Chapter 4

INTEGRATION

1. Preliminaries

Introduction.

Let $(X, \mathcal{F}, \mu)$ be a measure space. This chapter concerns the Lebesgue integration process $\int\limits_X f.d\mu$ of numerical measurable functions on X with respect to the measure μ. Such classes of functions have been introduced with their convergence properties in sections **1-3** of chapter **3**.

If X is the closed interval $[a, b]$ in the real system $\mathbb{R}$, it is also possible to define the Riemann integral $\int\limits_a^b f.dx$ of some function $f : [a, b] \longrightarrow \mathbb{R}$ (e.g continuous function).

If the Lebesgue integration process is applied to a sequence of Riemann integrable functions, it leads to a kind of convergence properties less restrictive and easier in applications than those needed in the Riemann process framework. Let us recall:

Classes of functions.1.1. (see sections 1-3 of chapter 3.)

$\mathcal{E} = \{s : X \longrightarrow \mathbb{R},\ s \text{ simple measurable}\}$

$\mathcal{E}_+ = \{s \in \mathcal{E} :\ s \text{ positive}\}$

$\mathcal{M}_+ = \{f : X \longrightarrow [0, \infty],\ \ f \ \ \text{measurable}\}$

$\mathcal{M}(\mathbb{R}) = \{f : X \longrightarrow \mathbb{R},\ \ f \ \ \text{measurable}\}$

$\mathcal{M}(\mathbb{C}) = \{f : X \longrightarrow \mathbb{C},\ \ f \ \ \text{measurable}\}$

Let us recall that if $f \in \mathcal{M}_+$, there is an increasing sequence s_n in $\mathcal{E}_+$ with: $\lim\limits_n s_n(x) = f(x), \forall x \in X$.

2. Integration in $\mathcal{E}_+$

Definition.2.1.

Let $s \in \mathcal{E}_+$ with $s(\cdot) = \sum\limits_1^n a_i.I_{A_i}(\cdot)$, where I_A is the Dirac function of the set A, and the sets $A_i, 1 \leq i \leq n$ form a partition of X in $\mathcal{F}$.

The integral of s with respect to μ is defined by:

$$\int\limits_X s.d\mu = \sum_1^n a_i.\mu(A_i)$$

with the convention $0 \cdot \infty = 0$.

Remark.2.2.

Suppose $s \in \mathcal{E}_+$ with $s(\cdot) = \sum\limits_{i=1}^n a_i.I_{A_i}(\cdot) = \sum\limits_{j=1}^m b_j.I_{B_j}(\cdot)$, where $\{A_i, 1 \leq i \leq n\}$ and $\{B_j, 1 \leq j \leq m\}$ are partitions of X. Then we have:

$A_i = \{x \in X : \ s(x) = a_i\}$ and $B_j = \{x \in X : \ s(x) = b_j\}$
so $a_i.I_{A_i \cap B_j}(\cdot) = b_j.I_{A_i \cap B_j}(\cdot)$ for $1 \leq i \leq n, 1 \leq j \leq m$.
$a_i.I_{A_i}(\cdot) = \sum_{j=1}^{m} a_i.I_{A_i \cap B_j}(\cdot)$ and $\sum_{i=1}^{n} a_i.I_{A_i}(\cdot) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i.I_{A_i \cap B_j}(\cdot)$
likewise $\sum_{j=1}^{m} b_j.I_{B_j}(\cdot) = \sum_{i=1}^{n}\sum_{j=1}^{m} b_j.I_{A_i \cap B_j}(\cdot)$ and the terms in the two double sums are equivalent so $\sum_{i=1}^{n} a_i.\mu(A_i) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i.\mu(A_i \cap B_j)$
and $\sum_{j=1}^{m} b_j.\mu(B_j) = \sum_{j=1}^{m}\sum_{i=1}^{n} b_j.\mu(A_i \cap B_j)$ then $\sum_{i=1}^{n} a_i.\mu(A_i) = \sum_{j=1}^{m} b_j.\mu(B_j)$
we deduce that the integral $\int_X s.d\mu = \sum_1^n a_i.\mu(A_i)$ is well defined.

Proposition.2.3.

Let s, t be in $\mathcal{E}_+$ and $c \geq 0$ then we have:

(1) $\int_X (s+t).d\mu = \int_X s.d\mu + \int_X t.d\mu$

$\int_X c.s.d\mu = c.\int_X s.d\mu$

(2) If $s \leq t$ then $\int_X s.d\mu \leq \int_X t.d\mu$

(3) If $E \in \mathcal{F}$ and $s(\cdot) = \sum_{i=1}^{n} a_i.I_{A_i}(\cdot)$ we have $s.I_E = \sum_{i=1}^{n} a_i.I_{A_i \cap E}(\cdot)$ and

$\int_X s.I_E.d\mu = \int_E s.d\mu = \sum_1^n a_i.\mu(A_i \cap E)$

Proof. Put $s(\cdot) = \sum_{i=1}^{n} a_i.I_{A_i}(\cdot)$, $t(\cdot) = \sum_{j=1}^{m} b_j.I_{B_j}(\cdot)$, then

(1) $s + t = \sum_{i.j}.(a_i + b_j).I_{A_i \cap B_j}$, $c.s = \sum_{i=1}^{n} ca_i.I_{A_i}$

$\int_X (s+t).d\mu = \sum_{i.j}.(a_i + b_j).\mu(A_i \cap B_j) = \sum_{i.j}.a_i.\mu(A_i \cap B_j) + \sum_{i.j}.b_j.\mu(A_i \cap B_j)$

but $\sum_{i=1}^{n} a_i.\sum_{j=1}^{m} \mu(A_i \cap B_j) = \sum_{i=1}^{n} a_i.\mu(A_i) = \int_X s.d\mu$

and $\sum_{j=1}^{m} b_j.\sum_{i=1}^{n} \mu(A_i \cap B_j) = \sum_{j=1}^{m} b_j.\mu(B_j) = \int_X t.d\mu$

so $\int_X (s+t).d\mu = \int_X s.d\mu + \int_X t.d\mu$, similarly $\int_X c.s.d\mu = c.\int_X s.d\mu$
(2) If $s \leq t$, then $t - s \geq 0$ and $t = s + (t - s)$

so $\int\limits_X t.d\mu = \int\limits_X s.d\mu + \int\limits_X (t-s)\,.d\mu \geq \int\limits_X s.d\mu$. Point (3) is obvious.■

Theorem.2.4.

