we have:

(1) $\int_X (s+t).d\mu = \int_X s.d\mu + \int_X t.d\mu$

$\int_X c.s.d\mu = c.\int_X s.d\mu$

(2) If $s \le t$ then $\int_X s.d\mu \le \int_X t.d\mu$

(3) If $E \in \mathcal{F}$ and $s(\cdot) = \sum_{i=1}^{n} a_i.I_{A_i}(\cdot)$ we have $s.I_E = \sum_{i=1}^{n} a_i.I_{A_i \cap E}(\cdot)$ and

$\int_X s.I_E.d\mu = \int_E s.d\mu = \sum_{1}^{n} a_i.\mu(A_i \cap E)$

Proof. Put $s(\cdot) = \sum_{i=1}^{n} a_i.I_{A_i}(\cdot)$, $t(\cdot) = \sum_{j=1}^{m} b_j.I_{B_j}(\cdot)$, then

(1) $s + t = \sum_{i.j}.(a_i + b_j).I_{A_i \cap B_j}$, $c.s = \sum_{i=1}^{n} ca_i.I_{A_i}$

$\int_X (s+t).d\mu = \sum_{i.j}.(a_i + b_j).\mu(A_i \cap B_j) = \sum_{i.j}.a_i.\mu(A_i \cap B_j) + \sum_{i.j}.b_j.\mu(A_i \cap B_j)$

but $\sum_{i=1}^{n} a_i.\sum_{j=1}^{m} \mu(A_i \cap B_j) = \sum_{i=1}^{n} a_i.\mu(A_i) = \int_X s.d\mu$

and $\sum_{j=1}^{m} b_j.\sum_{i=1}^{n} \mu(A_i \cap B_j) = \sum_{j=1}^{m} b_j.\mu(B_j) = \int_X t.d\mu$

so $\int_X (s+t).d\mu = \int_X s.d\mu + \int_X t.d\mu$, similarly $\int_X c.s.d\mu = c.\int_X s.d\mu$
(2) If $s \le t$, then $t - s \ge 0$ and $t = s + (t - s)$

so $\int\limits_X t.d\mu = \int\limits_X s.d\mu + \int\limits_X (t-s)\,.d\mu \geq \int\limits_X s.d\mu$. Point (3) is obvious.■

Theorem.2.4.

Let (s_n) be an increasing sequence in $\mathcal{E}_+$.
If $r \in \mathcal{E}_+$ is such that $r \leq \sup\limits_n .s_n$, then:

$$\int\limits_X r.d\mu \leq \sup\limits_n .\int\limits_X s_n.d\mu$$

Proof. Since s_n is increasing, the sequence $\int\limits_X s_n.d\mu$ is increasing in $[0,\infty]$ by Proposition **5.2.3**(2) so $\sup\limits_n .\int\limits_X s_n.d\mu$ exists in $[0,\infty]$. Let $0 < c < 1$ and put $E_n = \{s_n \geq cr\}$. Since $s_n \leq s_{n+1}$ we have $E_n \subset E_{n+1}$. On the other hand for $x \in X$ we have $c.r\,(x) < r\,(x) \leq \sup\limits_n .s_n\,(x)$, therefore there is n with $s_n\,(x) \geq c.r\,(x)$ and this gives $X = \bigcup\limits_n E_n$. Now put $r = \sum\limits_i \alpha_i.I_{A_i}$ and taking integrals, we obtain $\int\limits_X s_n.d\mu \geq \int\limits_X c.r.I_{E_n}.d\mu$ (since $s_n \geq c.r.I_{E_n}$ on X), then $\int\limits_X s_n.d\mu \geq c.\sum\limits_i \alpha_i.\mu\,(A_i \cap E_n)\,, \forall n$. This implies $\sup\limits_n .\int\limits_X s_n.d\mu \geq \lim\limits_n .\left(c.\sum\limits_i \alpha_i.\mu\,(A_i \cap E_n)\right) = c.\sum\limits_i \alpha_i.\mu\,(A_i) = c.\int\limits_X r.d\mu$, because $\mu\,(A_i \cap E_n)$ goes to $\mu\,(A_i)$ since E_n is increasing to X. Making $c \longrightarrow 1$ we get the proof.■

Corollary.

Let s_n, t_n be two increasing sequences in $\mathcal{E}_+$ such that $\sup\limits_n .s_n = \sup\limits_n .t_n$ then $\sup\limits_n .\int\limits_X s_n.d\mu = \sup\limits_n .\int\limits_X t_n.d\mu$

Proof. We have $\sup\limits_n .s_n = \sup\limits_n .t_n \Longrightarrow s_k \leq \sup\limits_n .t_n$, $\forall k$; from the Theorem we get $\int\limits_X s_k.d\mu \leq \sup\limits_n .\int\limits_X t_n.d\mu$, this gives $\sup\limits_k .\int\limits_X s_k.d\mu \leq \sup\limits_n .\int\limits_X t_n.d\mu$. By the same way we prove the reverse inequality.■

Now we are in a position to extend the integration process from the class $\mathcal{E}_+$ to the class $\mathcal{M}_+ = \{f : X \longrightarrow [0,\infty]\,,\ f\ \text{ measurable}\}$.

