Chapter 5

INTEGRATION IN PRODUCT SPACES
Product Measure and Fubini Theorem

In this chapter we give without proofs the most important results on product spaces useful in applications.Proofs are classical and in general simple.

1. Preliminaries and Notations

1.1 In all what follows, $(X, \mathcal{F}, \mu)$, $(Y, \mathcal{G}, \nu)$ will be fixed measure spaces.
1.2 Let us recall that the product $\sigma-$field $\mathcal{F} \otimes \mathcal{G}$ on $X \times Y$ is generated by the family $\{A \times B, \text{ with } A \in \mathcal{F}, B \in \mathcal{G}\}$, (Definition **3.4** Chapter **1**)
1.3 The set $\mathbb{R}$ will be endowed with its Borel $\sigma-$field $\mathcal{B}_{\mathbb{R}}$. The set $\mathbb{R}^2$ endowed with the σ-field $\mathcal{B}_{\mathbb{R}^2}$=$\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$(Theorem**2.9**Chap.**3**)

2. Product Measure

2.1 Definition
For any subset $E \subset X \times Y$ and any $(x, y) \in X \times Y$, define:
the section of E at x, $E_x = \{y \in Y, \quad (x, y) \in X \times Y\}$
the section of E at y, $E_y = \{x \in X, \quad (x, y) \in X \times Y\}$

2.2 Proposition
For every $E \in \mathcal{F} \otimes \mathcal{G}$, $E_x \in \mathcal{G}$ and $E_y \in \mathcal{F}$.

2.3 Theorem
Suppose that the measure μ and ν are $\sigma-$finite
then for every $E \in \mathcal{F} \otimes \mathcal{G}$, we have:
the function $x \longrightarrow \nu(E_x)$ is $\mathcal{F}$ measurable
the function $y \longrightarrow \mu(E_y)$ is $\mathcal{G}$ measurable
Moreover we have $\int\limits_X \nu(E_x)\, d\mu = \int\limits_Y \mu(E_y)\, d\nu$

Corollary.(Product measure)
Under the conditions of Theorem **1.6** the set function $\mu \otimes \nu$ defined on $\mathcal{F} \otimes \mathcal{G}$ by:

$$\mu \otimes \nu(E) = \int\limits_X \nu(E_x)\, d\mu = \int\limits_Y \mu(E_y)\, d\nu,\ E \in \mathcal{F} \otimes \mathcal{G}$$

is a $\sigma-$finite measure on $\mathcal{F} \otimes \mathcal{G}$. Moreover $\mu \otimes \nu$ is the unique $\sigma-$finite measure on $\mathcal{F} \otimes \mathcal{G}$ satisfying $\mu \otimes \nu(A \times B) = \mu(A).\nu(B)$ for every $A \in \mathcal{F}, B \in \mathcal{G}$.

3 Integration in Product Spaces

3.1 Definition Let $f : X \times Y \longrightarrow \mathbb{R}$ be any function and $(x, y) \in X \times Y$, define:
$f_x : Y \longrightarrow \mathbb{R}$ by $f_x(y) = f(x, y)$ (section of f at x)
$f_y : X \longrightarrow \mathbb{R}$ by $f_y(x) = f(x, y)$ (section of f at y)

3.2 Proposition

Let $f : X \times Y \longrightarrow \mathbb{R}$ be $\mathcal{F} \otimes \mathcal{G}$-measurable then

f_x is $\mathcal{G}$-measurable and f_y is $\mathcal{F}$-measurable

3.3 Theorem (Fubini)

Suppose that the measure μ and ν are $\sigma-$finite and $f : X \times Y \longrightarrow \mathbb{R}$ is $\mathcal{F} \otimes \mathcal{G}$-measurable positive then:

the function $x \longrightarrow \int\limits_Y f(x,y)\, d\nu$ is $\mathcal{F}$-measurable

the function $y \longrightarrow \int\limits_X f(x,y)\, d\mu$ is $\mathcal{G}$-measurable

and we have:

$$\int\limits_{X\times Y} f(x,y)\, d\mu \otimes \nu = \int\limits_X d\mu \int\limits_Y f(x,y)\, d\nu = \int\limits_Y d\nu \int\limits_X f(x,y)\, d\mu$$

3.4 Theorem (Fubini)

For every $f \in L_1(\mu \otimes \nu)$ we have:

