# Bochner Integration in Topological Vector Spaces

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Abstract

The subject of the present work deals with Bochner integrationl of bounded operators, acting on $L_1$-type spaces. More precisely, let $(S, \mathcal{F}, \mu)$ be a finite measure space and let $X$ be a Banach space or a locally convex space. We form the space $L_1(\mu, X)$ of all Bochner $\mu-$integrable functions $f : S \to X$, with an adequate topology. We perform integration for a class of bounded operators $T : L_1(\mu, X) \to X$, whose integral structure is similar to that of bounded functionals on $L_1(\mu)$.The main setting is the Bochner integration process with respect to finite abstract measure and the results obtained may be considered as generalizations of the classical Riesz Theorem.

Keywords:
Bochner integration process; Bounded operators; Vector measures

## Introduction

In the first part we consider a finite measure space $(S, \mathcal{F}, \mu)$, a Banach space $X$ and form the Banach space $L_1(\mu, X)$ of all Bochner $\mu-$integrable functions $f : S \to X$, with $L_1(\mu, X) = L_1(\mu)$ if $X = \mathbb{R}$. We introduce a class of linear bounded operators $T : L_1(\mu, X) \to X$, whose Bochner integral structure is much similar to that of bounded functionals on $L_1(\mu)$. We give two complete characterizations of this class. The first one, which may be considered as a Riesz type theorem, is obtained via integrals by functions in $L_\infty(\mu)$. Actually the identified class will be isometrically isomorphic to $L_\infty(\mu)$. The second characterization is more specific. It pertains to an operator valued measure, that will be attached to each operator of the class. This operator valued measure will be absolutely continuous with respect to $\mu$ and this property will be used to get another interesting characterization of the class under consideration.

In the second part, we assume that $X$ is a locally convex space whose topology is defined by a family $\{p_\alpha\}$ of continuous seminorms. We assume that $\{p_\alpha\}$ is separating, this means that for each nonzero $x \in X$ there is a $p_\alpha$ such that $p_\alpha(x) \neq 0$. Moreover we assume that $X$ is sequentially complete, that is, every Cauchy sequence in $X$ is convergent. The construction of the Bochner integral we give in this context is, as far as we know, new (for other approachs see $[1, 5, 13]$). Finally, arrangements are made so that each Part of this work is mostly selfcontained and can be read independently.

# Part 1.

## The Integral structure of some bounded operators on $L_1(\mu, X)$

### 1.Operators on the space of Bochner integrable functions

Let $(S, \mathcal{F}, \mu)$ be a finite measure space and let $X$ be a Banach space. We denote by $L_1(\mu, X)$ the Banach space of all Bochner $\mu-$integrable functions $f : S \to X$, with $L_1(\mu, X) = L_1(\mu)$ if $X = \mathbb{R}$. For all properties of the Bochner integral, we refer the reader to [6].
For $f \in L_1(\mu, X)$, we put:

$$(1.2) \qquad \|f\|_1 = \int_S \|f(s)\| \, d\mu(s)$$

Then it is well known that:

**1.3. Proposition:** *Formula* (1.2) *defines a norm on* $L_1(\mu, X)$, *for which* $L_1(\mu, X)$ *is a Banach space. Moreover the measurable simple functions* $s : S \to X$ *form a dense subspace of* $L_1(\mu, X)$. *This means that for each* $f \in L_1(\mu, X)$ *there is a sequence* $s_n$ *of simple functions such that* $\|f - s_n\|_1 \to 0$.

The starting point that has motivated the present work is contained in the following simple observation:

**1.4. Theorem:** *Fix a function* $g$ in $L_\infty(\mu)$ *( the space of all* $\mu-$*essentially bounded real functions on* $S$ *) and consider the operator* $T_g : L_1(\mu, X) \to X$ *defined by:*

$$(1.5) \quad f \in L_1(\mu, X), \qquad T_g(f) = \int_S fg \; d\mu:$$

*Then* $T_g$ *is linear bounded and satisfies* $\|T_g\| = \|g\|_\infty$.