Let (s_n) be an increasing sequence in $\mathcal{E}_+$.
If $r \in \mathcal{E}_+$ is such that $r \leq \sup\limits_n .s_n$, then:

$$\int\limits_X r.d\mu \leq \sup_n .\int\limits_X s_n.d\mu$$

Proof. Since s_n is increasing, the sequence $\int\limits_X s_n.d\mu$ is increasing in $[0,\infty]$ by Proposition **5.2.3**(2) so $\sup\limits_n .\int\limits_X s_n.d\mu$ exists in $[0,\infty]$. Let $0 < c < 1$ and put $E_n = \{s_n \geq cr\}$. Since $s_n \leq s_{n+1}$ we have $E_n \subset E_{n+1}$. On the other hand for $x \in X$ we have $c.r(x) < r(x) \leq \sup\limits_n .s_n(x)$, therefore there is n with $s_n(x) \geq c.r(x)$ and this gives $X = \bigcup\limits_n E_n$. Now put $r = \sum\limits_i \alpha_i.I_{A_i}$ and taking integrals, we obtain $\int\limits_X s_n.d\mu \geq \int\limits_X c.r.I_{E_n}.d\mu$ (since $s_n \geq c.r.I_{E_n}$ on X), then $\int\limits_X s_n.d\mu \geq c.\sum\limits_i \alpha_i.\mu(A_i \cap E_n), \forall n$. This implies $\sup\limits_n .\int\limits_X s_n.d\mu \geq \lim\limits_n .\left(c.\sum\limits_i \alpha_i.\mu(A_i \cap E_n)\right) = c.\sum\limits_i \alpha_i.\mu(A_i) = c.\int\limits_X r.d\mu$, because $\mu(A_i \cap E_n)$ goes to $\mu(A_i)$ since E_n is increasing to X. Making $c \longrightarrow 1$ we get the proof.■

Corollary.

Let s_n, t_n be two increasing sequences in $\mathcal{E}_+$ such that $\sup\limits_n .s_n = \sup\limits_n .t_n$ then $\sup\limits_n .\int\limits_X s_n.d\mu = \sup\limits_n .\int\limits_X t_n.d\mu$

Proof. We have $\sup\limits_n .s_n = \sup\limits_n .t_n \Longrightarrow s_k \leq \sup\limits_n .t_n$, $\forall k$; from the Theorem we get $\int\limits_X s_k.d\mu \leq \sup\limits_n .\int\limits_X t_n.d\mu$, this gives $\sup\limits_k .\int\limits_X s_k.d\mu \leq \sup\limits_n .\int\limits_X t_n.d\mu$. By the same way we prove the reverse inequality.■

Now we are in a position to extend the integration process from the class $\mathcal{E}_+$ to the class $\mathcal{M}_+ = \{f : X \longrightarrow [0,\infty]\,,\ f \ \text{ measurable}\}$.

3. Integration in $\mathcal{M}_+$

Definition.3.1.

Let $f \in \mathcal{M}_+$, we know by Theorem**. 5.6.** that for some increasing sequence s_n in $\mathcal{E}_+$ we have $\lim_n . s_n(x) = f(x)$, $\forall x \in X$.

We define the integral of f with respect to μ by $\int_X f.d\mu = \sup_n . \int_X s_n.d\mu$.

This integral is well defined, that is, it does not depend on the sequence s_n in $\mathcal{E}_+$ converging to f (corollary of Theorem**.2.4.**).

Definition.3.2.

Let $f \in \mathcal{M}_+$ and $E \in \mathcal{F}$. We define the integral of f over E by:

$$\int_E f.d\mu = \int_X f.I_E.d\mu$$

where $(f.I_E)(x) = f(x)$ for $x \in E$ and $(f.I_E)(x) = 0$ for $x \in E^c$

Proposition.3.3.

The integral in $\mathcal{M}_+$ has the following properties:
If $f, g \in \mathcal{M}_+$, $c \geq 0$, and $E, F \in \mathcal{F}$, then:

(1) $\int_X (f+g).d\mu = \int_X f.d\mu + \int_X g.d\mu$

$\int_X c.f.d\mu = c.\int_X f.d\mu$

(2) If $f \leq g$ then $\int_X f.d\mu \leq \int_X g.d\mu$ and $\int_E f.d\mu \leq \int_E g.d\mu$

(3) $E \subset F \Longrightarrow \int_E f.d\mu \leq \int_F f.d\mu$

(4) If $f = 0$ on E then $\int_E f.d\mu = 0$ even if $\mu(E) = \infty$.

(5) If $\mu(E) = 0$ then $\int_E f.d\mu = 0$ even if $f = \infty$ on E.

Proof. All properties are consequence of Definitions **3.1-3.2.**■

Theorem.3.4.