(a) $\int\limits_Y f(x,y)\, d\nu \in L_1(\mu)$ and $\int\limits_X f(x,y)\, d\mu \in L_1(\nu)$

(b) $\int\limits_{X\times Y} f(x,y)\, d\mu \otimes \nu = \int\limits_X d\mu \int\limits_Y f(x,y)\, d\nu = \int\limits_Y d\nu \int\limits_X f(x,y)\, d\mu$

3.5 Application. (Convolution of functions)

Let μ be the Lebesgue measure on $\mathbb{R}$,$\mathcal{B}_{\mathbb{R}}$ and $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions in $L_1(\mu)$, then:

$\int\limits_{\mathbb{R}} |f(x-y)| \,.\, |g(y)| \,.d\mu(y) < \infty$ for each x

Let us define the convolution of f and g by the function $h : \mathbb{R} \longrightarrow \mathbb{R}$:

$h(x) = \int\limits_{\mathbb{R}} f(x-y)\,.g(y)\,.d\mu(y)$

we denote h by $h = f * g$

Since $\left| \int\limits_{\mathbb{R}} f(x-y)\,.g(y)\,.d\mu(y) \right| \leq \int\limits_{\mathbb{R}} |f(x-y)| \,.\, |g(y)| \,.d\mu(y) < \infty$ we deduce that $h \in L_1(\mu)$

3.6 Lemma

Under the definition above we have $\|f * g\| \leq \|f\| \,.\, \|g\|$.■

4 Convolution of Measures

4.1 Definition
Let us consider on the set $\mathbb{R}^2$ endowed with the σ-field $\mathcal{B}_{\mathbb{R}^2}=\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$, the transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $T(x, y) = x + y$ which is measurable because continuous. Let $\mu_1 \otimes \mu_2$ be the product of two finite measures μ_1, μ_2 defined on $\mathbb{R}$, $\mathcal{B}_{\mathbb{R}}$. The convolution $\mu_1 * \mu_2$ of the measures μ_1, μ_2 is the measure on $\mathcal{B}_{\mathbb{R}}$ given by: $B \in \mathcal{B}_{\mathbb{R}}$, $(\mu_1 * \mu_2)(B) = (\mu_1 \otimes \mu_2)\left(T^{-1}(B)\right)$. Then we have:

4.2 Proposition Let $B \in \mathcal{B}_{\mathbb{R}}$ and define:
$\left[T^{-1}(B)\right]_x = \{y \in \mathbb{R}, x + y \in B\} = B - x$
$\left[T^{-1}(B)\right]_y = \{x \in \mathbb{R}, x + y \in B\} = B - y$
then we get: $(\mu_1 * \mu_2)(B) = \int_{\mathbb{R}} .\mu_2(B - x) .\mu_1(dx) = \int_{\mathbb{R}} .\mu_1(B - y) .\mu_2(dy)$
by applying Fubini Theorem and the relation $(\mu_1 * \mu_2)(B) = (\mu_1 \otimes \mu_2)\left(T^{-1}(B)\right) = \int_{X\times Y} .I_{T^{-1}(B)}(x, y) . (\mu_1 \otimes \mu_2)(dx, dy)$.
Moreover if we take a function $f : \mathbb{R} \longrightarrow \mathbb{C}$ integrable with respect to $\mu_1 * \mu_2$ we obtain the following nice relation:
$\int_{\mathbb{R}} f(t) . (\mu_1 * \mu_2)(dt) = \int_{\mathbb{R}} \mu_2(dy) \int_{\mathbb{R}} f(x + y) .\mu_1(dx) = \int_{\mathbb{R}} \mu_1(dx) \int_{\mathbb{R}} f(x + y) .\mu_2(dy)$

4.3 Proposition With the definitions above we have:
(1) $\mu_1 * \mu_2 = \mu_2 * \mu_1$
(2) $(\mu_1 * \mu_2)(\mathbb{R}) = (\mu_1 \otimes \mu_2)\left(T^{-1}(\mathbb{R})\right) = (\mu_1 \otimes \mu_2)\left(\mathbb{R}^2\right) = \mu_1(\mathbb{R}) .\mu_2(\mathbb{R})$
(3) $\mu_1 * \delta_0 = \delta_0 * \mu_1 = \mu_1$, δ_0 is the Dirac measure at 0.■