**Proof:** Since $\|f(s)g(s)\| \leq \|f(s)\| \|g\|_\infty$ $\quad \mu-$ a.e. we deduce from (1.5), $\|T_g(f)\| \leq \|g\|_\infty \cdot \int_S \|f(s)\| \, d\mu(s) = \|g\|_\infty \cdot \|f\|_1$. So the operator $T_g$ is bounded and $\|T_g\| \leq \|g\|_\infty$. To prove the reverse inequality, apply $T_g$ to a function $f \in L_1(\mu, X)$ of the form $f = \varphi.x$, where $\varphi \in L_1(\mu)$, such that $\|\varphi\|_1 = 1$ and $x$ fixed in $X$ with $\|x\| = 1$. We get $\|f\|_1 = \|\varphi\|_1 = 1$ and $T_g(f) = \int_S \varphi g \; x d\mu = \left(\int_S \varphi g . d\mu\right).x$, by standard integration tools.
So we deduce $\|T_g(f)\| = \left|\int_S \varphi g . d\mu\right| \leq \|T_g\|$ and then
$Sup\left\{\left|\int_S \varphi g . d\mu\right|, \varphi \in L_1(\mu), \; \|\varphi\|_1 = 1\right\} \leq \|T_g\|$. But the LHS of the preceding inequality is equal to $\|g\|_\infty$ by the Riesz duality theorem for $L_1(\mu)$. So we get $\|g\|_\infty \leq \|T_g\|$ and then $\|T_g\| = \|g\|_\infty$ .■

**1.6. Remark:** Another way to put the conclusion of **Theorem 1.4** is the following:

*The map* $\Phi : g \to T_g$ *from* $L_\infty(\mu)$ *into* $\mathcal{L}(L_1(\mu, X), X)$, *the space of bounded operators* $T : L_1(\mu, X) \to X$, *is a linear isometry.*

We can wonder whether $\Phi$ is onto. This is certainly true if $X = \mathbb{R}$ by the Riesz duality theorem for $L_1(\mu)$. But if dimension of $X$ is greater than one, the following example shows that not all operators in $\mathcal{L}(L_1(\mu, X), X)$ can be written in the form (1.5) for some $g$ in $L_\infty(\mu)$.

**1.7. Example:** Let $X = \mathbb{R}^2$, equipped with the norm: $z = (z_1, z_2)$, $\|z\| = |z_1| + |z_2|$. If $f = (f_1, f_2) : S \to \mathbb{R}^2$ is Bochner $\mu-$integrable with the Borel $\sigma-$field on $\mathbb{R}^2$, then $f_1, f_2 : S \to \mathbb{R}$ are $\mu-$integrable and $\displaystyle\int_S f\, d\mu = \left(\int_S f_1\, d\mu, \int_S f_2\, d\mu\right)$. Note also that $\|f(s)\| = |f_1(s)| + |f_2(s)|$, so that $\|f\|_1 = \displaystyle\int_S |f_1|\, d\mu + \int_S |f_2|\, d\mu$. Now define the operator $T : L_1(\mu, \mathbb{R}^2) \to \mathbb{R}^2$, by $Tf = T(f_1, f_2) = \left(\displaystyle\int_S f_1\, d\mu, \alpha \int_S f_2\, d\mu\right)$, where $0 < \alpha < 1$ is a fixed constant. It is clear that $T$ is linear and we have $\|Tf\| = \left|\displaystyle\int_S f_1\, d\mu\right| + \alpha \left|\int_S f_2\, d\mu\right| \leq \|f\|_1$, so that $T$ is bounded. If there were a $g \in L_\infty(\mu)$ such that $T(f) = \int_S fg\ d\mu$, we would have $\displaystyle\int_S f_1\, d\mu = \int_S f_1\, g.d\mu$ and $\alpha \displaystyle\int_S f_2\, d\mu = \int_S f_2.g.d\mu$, for all $\mu-$integrable functions $f_1, f_2$. Taking $f_1, f_2$ both characteristic functions of sets in $\mathcal{F}$, this would imply $g = 1$, $\mu-$a.e and $g = \alpha$, $\mu-$a.e. This is impossible by the choice of $\alpha$. Consequently the operator $T$ cannot be written in the form (1.5).