Let $f \in \mathcal{M}_+$ then we have:

$$\int_X f.d\mu = \sup.\left\{ \int_X s.d\mu : \ s \in \mathcal{E}_+ \text{ and } s \leq f \right\}$$

Proof. If $s \in \mathcal{E}_+$ and $s \leq f$ then $\int_X s.d\mu \leq \int_X f.d\mu$

so $\sup_n .\left\{\int_X s.d\mu : \ s \in \mathcal{E}_+ \text{ and } s \leq f\right\} \leq \int_X f.d\mu.$

But by Definition **5.3.1.**we have $\int_X f.d\mu = \sup_n .\left\{\int_X s_n.d\mu, \ s_n \in \mathcal{E}_+ \text{ and } s_n \leq f\right\}$ from this we deduce the proof of the Theorem.■

Theorem.3.5. (Beppo-Levy monotone convergence Theorem)

Let (f_n) be an increasing sequence in $\mathcal{M}_+$, then:

$\lim_n f_n = f \in \mathcal{M}_+$ and $\int_X f.d\mu = \lim_n \int_X f_n \, d\mu$, in other words:

$$\lim_n \int_X f_n \, d\mu = \int_X \lim_n f_n \, d\mu$$

Proof. We know that $\lim_n f_n = f \in \mathcal{M}_+$ (see chapter 4, section 2) and since (f_n)

is increasing we have $\int_X f_n \, d\mu \leq \int_X f_{n+1} \, d\mu \leq \int_X f.d\mu$, $\forall n$. So $a = \lim_n \int_X f_n \, d\mu$ exists

and $a \leq \int_X f.d\mu$. Let $s \in \mathcal{E}_+$ with $s \leq f$ and for $0 < c < 1$ put $E_n = \{f_n \geq c.s\}$.

We have $E_n \subset E_{n+1}$ since $f_n \leq f_{n+1}$ and $\cup_n E_n = X$ because $c.s < f = \sup_n f_n$.

On the other hand $f_n \geq 0 \Longrightarrow f_n \geq c.s.I_{E_n}, \forall n$.

Now put $s = \sum_i \alpha_i.I_{A_i}$ and taking integrals, we obtain $\int_X f_n.d\mu \geq \int_X c.s.I_{E_n}.d\mu$

(since $f_n \geq c.s.I_{E_n}$ on X), then $\int_X f_n.d\mu \geq c.\sum_i \alpha_i.\mu(A_i \cap E_n), \forall n$. This implies

$a = \lim_n . \int_X f_n.d\mu \geq \lim_n . \left(c.\sum_i \alpha_i.\mu(A_i \cap E_n)\right) = c.\sum_i \alpha_i.\mu(A_i) = c.\int_X s \, d\mu$, because $\mu(A_i \cap E_n)$ goes to $\mu(A_i)$ since E_n is increasing to X. Making $c \longrightarrow 1$ we get $a \geq \int_X s \, d\mu$ for all $s \in \mathcal{E}_+$ with $s \leq f$, so $a \geq \sup\left\{\int_X s \, d\mu, s \in \mathcal{E}_+, s \leq f\right\} = \int_X f.d\mu$ by Theorem.5.3.4, then $a = \int_X f.d\mu$.■

Remark. Theorem.**3.5.**is not valid in general for decreasing sequences (f_n) as is shown by the following example: let $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu)$ be the Borel measure space and $f_n = I_{]n,\infty[}$, then f_n decreases to 0 but $\lim_n . \int_X f_n.d\mu = \infty$.■

Lemma 3.6. (Fatou Lemma)

Let (f_n) be any sequence in $\mathcal{M}_+$, then:

$$\int_X \liminf_n f_n \, d\mu \leq \liminf_n \int_X f_n \, d\mu$$

Proof. Put $F_k = \inf_{n\geq k} f_n$ then F_k is increasing in $\mathcal{M}_+$ to $\liminf_n f_n$,

so by Theorem.5.3.5, $\lim_k . \int_X F_k.d\mu = \int_X \liminf_n f_n \, d\mu$.

But $F_k \leq f_n, \forall n \geq k$, which implies $\int_X F_k.d\mu \leq \inf_{n\geq k} \int_X f_n \, d\mu$ and then

making $k \longrightarrow \infty$ we get $\lim_k . \int_X F_k.d\mu = \int_X \liminf_n f_n \, d\mu \leq \lim_k \inf_{n\geq k} \int_X f_n \, d\mu =$

$\liminf_n \int_X f_n \, d\mu.$ ■

4. Exercises

29.(a) Let $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ be the counting measure on $\mathbb{N}$.

If $f : \mathbb{N} \longrightarrow [0, \infty[$ is given by $f(i) = a_i \; i \in \mathbb{N}$ prove that:

$$\int_{\mathbb{N}} f.d\mu = \sum_i a_i$$

(b) Let $\mu = \delta_{x_0}$ be the Dirac measure on the power set $\mathcal{P}(X)$ of X.

then for any $f : X \longrightarrow [0, \infty[$, $\int_X f.d\mu = f(x_0)$.

30. Let (f_n) be any sequence in $\mathcal{M}_+$, prove that $\sum_n f_n \in \mathcal{M}_+$ and:

$$\int_X \sum_n f_n \, d\mu = \sum_n \int_X f_n.d\mu$$

(Hint $\sum_1^n f_i$ increases to $\sum_n f_n$ and use Theorem.**3.5**).

31. Let $f \in \mathcal{M}_+$

(a) Prove that the set function $\nu : A \longrightarrow \int_A f.d\mu$, defined on $\mathcal{F}$ is a positive measure

(b) If $g \in \mathcal{M}_+$ prove that $\int_X g.d\nu = \int_X f.g.d\mu$

(Hint: check (b) for $g \in \mathcal{E}_+$ and apply Theorem **3.5** for $g \in \mathcal{M}_+$)

32. Let (f_n) be a sequence in $\mathcal{M}_+$ with $\lim_n f_n(x) = f(x)$, $\forall x \in X$ for some $f \in \mathcal{M}_+$.Suppose $\sup_n \int_X f_n.d\mu < \infty$, and prove that $\int_X f.d\mu < \infty$

(Apply Fatou Lemma **3.6**)

33. Let (f_n) be a decreasing sequence in $\mathcal{M}_+$ such that

$\int_X f_{n_0}.d\mu < \infty$, for some $n_0 \geq 1$

Prove that $\lim_n \int_X f_n\, d\mu = \int_X \lim_n f_n\, d\mu$

(Hint: apply Theorem **3.5** to the increasing positive sequence $(f_{n_0} - f_n)$ $n \geq n_0$)

34. Let the interval $]0,1[$ of real numbers be endowed with Lebesgue measure. Apply Fatou Lemma to the following sequence:

$f_n(x) = n, 0 \leq x \leq \frac{1}{n}$ and $f_n(x) = 0, 1 > x > \frac{1}{n}$.