3. Integration in $\mathcal{M}_+$

Definition.3.1.

Let $f \in \mathcal{M}_+$, we know by Theorem**. 5.6.** that for some increasing sequence s_n in $\mathcal{E}_+$ we have $\lim\limits_n . s_n(x) = f(x)$, $\forall x \in X$.

We define the integral of f with respect to μ by $\int\limits_X f.d\mu = \sup\limits_n . \int\limits_X s_n.d\mu.$

This integral is well defined, that is, it does not depend on the sequence s_n in $\mathcal{E}_+$ converging to f (corollary of Theorem**.2.4.**).

Definition.3.2.

Let $f \in \mathcal{M}_+$ and $E \in \mathcal{F}$. We define the integral of f over E by:

$$\int\limits_E f.d\mu = \int\limits_X f.I_E.d\mu$$

where $(f.I_E)(x) = f(x)$ for $x \in E$ and $(f.I_E)(x) = 0$ for $x \in E^c$

Proposition.3.3.

The integral in $\mathcal{M}_+$ has the following properties:

If $f, g \in \mathcal{M}_+$, $c \geq 0$, and $E, F \in \mathcal{F}$, then:

(1) $\int\limits_X (f+g).d\mu = \int\limits_X f.d\mu + \int\limits_X g.d\mu$

$\int\limits_X c.f.d\mu = c.\int\limits_X f.d\mu$

(2) If $f \leq g$ then $\int\limits_X f.d\mu \leq \int\limits_X g.d\mu$ and $\int\limits_E f.d\mu \leq \int\limits_E g.d\mu$

(3) $E \subset F \Longrightarrow \int\limits_E f.d\mu \leq \int\limits_F f.d\mu$

(4) If $f = 0$ on E then $\int\limits_E f.d\mu = 0$ even if $\mu(E) = \infty$.

(5) If $\mu(E) = 0$ then $\int\limits_E f.d\mu = 0$ even if $f = \infty$ on E.

Proof. All properties are consequence of Definitions **3.1-3.2.**■

Theorem.3.4.

Let $f \in \mathcal{M}_+$ then we have:

$$\int\limits_X f.d\mu = \sup . \left\{ \int\limits_X s.d\mu : \ s \in \mathcal{E}_+ \text{ and } s \leq f \right\}$$

Proof. If $s \in \mathcal{E}_+$ and $s \leq f$ then $\int\limits_X s.d\mu \leq \int\limits_X f.d\mu$

so $\sup_n . \left\{ \int_X s.d\mu : \; s \in \mathcal{E}_+ \text{ and } s \leq f \right\} \leq \int_X f.d\mu.$

But by Definition **5.3.1.**we have $\int_X f.d\mu = \sup_n . \left\{ \int_X s_n.d\mu, \; s_n \in \mathcal{E}_+ \text{ and } s_n \leq f \right\}$ from this we deduce the proof of the Theorem.■

Theorem.3.5. (Beppo-Levy monotone convergence Theorem)

Let (f_n) be an increasing sequence in $\mathcal{M}_+$, then:

$\lim_n f_n = f \in \mathcal{M}_+$ and $\int_X f.d\mu = \lim_n \int_X f_n \, d\mu$, in other words:

$$\lim_n \int_X f_n \, d\mu = \int_X \lim_n f_n \, d\mu$$

Proof. We know that $\lim_n f_n = f \in \mathcal{M}_+$ (see chapter 4, section 2) and since (f_n) is increasing we have $\int_X f_n \, d\mu \leq \int_X f_{n+1} \, d\mu \leq \int_X f.d\mu$, $\forall n$. So $a = \lim_n \int_X f_n \, d\mu$ exists and $a \leq \int_X f.d\mu$. Let $s \in \mathcal{E}_+$ with $s \leq f$ and for $0 < c < 1$ put $E_n = \{f_n \geq c.s\}$.

We have $E_n \subset E_{n+1}$ since $f_n \leq f_{n+1}$ and $\cup_n E_n = X$ because $c.s < f = \sup_n f_n$.

On the other hand $f_n \geq 0 \Longrightarrow f_n \geq c.s.I_{E_n}, \forall n$.

