SOLUTIONS TO SOME EXERCISES

24. (a) Prove that in any measure space the uniform convergence implies the convergence in measure.
(b) In the counting measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ the uniform convergence is equivalent to the convergence in measure.

solution.
(a) Let $f_n, f : X \longrightarrow \mathbb{R}$ be measurable in the space $(X, \mathcal{F}, \mu)$ such that f_n converges uniformly to f
then we have $\forall \epsilon > 0, \exists N_\epsilon$ such that $\forall n \geq N_\epsilon, \ \ |f_n(x) - f(x)| < \epsilon$ for all $x \in X$
this implies $\{x : |f_n(x) - f(x)| > \epsilon\} = \phi, \forall n \geq N_\epsilon$
that is $\lim\limits_n \mu(|f_n - f| > \epsilon) = 0$ so f_n converges in measure to f . The result is true if f_n converges uniformly $\mu - a.e$ to f.
(b) use the fact that for the counting measure we have:
$A \subset \mathbb{N}$ and $\mu(A) = 0 \Longrightarrow A = \phi$.

25. In the space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ consider the sequence of indicator functions $f_n = I_{\{1,2,\ldots,n\}}$; prove that f_n converges $\mu{-}a.e$ but does not converge in measure.

solution.
The sets $\{1, 2, ..., n\}$ increase to $\mathbb{N}$ as $n \longrightarrow \infty$ and so $I_{\{1,2,\ldots,n\}}$ converges to 1 for any $x \in \mathbb{N}$.
On the other hand for $\epsilon > 0$ $\left\{\left|I_{\{1,2,\ldots,n\}} - 1\right| > \epsilon\right\} = \{x \in \mathbb{N} : x > n\}$
$= \{n + 1, n + 2, n + 3, ...\}$ which gives $\mu\left\{\left|I_{\{1,2,\ldots,n\}} - 1\right| > \epsilon\right\} = \infty \ \ \forall n.$■

26. Let $f_n, f \in \mathcal{M}(X, \mathbb{R})$ and suppose f_n converges pointwise to f and there is a positive measurable function g satisfying $\lim\limits_n \mu\{g > \epsilon_n\} = 0$ for some sequence of positive numbers ϵ_n with $\lim\limits_n \epsilon_n = 0$. Then if $|f_n| \leq g, \forall n$, prove that f_n converges in measure to f.

solution.
We have to prove that $n \longrightarrow \infty \Longrightarrow \mu(|f_n - f| > \epsilon) \longrightarrow 0, \forall \epsilon > 0$
Since $|f_n| \leq g$ and f_n converges pointwise to f we deduce that $|f| \leq g$
so $|f_n - f| \leq 2g$. Let $\epsilon > 0$, since $\lim\limits_n \epsilon_n = 0$ there is $N \geq 1$ with $2\epsilon_n < \epsilon, \forall n \geq N$.
Now we have $(|f_n - f| > \epsilon) \subset (2g > \epsilon) \subset (2g > 2\epsilon_n) = (g > \epsilon_n), \forall n \geq N$
we deduce that $\lim\limits_n \mu(|f_n - f| > \epsilon) \leq \lim\limits_n \mu\{g > \epsilon_n\} = 0$. So f_n converges in measure to f.■

27. Let $f : X \longrightarrow \mathbb{R}$ be measurable in the space $(X, \mathcal{F}, \mu)$ and put:
$M(f) = \inf\{\alpha \geq 0 : \ \ \mu\{|f| > \alpha\} = 0\}$,Prove that $|f| \leq M(f) \ \ \mu - a.e.$
Prove that $\lim\limits_n M(f_n - f) = 0$ iff $\lim\limits_n f_n = f$ uniformly $\mu - a.e.$

solution.
We have to prove that $\mu\{|f| > M(f)\} = 0$
If $M(f) = \infty$ the result is true.
Suppose $M(f)$ finite then we have $\{|f| > M(f)\} = \bigcup\limits_n \left\{|f| > M(f) + \dfrac{1}{n}\right\}$

but $M(f) < M(f) + \dfrac{1}{n} \Longrightarrow \exists \alpha_n \in \{\alpha \geq 0 : \quad \mu\{|f| > \alpha\} = 0\}$
with $M(f) < \alpha_n < M(f) + \dfrac{1}{n}$ so $\left\{|f| > M(f) + \dfrac{1}{n}\right\} \subset \{|f| > \alpha_n\}$ and then $\mu\left\{|f| > M(f) + \dfrac{1}{n}\right\} \leq \mu\{|f| > \alpha_n\} = 0, \forall n,$ we deduce $\mu\{|f| > M(f)\} = \mu\left(\underset{n}{\cup}\left\{|f| > M(f) + \dfrac{1}{n}\right\}\right) \leq \sum\limits_n \mu\left\{|f| > M(f) + \dfrac{1}{n}\right\} = 0.$■