5. Integration of Complex Functions

Definition.5.1.

Let $\mathcal{L}_1(\mu)$ be the subset of $\mathcal{M}(X,\mathbb{C})$ defined by:

$$\mathcal{L}_1(\mu) = \left\{ f \in \mathcal{M}(X,\mathbb{C}) : \int_X |f|.d\mu < \infty \right\}$$

where $\mathcal{M}(X,\mathbb{C}) = \{f : X \longrightarrow \mathbb{C} \ \ f \text{ measurable}\}$ (see Definitions **1.1** and **1.2**)
if $f = u + iv \in \mathcal{L}_1(\mu)$ we define the integral of f by:

$$\int_X f.d\mu = \int_X u.d\mu + i\int_X v.d\mu = \int_X u^+.d\mu - \int_X u^-.d\mu + i\int_X v^+.d\mu - i\int_X v^-.d\mu$$

this integral is well defined since u^+, u^-, v^+, v^- are less then $|f|$.

If f is real valued, we have $v = 0$ and $\int_X f.d\mu = \int_X u^+.d\mu - \int_X u^-.d\mu$

Definition.5.2.

If $f \in \mathcal{M}\left(X, \overline{\mathbb{R}}\right)$ we define the integral of f by: $\int_X f.d\mu = \int_X f^+.d\mu - \int_X f^-.d\mu$

provided that $\int_X f^+.d\mu < \infty$ or $\int_X f^-.d\mu < \infty$

Proposition.5.3.

$\mathcal{L}_1(\mu)$ is a vector space on the field $\mathbb{C}$ and we have

$$\int_X (\alpha f + \beta g).d\mu = \alpha\int_X f.d\mu + \beta\int_X g.d\mu$$

Proof. Use the following facts:
$|\alpha f+\beta g| \leq |\alpha| . |f| + |\beta| . |g|$ and
$f = u + iv = u^+ - u^- + iv^+ - iv^-$, $g = z + iw = z^+ - z^- + iw^+ - iw^-$
then apply Definition **5.1.**■

Lemma.5.4.

Let f, g be in $\mathcal{L}_1(\mu)$ such that $f = g\ \mu - a.e.$ then $\int\limits_X f.d\mu = \int\limits_X g.d\mu$

Proof. Let $E = \{x : f(x) = g(x)\}$ then $\mu(E^c) = 0$
on the other hand we have $\int\limits_{E^c} f.d\mu = \int\limits_{E^c} g.d\mu = 0$ by point (5) Proposition **3.3** applied to the integrals of f^+, f^-, g^+, g^-, since $f.I_E = g.I_E$ we deduce that $\int\limits_E f.d\mu = \int\limits_E g.d\mu$ that is $\int\limits_X f.d\mu = \int\limits_X g.d\mu$.■

By the same way one can prove:

Proposition.5.5.

(1) If f, g are real valued in $\mathcal{L}_1(\mu)$ and $f \leq g.\mu - a.e.$ then $\int\limits_X f.d\mu \leq \int\limits_X g.d\mu$

(2) $\left|\int\limits_X f.d\mu\right| \leq \int\limits_X |f|.d\mu$ for all f in $\mathcal{L}_1(\mu)$.

(3) If $\in \mathcal{M}_+$ and $\int\limits_E f.d\mu = 0$ then $f = 0\ \mu - a.e.$ on E

(4) If $f \in \mathcal{L}_1(\mu)$ and $\int\limits_E f.d\mu = 0$ for all $E \in \mathcal{F}$ then $f = 0\ \mu - a.e.$

(5) If $f \in \mathcal{M}\left(X, \overline{\mathbb{R}}\right)$ and $\int\limits_X |f|.d\mu < \infty$ then $\mu\{|f| = +\infty\} = 0$,
i.e f is finite $\mu - a.e.$

Corollary.
Let f, g be in $\mathcal{L}_1(\mu)$:
(a) $\int\limits_E f.d\mu = \int\limits_E g.d\mu\ \forall E \in \mathcal{F} \Longrightarrow f = g.\mu - a.e.$

(b) If f, g are real valued then $\int\limits_E f.d\mu \leq \int\limits_E g.d\mu$, $\forall E \in \mathcal{F} \Longrightarrow f \leq g.\mu - a.e.$

6. The Banach Space $L_1(\mu)$

Definition 6.1
The binary relation $f = g \ \ \mu - a.e$ is an equivalence relation on $\mathcal{L}_1(\mu)$
Let $L_1(\mu)$ be the quotient of $\mathcal{L}_1(\mu)$ by this equivalence relation, that is $L_1(\mu)$ is the set of equivalence classes in $\mathcal{L}_1(\mu)$.
It is well known that $L_1(\mu)$ is a vector space on $\mathbb{R}$ with the operations defined by: class(x) +class(y) =class$(x+y)$ and α.class(x) =class$(\alpha.x)$.
In the sequel we consider elements of $L_1(\mu)$ as functions although they are classes of functions.
If $f \in L_1(\mu)$, formula $\|f\| = \int_X |f|\, d\mu$ defines a norm on $L_1(\mu)$

Theorem.6.2
Endowed with the norm $\|f\| = \int_X |f|\, d\mu$ the space $L_1(\mu)$ is a Banach space.