Now put $s = \sum_i \alpha_i.I_{A_i}$ and taking integrals, we obtain $\int_X f_n.d\mu \geq \int_X c.s.I_{E_n}.d\mu$ (since $f_n \geq c.s.I_{E_n}$ on X), then $\int_X f_n.d\mu \geq c.\sum_i \alpha_i.\mu\left(A_i \cap E_n\right), \forall n$. This implies $a = \lim_n . \int_X f_n.d\mu \geq \lim_n . \left(c.\sum_i \alpha_i.\mu\left(A_i \cap E_n\right) \right) = c.\sum_i \alpha_i.\mu\left(A_i\right) = c.\int_X s \, d\mu$, because $\mu\left(A_i \cap E_n\right)$ goes to $\mu\left(A_i\right)$ since E_n is increasing to X. Making $c \longrightarrow 1$ we get $a \geq \int_X s \, d\mu$ for all $s \in \mathcal{E}_+$ with $s \leq f$, so $a \geq \sup \left\{ \int_X s \, d\mu, s \in \mathcal{E}_+, s \leq f \right\} = \int_X f.d\mu$ by Theorem.5.3.4, then $a = \int_X f.d\mu$.■

Remark. Theorem.**3.5.**is not valid in general for decreasing sequences (f_n) as is shown by the following example: let $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu)$ be the Borel measure space and $f_n = I_{]n,\infty[}$, then f_n decreases to 0 but $\lim_n . \int_X f_n.d\mu = \infty$.■

Lemma 3.6. (Fatou Lemma)

Let (f_n) be any sequence in $\mathcal{M}_+$, then:

$$\int\limits_X \liminf_n f_n \, d\mu \leq \liminf_n \int\limits_X f_n \, d\mu$$

Proof. Put $F_k = \inf\limits_{n\geq k} f_n$ then F_k is increasing in $\mathcal{M}_+$ to $\liminf\limits_n f_n$,

so by Theorem.5.3.5, $\lim\limits_k . \int\limits_X F_k . d\mu = \int\limits_X \liminf\limits_n f_n \, d\mu$.

But $F_k \leq f_n, \forall n \geq k$, which implies $\int\limits_X F_k . d\mu \leq \inf\limits_{n\geq k} \int\limits_X f_n \, d\mu$ and then

making $k \longrightarrow \infty$ we get $\lim\limits_k . \int\limits_X F_k . d\mu = \int\limits_X \liminf\limits_n f_n \, d\mu \leq \lim\limits_k \inf\limits_{n\geq k} \int\limits_X f_n \, d\mu =$

$\liminf\limits_n \int\limits_X f_n \, d\mu$.■

4. Exercises

29.(a) Let $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ be the counting measure on $\mathbb{N}$.

If $f : \mathbb{N} \longrightarrow [0, \infty[$ is given by $f(i) = a_i \ i \in \mathbb{N}$ prove that:

$$\int\limits_{\mathbb{N}} f . d\mu = \sum_i a_i$$

(b) Let $\mu = \delta_{x_0}$ be the Dirac measure on the power set $\mathcal{P}(X)$ of X.

then for any $f : X \longrightarrow [0, \infty[$, $\int\limits_X f . d\mu = f(x_0)$.

30. Let (f_n) be any sequence in $\mathcal{M}_+$, prove that $\sum\limits_n f_n \in \mathcal{M}_+$ and:

$$\int\limits_X \sum_n f_n \, d\mu = \sum_n \int\limits_X f_n . d\mu$$

(Hint $\sum\limits_1^n f_i$ increases to $\sum\limits_n f_n$ and use Theorem.5.3.5).

31. Let $f \in \mathcal{M}_+$

(a) Prove that the set function $\nu : A \longrightarrow \int\limits_A f . d\mu$, defined on $\mathcal{F}$ is a positive measure

(b) If $g \in \mathcal{M}_+$ prove that $\int\limits_X g . d\nu = \int\limits_X f . g . d\mu$

(Hint: check (b) for $g \in \mathcal{E}_+$ and apply Theorem **3.5** for $g \in \mathcal{M}_+$)

32. Let (f_n) be a sequence in $\mathcal{M}_+$ with $\lim\limits_n f_n(x) = f(x)$, $\forall x \in X$ for some $f \in \mathcal{M}_+$

Suppose $\sup_n \int_X f_n . d\mu < \infty$, and prove that $\int_X f . d\mu < \infty$

(Apply Fatou Lemma **3.6**)

33. Let (f_n) be a decreasing sequence in $\mathcal{M}_+$ such that

$\int_X f_{n_0} . d\mu < \infty$, for some $n_0 \geq 1$

Prove that $\lim_n \int_X f_n \, d\mu = \int_X \lim_n f_n \, d\mu$

(Hint: apply Theorem **3.5** to the increasing positive sequence $(f_{n_0} - f_n)$ $n \geq n_0$)

34. Let the interval $]0,1[$ of real numbers be endowed with Lebesgue measure. Apply Fatou Lemma to the following sequence:

$f_n(x) = n, 0 \leq x \leq \frac{1}{n}$ and $f_n(x) = 0, 1 > x > \frac{1}{n}$.