28 Let $f_n, f : X \longrightarrow \mathbb{R}$ be measurable functions in the space $(X, \mathcal{F}, \mu)$ and suppose that f_n converges in measure to f ; if $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a uniformly continuous function prove that the sequence $g \circ f_n$ converges in measure to $g \circ f$

solution.
We have to prove that $n \longrightarrow \infty \Longrightarrow \mu(|g \circ f_n - g \circ f| > \epsilon) \longrightarrow 0, \forall \epsilon > 0$
g uniformly continuous implies:
$(*) \quad \forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall (x, y) \in \mathbb{R} \times \mathbb{R} \qquad |x - y| < \delta \Longrightarrow |g(x) - g(y)| < \epsilon$
$(**) \quad f_n$ converges in measure to $f \Longrightarrow \mu(|f_n - f| > \alpha) \longrightarrow 0, \forall \alpha > 0$
$(*) \Longrightarrow \forall \epsilon > 0 \;\; \exists \delta > 0$ such that $(|g \circ f_n - g \circ f| > \epsilon) \subset (|f_n - f| > \delta)$
then applying μ we get $\mu(|g \circ f_n - g \circ f| > \epsilon) \leq \mu(|f_n - f| > \delta)$
$(**) \Longrightarrow \lim\limits_n \mu(|f_n - f| > \delta) = 0$ so we deduce
$\lim\limits_n \mu(|g \circ f_n - g \circ f| > \epsilon) = 0, \forall \epsilon > 0.$■

29.(a) Let $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ be the counting measure on $\mathbb{N}$.
If $f : \mathbb{N} \longrightarrow [0, \infty[$ is given by $f(i) = a_i \ i \in \mathbb{N}$ prove that:

$$\int_{\mathbb{N}} f.d\mu = \sum_i a_i$$

(b) Let $\mu = \delta_{x_0}$ be the Dirac measure on the power set $\mathcal{P}(X)$ of X.

then for any $f : X \longrightarrow [0, \infty[, \ \int_X f.d\mu = f(x_0)$.

solution. $f : \mathbb{N} \longrightarrow [0, \infty[$

(a) Suppose f simple function of the form $\sum_1^n a_i.I_{\{i\}}$ then $\int_{\mathbb{N}} f.d\mu = \sum_1^n a_i.\mu\{i\}$

but $\mu\{i\} = 1$ since μ is the counting measure so $\int_{\mathbb{N}} f.d\mu = \sum_1^n a_i$

now take f of the form $f = .\sum_i a_i.I_{\{i\}}$ which is the limit pointwise of the increasing sequence $\varphi_n = \sum_1^n a_i.I_{\{i\}}$, by Beppo-Levy theorem we get

$$\int_{\mathbb{N}} f.d\mu = \lim_n \int_{\mathbb{N}} \varphi_n.d\mu = \lim_n \sum_1^n a_i = \sum_i a_i.$$

(b) Recall that Dirac measure is defined on $\mathcal{P}(X)$ by

$$\delta_{x_0}(A) = I_A(x_0) = \left\{ \begin{array}{c} 1 \text{ if } x_0 \in A \\ 0 \text{ if } x_0 \notin A \end{array} \right\}$$

so we have $\delta_{x_0}(A) = \int_X I_A.d\delta_{x_0}$ and generalize this formula by usual procedure to get for any $f : X \longrightarrow [0, \infty[, \ \int_X f.d\delta_{x_0} = f(x_0)$.■

30. Let (f_n) be any sequence in $\mathcal{M}_+$, prove that $\sum_n f_n \in \mathcal{M}_+$ and:

$$\int_X \sum_n f_n \, d\mu = \sum_n \int_X f_n.d\mu$$

solution.
$\sum_1^n f_i$ increases to $\sum_n f_n$ and use Beppo-Levy Theorem, see the recall.■

31. Let $f \in \mathcal{M}_+$

(a) Prove that the set function $\nu : A \longrightarrow \int_A f.d\mu$, defined on $\mathcal{F}$ is a positive measure