Proof. Let (f_n) be a Cauchy sequence in $L_1(\mu)$ then we have:

$$\forall j \geq 1, \exists N_j \geq 1 \text{ such that } n, m \geq N_j \implies \|f_n - f_m\| < \frac{1}{2^j}$$

let us define the strictly increasing subsequence $n_1 < n_2 < n_3 < ....$ by the following recipe:

$$n_1 = N_1, n_2 = \max(n_1 + 1, N_2), ..., n_j = \max(n_{j-1} + 1, N_j), ...$$

then we have: $\left\|f_{n_{j+1}} - f_{n_j}\right\| < \frac{1}{2^j}, ..\forall j = 1, 2, ...$

now consider the functions: $g_k = \sum_{j=1}^{k} \left|f_{n_{j+1}} - f_{n_j}\right|$ and $g = \sum_{j=1}^{\infty} \left|f_{n_{j+1}} - f_{n_j}\right|$

$$\|g_k\| \leq \sum_{j=1}^{k} \left\|f_{n_{j+1}} - f_{n_j}\right\| \leq \sum_{j=1}^{k} \frac{1}{2^j} \leq \sum_{j=1}^{\infty} \frac{1}{2^j} < 1 \text{ and also } \|g\| < 1$$

so g is integrable $\Longrightarrow g$ is finite $\mu - a.e$
let us define the function $f : X \longrightarrow \mathbb{R}$ by $f(x) = f_{n_1}(x) + \sum_{j=1}^{\infty} \left(f_{n_{j+1}}(x) - f_{n_j}(x)\right)$
then

we have obviously $f(x) = \lim_{i \longrightarrow \infty} .f_{n_j}(x)$

now let us observe that the sequence $\left(f_{n_j}\right)$ is cauchy since it is a subsequence of (f_n) which is cauchy so

$$\forall \epsilon > 0, ..\exists N_\epsilon \geq 1 : n_j, m \geq N \Longrightarrow \left\|f_{n_j} - f_m\right\| = \int_X \left|f_{n_j} - f_m\right| .d\mu < \epsilon$$

by Fatou lemma **3.6** applied for n_j we get $\int_X .\liminf_{n_j} \left|f_{n_j} - f_m\right| .d\mu = \int_X |f - f_m| .d\mu \leq$ $\liminf_{n_j} . \int_X . \left|f_{n_j} - f_m\right| .d\mu \leq \limsup_{n_j} \int_X . \left|f_{n_j} - f_m\right| .d\mu < \epsilon$. So $f \in L_1(\mu)$ and $\lim_m . \int_X |f - f_m| .d\mu = 0.$■

Now we give one of the most famous convergence theorem of Lebesgue integration theory

Theorem.6.3 (Lebesgue's dominated convergence theorem)
Let (f_n) be a sequence in $L_1(\mu)$ such that:
(a) f_n converges $\mu - a.e$ to a function f
(b) there is g in $L_1(\mu)$ such that $\forall n \geq 1$ $|f_n| \leq |g|$ $\mu - a.e$
Then the function f is in $L_1(\mu)$ and $\lim_n \int_X |f_n - f| \, d\mu = 0$
in particular $\lim_n \int_X f_n \, d\mu = \int_X f \, d\mu$

Proof. Put $E = \{x : f_n(x) \text{ converges to } f(x)\} \cup \left\{ \underset{n}{\cup} \{|f_n| \leq |g|\} \right\}$
then $\mu(E^c) = 0$
We can assume that f_n converges everywhere to a function f
and that $|f_n| \leq |g|$ everywhere $\forall n \geq 1$
(if necessary replace f_n by $F_n = f_n I_E$ and g by $G = gI_E$)
first since $|f_n| \leq |g|$ everywhere $\forall n \geq 1$ and f_n converges everywhere to f we deduce that
$|f| \leq |g|$ and $|f_n - f| \leq 2g$ so $2g - |f_n - f| \geq 0$
applying Fatou lemma **3.6** to the function $2g - |f_n - f|$ we get:
$\int_X .\liminf_n [2g - |f_n - f|] \, .d\mu = \int_X \left[2g. - \limsup_n |f_n - f| \right] .d\mu = \int_X 2g.d\mu \leq$ $\liminf_n . \int_X [2g - |f_n - f|] \, .d\mu = \int_X 2g.d\mu - .\limsup_n . \int_X |f_n - f| \, .d\mu$ and so $\int_X 2g.d\mu \leq$ $\int_X 2g.d\mu - .\limsup_n . \int_X |f_n - f| \, .d\mu$, this gives $0 \leq -.\limsup_n . \int_X |f_n - f| \, .d\mu$ that is $\limsup_n . \int_X |f_n - f| \, .d\mu = 0.$■

Theorem.6.4 (Bounded convergence theorem)
Suppose $\mu(X) < \infty$. Let (f_n) be a sequence in $L_1(\mu)$ such that
$|f_n| \leq M$ $\mu - a.e$ for some constant $M > 0$ then the conclusions of Theorem **6.3** are valid.

Application.6.5 (continuity of integrals depending on a parameter)

Let T be an interval of $\mathbb{R}$ and $f : X \times T \longrightarrow \mathbb{R}$ a function such that:
(a) for each $t \in T$ the function $x \longrightarrow f(x,t)$ is in $L_1(\mu)$
(b) there is g in $L_1(\mu)$such that$|f(x,t)| \leq |g(x)|$ $\mu - a.e$ for all $t \in T$

then we have $\lim_{t \to t_0} \int_X f(x,t) \, d\mu = \int_X f(x,t_0) \, d\mu$

Application.6.6 (Derivative of integrals depending on a parameter)

Let T be an open set of $\mathbb{R}$ and $f : X \times T \longrightarrow \mathbb{R}$ a function such that:
(a) for each $t \in T$ the function $x \longrightarrow f(x,t)$ is in $L_1(\mu)$
(b) the function $t \longrightarrow f(x,t)$ derivable on T for each $x \in X$
(c) there is $g \in L_1(\mu)$ $\left| \frac{d}{dt} f(x,t) \right| \leq |g(x)|$ $\mu - a.e$ for all $t \in T$

Then the function $t \longrightarrow \int\limits_X f(x,t)\, d\mu$ is differentiable on T

and $\frac{d}{dt}\int\limits_X f(x,t)\, d\mu = \int\limits_X \frac{d}{dt} f(x,t)\, d\mu$

Application.6.7 (Change of variable formula)
Let $(X, \mathcal{F}, \mu)$ be a measure space and let $(Y, \mathcal{G})$ be a measurable space:
If $\varphi : X \longrightarrow Y$ is a measurable mapping from $(X, \mathcal{F})$ into $(Y, \mathcal{G})$ then:
(1) the set function $\nu : \mathcal{G} \longrightarrow [0, \infty]$ given by $G \in \mathcal{G}, \nu(G) = \mu\left(\varphi^{-1}(G)\right)$
is a measure on $(Y, \mathcal{G})$
(2) for every function $g : Y \longrightarrow \mathbb{C}$, $\nu-$integrable the function $g \circ \varphi$ is $\mu-$integrable
and

$$(*) \int\limits_Y g.d\nu = \int\limits_X g \circ \varphi.d\mu$$

$$(**) \int\limits_E g.d\nu = \int\limits_{\varphi^{-1}(E)} g \circ \varphi.d\mu \ \forall E \in \mathcal{G}.$$

As a particular case take $(Y, \mathcal{G}) = (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ and $\varphi : X \longrightarrow \mathbb{R}$, $\mu-$integrable
put $\nu(B) = \hat{\mu}(B) = \mu\left(\varphi^{-1}(B)\right)$ for $B \in \mathcal{B}_{\mathbb{R}}$

then we get from$(**)$: $\int\limits_{\varphi^{-1}(B)} \varphi.d\mu = \int\limits_B t.d\hat{\mu}$

Application.6.8
Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f \in \mathcal{M}_+$ then

the set function $\nu : \mathcal{F} \longrightarrow [0, \infty]$ given by: $A \in \mathcal{F}, \nu(A) = \int\limits_A f.d\mu$

is a positive measure on $\mathcal{F}$ and we have:

$$\int\limits_X g.d\nu = \int\limits_X f.g.d\mu, \text{ for every } g \in \mathcal{M}_+.$$

7. The L_p-Spaces

Let $(X, \mathcal{F}, \mu)$ be a measure space. This section concerns a short description of the L_p-spaces with some important convexity inequalities.

Definition 7.1
Let $\mathcal{L}_p(\mu)$ be the subset of $\mathcal{M}(X, \mathbb{C})$ defined by:

$$\mathcal{L}_p(\mu) = \left\{ f \in \mathcal{M}(X, \mathbb{C}) : \int\limits_X |f|^p .d\mu < \infty \right\}$$

for some real number $0 < p < \infty$.

Definition 7.2
Two real positive numbers $0 < p, q < 1$ such that $p + q = pq$ or equivalently $\frac{1}{p} + \frac{1}{q} = 1$ are called conjugate exponents. If $p \longrightarrow 1$ then $q \longrightarrow \infty$ so $1, \infty$ are considered as conjugate exponents.

Theorem 7.3
Let $f, g \in \mathcal{M}_+$ and let $0 < p, q < 1$ be conjugate exponents then we have:

(1) Hölder's inequality: $\int\limits_X f.g.d\mu \leq \left\{ \int\limits_X f^p.d\mu \right\}^{\frac{1}{p}} . \left\{ \int\limits_X g^q.d\mu \right\}^{\frac{1}{q}}$

(2) Minkowski's inequality: $\left\{ \int\limits_X (f+g)^p .d\mu \right\}^{\frac{1}{p}} \leq \left\{ \int\limits_X f^p.d\mu \right\}^{\frac{1}{p}} + \left\{ \int\limits_X g^p.d\mu \right\}^{\frac{1}{p}}$

Remark: Using Minkowski's inequality it is not difficult to prove that $\mathcal{L}_p(\mu)$ is a vector space over $\mathbb{C}$.

Definition 7.4 Let $0 < p < \infty$ be a positive real number
The binary relation $f = g \ \ \mu - a.e$ is an equivalence relation on $\mathcal{L}_p(\mu)$
Let $L_p(\mu)$ be the quotient of $\mathcal{L}_p(\mu)$ by this equivalence relation, that is $L_p(\mu)$ is the set of equivalence classes in $\mathcal{L}_p(\mu)$.
It is well known that $L_p(\mu)$ is a vector space on $\mathbb{R}$ with the operations defined by: class(x) +class(y) =class$(x+y)$ and α.class(x) =class$(\alpha.x)$.
In the sequel we consider elements of $L_p(\mu)$ as functions although they are classes of functions.

Theorem 7.5
If $f \in L_p(\mu)$, formula $\|f\|_p = \left\{ \int_X |f|^p \, d\mu \right\}^{\frac{1}{p}}$
defines a norm on $L_p(\mu)$ and with respect to this norm $L_p(\mu)$ is a Banach space. (mimic the proof made for L_1 Theorem **6.2**)

Definition 7.6 The Hilbert Space $L_2(\mu)$
For $p = 2$ it is not difficult to see that the norm $\|f\|_2 = \left\{ \int_X |f|^2 \, d\mu \right\}^{\frac{1}{2}}$ is induced by the inner product $\langle f, g \rangle = \int_X f.\bar{g}.d\mu$, which makes $L_2(\mu)$ a Hilbert space.

8. The Space L_∞

Definition 8.1 Let $(X, \mathcal{F}, \mu)$ be a measure space.
Let $f \in \mathcal{M}_+$ we define the essential supremum of f by:
$$ess - \sup f = \left\{ \begin{array}{c} \alpha \geq 0 : \mu[f > \alpha] = 0 \\ \infty \text{ if } \mu[f > \alpha] > 0, \forall \alpha \geq 0 \end{array} \right\}$$

if $f \in \mathcal{M}(X, \mathbb{C})$ we put $N_\infty(f) = ess - \sup|f|$

Remark.