(b) If $g \in \mathcal{M}_+$ prove that $\int_X g.d\nu = \int_X f.g.d\mu$

solution.
(a) Let (A_n) be a pairwise disjoint sequence of sets in $\mathcal{F}$

we have to prove that $\int\limits_{\underset{n}{\cup} A_n} f.d\mu = \sum\limits_n \int\limits_{A_n} f.d\mu$

since the sets A_n are pairwise disjoint we have $I_{\underset{n}{\cup} A_n} = \sum\limits_n I_{A_n}$ and $f \geq 0$ then

$f.I_{\underset{n}{\cup} A_n} = \sum\limits_n f.I_{A_n}$, so we get $\int\limits_X f.I_{\underset{n}{\cup} A_n}.d\mu = \int\limits_X \sum\limits_n f.I_{A_n}.d\mu = \sum\limits_n \int\limits_{A_n} f.d\mu$

where the last equality comes from **Beppo-Levy** Theorem **3.5** (see recall 1)
(b) check (b) for $g \in \mathcal{E}_+$ and apply Beppo-Levy Theorem for $g \in \mathcal{M}_+$.■

32. Let (f_n) be a sequence in $\mathcal{M}_+$ with $\lim\limits_n f_n(x) = f(x)$, $\forall x \in X$ for some $f \in \mathcal{M}_+$.Suppose $\sup\limits_n \int\limits_X f_n.d\mu < \infty$, and prove that $\int\limits_X f.d\mu < \infty$

solution.

(Apply **Fatou Lemma 3.6** see recall 1)

$\int\limits_X \liminf\limits_n f_n \, d\mu \leq \liminf\limits_n \int\limits_X f_n \, d\mu$ with $\liminf\limits_n f_n = \lim\limits_n f_n(x) = f(x)$, $\forall x \in X$ for some $f \in \mathcal{M}_+$ so $\int\limits_X f.d\mu \leq \liminf\limits_n \int\limits_X f_n \, d\mu \leq \sup\limits_n \int\limits_X f_n.d\mu < \infty$.■

33. Let (f_n) be a decreasing sequence in $\mathcal{M}_+$ such that

$\int\limits_X f_{n_0}.d\mu < \infty$, for some $n_0 \geq 1$

Prove that $\lim\limits_n \int\limits_X f_n \, d\mu = \int\limits_X \lim\limits_n f_n \, d\mu$

solution.
apply Theorem **3.5** (Recall 1) to the increasing positive sequence $(f_{n_0} - f_n)$ $n \geq n_0$
indeed we have $f_{n+1} \leq f_n \implies f_{n_0} - f_n \leq f_{n_0} - f_{n+1}, \forall n \geq n_0$ and so $\lim\limits_n (f_{n_0} - f_n) = f_{n_0} - f$

by Theorem **3.5** we deduce $\lim\limits_n \int\limits_X (f_{n_0} - f_n).d\mu = \int\limits_X f_{n_0}.d\mu - \lim\limits_n \int\limits_X f_n.d\mu =$

$\int\limits_X f_{n_0}.d\mu - \int\limits_X f.d\mu$ since $f \in \mathcal{M}_+$

by the fact $\int\limits_X f_{n_0}.d\mu < \infty$, we get $\lim\limits_n \int\limits_X f_n.d\mu = \int\limits_X f.d\mu$ ■

34. Let the interval $]0,1[$ of real numbers be endowed with Lebesgue measure. (Apply **Fatou Lemma 3.6** see recall 1) to the following sequence:

$f_n(x) = n, 0 \leq x \leq \frac{1}{n}$ and $f_n(x) = 0, \frac{1}{n} < x < 1$.

solution.

$\int\limits_X \liminf_n f_n \, d\mu \leq \liminf_n \int\limits_X f_n \, d\mu$ with $\liminf_n f_n = \sup_n . \inf_{k \geq n} f_k = 0$ and $\int\limits_X f_n \, d\mu = 1, \forall n$

whence $0 \leq \liminf_n \int\limits_X f_n \, d\mu \leq 1.$■

35 (continuity of integrals depending on a parameter)

Let T be an interval of $\mathbb{R}$ and $f : X \times T \longrightarrow \mathbb{R}$ a function such that:

(a) for each $t \in T$ the function $x \longrightarrow f(x,t)$ is in $L_1(\mu)$

(b) there is g in $L_1(\mu)$such that$|f(x,t)| \leq |g(x)|$ $\mu - a.e$ for all $t \in T$

if $\lim_{t \to t_0} .f(x,t) = f(x,t_0)$ then we have $\lim_{t \to t_0} \int\limits_X f(x,t) \, d\mu = \int\limits_X f(x,t_0) \, d\mu$

solution.

Consider the function $h : T \longrightarrow \mathbb{R}$ given by $h(t) = \int\limits_X f(x,t) \, d\mu$

we have to prove that $\lim_{t \to t_0} h(t) = h(t_0)$

that is the function h is continuous on T which is equivalent to: for any sequence (t_n) with $\lim_n t_n = t_0$ we have $\lim_n h(t_n) = h(t_0)$

let us observe that the functions u_n defined by $u_n(x) = f(x,t_n)$

satisfies **Theorem.3.7** by (b) and $\lim_n .u_n(x) = f(x,t_0)$, so $\int\limits_X u_n.d\mu = h(t_n)$

converges to $\int\limits_X \lim_n .u_n(x).d\mu = \int\limits_X f(x,t_0).d\mu = h(t_0)$.■

36 (Derivative of integrals depending on a parameter)

Let T be an open set of $\mathbb{R}$ and $f : X \times T \longrightarrow \mathbb{R}$ a function such that:

(a) for each $t \in T$ the function $x \longrightarrow f(x,t)$ is in $L_1(\mu)$

(b) the function $t \longrightarrow f(x,t)$ derivable on T for each $x \in X$

(c) there is $g \in L_1(\mu)$ $\left| \frac{d}{dt} f(x,t) \right| \leq |g(x)|$ $\mu - a.e$ for all $t \in T$

Then the function $t \longrightarrow \int\limits_X f(x,t) \, d\mu$ is differentiable on T

and $\frac{d}{dt} \int\limits_X f(x,t) \, d\mu = \int\limits_X \frac{d}{dt} f(x,t) \, d\mu$

solution.

Let (t_n) be a sequence with $\lim_n t_n = t$ and define the sequence (g_n) of functions by

$g_n(x) = \dfrac{f(x,t_n) - f(x,t)}{t_n - t}$ then $\lim_n g_n(x) = \dfrac{d}{dt} f(x,t)$. By the Mean Value Theorem

there is $\theta_n(x)$ between t_n and t such that $g_n(x) = \dfrac{d}{dt} f(x, \theta_n(x))$.

Now we have $\lim_n t_n = t$ so $\lim_n \theta_n(x) = t$ and $\lim_n g_n(x) = \dfrac{d}{dt} f(x,t)$. But $|g_n(x)| \leq |g(x)|$ by (c) then

we can apply **Theorem.3.7** to $g_n(x)$ with

$$\int_X g_n(x).d\mu = \frac{\int_X f(x,t_n).d\mu - \int_X f(x,t).d\mu}{t_n - t}$$

to get $\lim_n \int_X g_n(x).d\mu = \dfrac{d}{dt}\int_X f(x,t)\ d\mu = \int_X \lim_n g_n(x)\ d\mu = \int_X \dfrac{d}{dt} f(x,t)\ d\mu.$ ■

37 (Change of variable formula)

Let $(X, \mathcal{F}, \mu)$ be a measure space and let $(Y, \mathcal{G})$ be a measurable space:

If $\varphi : X \longrightarrow Y$ is a measurable mapping from $(X, \mathcal{F})$ into $(Y, \mathcal{G})$ then:

(1) the set function $\nu : \mathcal{G} \longrightarrow [0, \infty]$ given by $G \in \mathcal{G}, \nu(G) = \mu\left(\varphi^{-1}(G)\right)$ is a measure on $(Y, \mathcal{G})$

(2) for every function $g : Y \longrightarrow \mathbb{C}$, $\nu-$integrable the function $g \circ \varphi$ is $\mu-$integrable and

$$(*) \int_Y g.d\nu = \int_X g \circ \varphi.d\mu$$

$$(**) \int_E g.d\nu = \int_{\varphi^{-1}(E)} g \circ \varphi.d\mu \ \forall E \in \mathcal{G}.$$

solution.

Apply usual procedure:

start with g simple then g in $\mathcal{M}_+$ and finally g integrable for ν. ■

38 Measure defined by an integral. (see exercise **31 for the proof)**

Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f \in \mathcal{M}_+$ then

(a) the set function $\nu : \mathcal{F} \longrightarrow [0, \infty]$ given by: $A \in \mathcal{F}, \nu(A) = \int_A f.d\mu$

is a positive measure on $\mathcal{F}$ and we have:

(b) $\int_X g.d\nu = \int_X f.g.d\mu$, for every $g \in \mathcal{M}_+$.