For $f \in \mathcal{M}(X, \mathbb{C})$ we have:

$\alpha \in \{\alpha \geq 0 : \mu[|f| > \alpha] = 0\} \iff |f| \leq \alpha \quad \mu - a.e$

Lemma.8.2

For $f \in \mathcal{M}(X, \mathbb{C})$ we have:

$\mu[|f| > N_\infty(f)] = 0$, that is $|f| \leq N_\infty(f) \quad \mu - a.e$

Definition 8.3

Let $\mathcal{L}_\infty(\mu)$ be the subset of $\mathcal{M}(X, \mathbb{C})$ defined by:

$\mathcal{L}_\infty(\mu) = \{f \in \mathcal{M}(X, \mathbb{C}) : N_\infty(f) < \infty\}$

It is easy to prove that the binary relation $f = g \quad \mu - a.e$ is an equivalence relation on $\mathcal{L}_\infty(\mu)$ and $N_\infty(f) = N_\infty(g)$ if $f = g \quad \mu - a.e$
Let $L_\infty(\mu)$ be the quotient of $\mathcal{L}_\infty(\mu)$ by this equivalence relation, that is $L_\infty(\mu)$ is the set of equivalence classes in $\mathcal{L}_\infty(\mu)$.
Also one can prove that $L_\infty(\mu)$ is a vector space on $\mathbb{R}$ with the operations defined by: class(f) +class(g) =class($f + g$) and α.class(f) =class($\alpha.f$).
In the sequel we consider elements of $L_\infty(\mu)$ as functions although they are classes of functions and

Definition 8.3

For any f in $L_\infty(\mu)$ define $\|f\|_\infty$ by $N_\infty(h)$ where h is any function satisfying $f = h \ \ \mu - a.e$ then $L_\infty(\mu)$ is a vector space on $\mathbb{C}$ and $\|f\|_\infty$ is a norm on $L_\infty(\mu)$:

Theorem 8.4

$L_\infty(\mu)$ endowed with the norm $\|f\|_\infty$ defined above is a Banach space.

An important property of the sequences (f_n) in the spaces L_p is the following:

Theorem 8.5

Let (f_n) be a cauchy sequence in L_p that is a sequence (f_n) satisfying $\lim\limits_{m,n}. \|f_n - f_m\|_p = 0$ then:

(1) For $1 \leq p < \infty$, the sequence (f_n) contains a subsequence $\left(f_{n_j}\right)$ converging $\mu - a.e$ to a function $f \in L_p$

(2) For $p = \infty$ the sequence (f_n) itself converges uniformly $\mu - a.e$ to a function $f \in L_\infty$.

9. Duality of the L_p-Spaces

Recall.

1 Let X, Y be normed spaces. A linear operator T from a normed space X into a normed space Y is said to be bounded if there is a constant $M > 0$ such that:

$$\|T(x)\| \leq M. \|x\|, \forall x \in X$$

This definition means that if B is a bounded subset of X, the set $\{T(x), x \in B\}$ is bounded in Y. For instance if $B = \{x : \|x\| \leq 1\}$ then $\|T(x)\| \leq M, \forall x \in B$.

2 Let T be a bounded operator from X into Y. Define:

$\|T\| = \sup\left\{\dfrac{\|T(x)\|}{\|x\|} : x \in X, x \neq 0\right\}$

$m_1 = \sup\{\|T(x)\| : x \in X, \|x\| = 1\}$

$m_2 = \sup\{\|T(x)\| : x \in X, \|x\| < 1\}$

$m_3 = \sup\{\|T(x)\| : x \in X, \|x\| \leq 1\}$

Then $m_1 = m_2 = m_3 = \|T\| < \infty$ and we have:

$\|T(x)\| \leq \|T\| \|x\|, \forall x \in X$

3 If X is a normed space the strong dual of X is the Banach space X^* of continuous linear functionals on X. If $x \in X$ and $x^* \in X^*$, we denote $x^*(x)$ by $\langle x^*, x\rangle$.

Definition 9.1

Let $(X, \mathcal{F}, \mu)$ be a measure space and let $1 \leq p, q \leq \infty$ be conjugate exponents. For g fixed in L_q let us define the functional φ_g on L_p by:

$$\varphi_g : L_p \longrightarrow \mathbb{C}, \quad f \in L_p \quad \varphi_g(f) = \int_X f.g.d\mu$$

It is clear that φ_g is well defined and we have:

Theorem 9.2

(a) φ_g is linear continuous on L_p for any $1 \leq p \leq \infty$.

Moreover if $p > 1$ we have $\left\|\varphi_g\right\| = \|g\|_q$

where $\left\|\varphi_g\right\| = \sup\left\{\left\|\varphi_g(f)\right\| : f \in L_p, \|f\| \leq 1\right\}$

(b) If μ is $\sigma-$finite (Definition **3.3** Chapter **2**) then we have

$\left\|\varphi_g\right\| = \|g\|_\infty$ for $p = 1$.

Theorem 9.3 (L_p **Duality**)

Let $(X, \mathcal{F}, \mu)$ be a measure space with μ $\sigma-$finite

and let $\varphi : L_p \longrightarrow \mathbb{C}$ be a continuous linear functional on L_p

If $1 \leq p < \infty$ there is a unique $g \in L_q$, for q conjugate exponent of p such that

$$\varphi(f) = \int_X f.g.d\mu \ \forall f \in L_p \text{ and } \|\varphi\| = \|g\|_q$$

In other words the strong dual $(L_p)^*$ of L_p is linearly isometric to L_q for q conjugate exponent of p.

Remark

(a) For $p = 1$ **Theorem 9.3** is not true in general if μ is not $\sigma-$finite as is shown by the following example:

take $X = \{a, b\}$, $\mu(a) = 1$, $\mu(\phi) = 0$, $\mu(b) = \mu(X) = \infty$

then μ is not $\sigma-$finite. In this case we have

$L_1 = \{f : \{a, b\} \longrightarrow \mathbb{C}, \text{such that } f(b) = 0\} = \mathbb{C}$

so $L_1 = (L_1)^* = \mathbb{C}$, but $L_\infty = \left\{\begin{array}{c} f : \{a, b\} \longrightarrow \mathbb{C}, \text{such that} \\ \sup(f(a), f(b)) < \infty \end{array}\right\} = \mathbb{C}^2$.

(b) The Theorem **9.3** is not true in general for the space L_∞ even if μ is finite in other words we have $L_1 \subset (L_\infty)^*$ and the inclusion is strict in general. Here is an example:

(c) Let $[0, 1]$ the unit interval endowed with the Lebesgue μ and let $C[0, 1]$ be the space of real continuous functions on $[0, 1]$ equipped with the uniform norm $\|f\| = \sup\{|f(x)|, x \in [0, 1]\}$. Let us observe that if f, g are continuous and satisfying $f = g \quad \mu - a.e$ then $f = g$ everywhere, indeed let $F \subset [0,1]$ be measurable with $\mu(F) = 0$ and $f(x) = g(x)\ \forall x \in [0,1] \backslash F$, so the set $A = \{x \in [0,1] : |f(x) - g(x)| > 0\} = F$, but A is open by the continuity of f,g, then since $\mu(F) = 0$ the equality $A = F$ implies $F = \phi$ and so $f = g$ everywhere on $[0,1]$. Consequently the class of f for the equivalence relation $f = g \quad \mu - a.e$ is reduced to only f. Since any $f \in C[0,1]$ is bounded we have $C[0,1] \subset L_\infty$.

Now let us consider the linear functional $\varphi : C[0,1] \longrightarrow \mathbb{R}$ given by $\varphi(f) = f(0)$, φ is continuous since $|\varphi(f)| \le \|f\| = \sup\{|f(x)|, x \in [0,1]\}$ and $\|\varphi\| \le 1$. By Hahn-Banach Theorem, φ can be extented to a continuous linear functional on all of L_∞; if there were some $g \in L_1$ such that $\varphi(f) = \int_{[0,1]} f.g.d\mu\ \forall f \in L_\infty$, we would have $f(0) = \int_{[0,1]} f.g.d\mu\ \forall f \in C[0,1]$.

Taking $f(x) = \cos(nx)$ we get $f(0) = 1 = \int_{[0,1]} \cos(nx).g.d\mu \quad \forall n \ge 1$, this leads to a contradiction since by the **Riemann-Lebesgue Lemma** ,(see Theorem **10.6** below) we have $\lim_{n \longrightarrow \infty}. \int_a^b f(x)\cos(nx).d\,x = 0$.

10. Riemann Integral and Lebesgue Integral

In this section we consider a **bounded** function $f : [a,b] \longrightarrow \mathbb{R}$,defined on the interval $[a,b]$ with values in $\mathbb{R}$.

10.1 Definition (Darboux sums)
Let $\pi = \{I_1, I_2, ..., I_n\}$ be a finite partition of $[a,b]$ into intervals.
Put $m = \inf\{f(x), x \in [a,b]\}$ and $M = \sup\{f(x), x \in [a,b]\}$
$m_k = \inf\{f(x), x \in I_k\}$ and $M_k = \sup\{f(x), x \in I_k\}$, $1 \le k \le n$.
We define the lower and upper Darboux sums of f
with respect to the partition π by:

$$\underline{S}_\pi(f) = \sum_{k=1}^{k=n} m_k.\lambda(I_k) \text{ and } \overline{S}_\pi = \sum_{k=1}^{k=n} M_k.\lambda(I_k)$$

where $\lambda(I)$ is the lengh of the interval I.

10.2 Definition (Lower integral and Upper integral)
The Lower integral of f is defined by:

$\underline{S}(f) = \sup \underline{S}_\pi(f)$

The Upper integral of f is defined by:

$\overline{S}(f) = \inf \overline{S}_\pi$

where the sup and inf are taken over the finite partitions π of $[a,b]$.
It is clear that $\underline{S}(f) \le \overline{S}(f)$. We say that f is integrable if $\underline{S}(f) = \overline{S}(f)$.

We define the Riemann integral of f on $[a,b]$ by $\int_a^b f(x)\,dx = \underline{S}(f) = \overline{S}(f)$.

10.3 Theorem
A **bounded** function $f : [a, b] \longrightarrow \mathbb{R}$ is Riemann integrable if and only if it is continuous $\mu - a.e$, in this case the Riemann integral is equal to the Lebesgue integral, that is we have:

$$\int_a^b f(x)\; dx = \int_{[a,b]} f \; d\mu, \text{ where } \mu \text{ is the Lebesgue measure on } [a,b].$$

10.4 Theorem
Let f_n:$[a, b] \to \mathbb{R}$ be Riemann integrable functions and assume that f_n converges uniformly to f on $[a, b]$. Then f is Riemann integrable

and $\lim_n \int_a^b f_n \; dx = \int_a^b f \; dx$

If we replace uniform convergence by pointwise convergence, then the above Theorem shows that the limit function f does not have to be Riemann integrable. Therefore the above theorem is not true if we replace uniform convergence by pointwise convergence. There is however a version of the above theorem for pointwise convergence if we add the hypothesis that the limit function is Riemann integrable. This theorem is called **Arzela's Theorem** for the Riemann integral, which is a special case of the Bounded Convergence Theorem of Lebesgue for the Lebesgue integral.

10.5 Theorem (Arzela's Theorem). Let f, f_n:$[a, b] \to \mathbb{R}$ be Riemann integrable functions and assume that f_n converges pointwise to f on $[a, b]$. If there exists M such that $|f_n(x)| \leq M$ for all $n \geq 1$.Then $\lim_n \int_a^b f_n \; dx = \int_a^b f \; dx$.

10.6 Theorem (Riemann-Lebesgue Lemma)
If f is an intégrable function on the interval $[a, b]$, then :

$$\lim_{n \longrightarrow \infty} . \int_a^b f(x) \cos(nx) .dx = 0 \text{ and } \lim_{n \longrightarrow \infty} . \int_a^b f(x) \sin(nx) .dx = 0$$

The proof is easy if f is bounded or if f is C^1 using intégration by parts.