The aim is to characterize those bounded operators $T : L_1(\mu, X) \to X$ that have integral form (1.5) with a function $g \in L_\infty(\mu)$. This amounts to describe the range of the operator $\Phi$ in **remark 1.6**. In section 2 we give the ingredients of this characterization which allows a representation of operators on the space $L_1(\mu, X)$, much simpler than those given in [10]. In section3 we prove integral representations by operator valued measures, for operators introduced in section 2. This leads to a rather precise description of such operators.

## 2 A Characterizing class

In this section we want to identify those operators $T \in \mathcal{L}(L_1(\mu, X), X)$, for which there is $g \in L_\infty(\mu)$ such that $T = T_g$. Let $X^*$ be the topological dual of $X$. For each $x^* \in X^*$ consider the operator $\varphi_{x^*} : L_1(\mu, X) \to L_1(\mu)$, given by:

$$(2.1) \qquad f \in L_1(\mu, X), \quad \varphi_{x^*} f = x^* \circ f$$

where $(x^* \circ f)(t) = x^*(f(t))$, $t \in S$.
We collect some facts about $\varphi_{x^*}$ for later use:

**2.2. Proposition:** *(a) $\varphi_{x^*}$ is linear bounded and $\|\varphi_{x^*}\| = \|x^*\|$.*
*(b) $\varphi_{x^*}$ is onto for each $x^* \neq 0$.*
*(c) There exist $y^* \in X^*$ such that for each $h \in L_1(\mu)$ there is $f \in L_1(\mu, X)$ with $\|f\|_1 = \|h\|_1$ and $\varphi_{y^*} f = h$.*

**Proof:** $(a)$ $\|\varphi_{x^*} f\| = \displaystyle\int_S |x^* \circ f| \, d\mu \leq \|x^*\| \int_S \|f(s)\| \, d\mu(s) = \|x^*\| \|f\|_1$.
So $\varphi_{x^*}$ is bounded and $\|\varphi_{x^*}\| \leq \|x^*\|$. To see the reverse inequality apply $\varphi_{x^*}$ to a function $f \in L_1(\mu, X)$ of the form $f(\bullet) = g(\bullet).x$, with $g \in L_1(\mu)$ such that $\|g\|_1 = 1$ and $x$ fixed in $X$ with $\|x\| = 1$. We get $\|f\|_1 = 1$ and $\|\varphi_{x^*} f\| = \displaystyle\int_S |x^* \circ f| \, d\mu = |x^*(x)|$. Thus $|x^*(x)| \leq \|\varphi_{x^*}\|$ for every $x \in X$ with $\|x\| = 1$. Consequently $\|x^*\| = Sup\{|x^*(x)|, x \in X, \ \|x\|_1 = 1\} \leq \|\varphi_{x^*}\|$.
$(b)$ Let $x^* \neq 0$ and choose $x \in X$ such that $x^*(x) = 1$. Now if $h \in L_1(\mu)$ put $f = h.x$, then clearly we have $\varphi_{x^*} f = h$..
$(c)$ Choose $x \in X$ with $\|x\| = 1$, then choose $y^* \in X^*$ such that $y^*(x) = \|x\| = 1$, $\|y^*\| = 1$, this is possible by Hahn-Banach theorem . If $h \in L_1(\mu)$, the function $f = h.x$ is in $L_1(\mu, X)$ and fits the conclusion.■

The following class of operators will play an essential role for the characterization we need:

**2.3. Definition:** Let $\mathfrak{D}$ be the class of linear bounded operators $T \in \mathcal{L}(L_1(\mu, X), X)$ satisfying the following condition:

$$(2.4) \qquad x^*, y^* \in X^*, \ f, g \in L_1(\mu, X) : \varphi_{x^*} f = \varphi_{y^*} g \Longrightarrow x^* T f = y^* T g$$

It is easy to check that $\mathfrak{D}$ is a closed subspace of $\mathcal{L}(L_1(\mu, X), X)$. Note also that every $T_g$ as defined by (1.5) is in $\mathfrak{D}$.
The important fact about $\mathfrak{D}$ is:

**2.5. Theorem:** *Let $T$ be an operator in $\mathfrak{D}$, then there exists a unique bounded linear functional $V : L_1(\mu) \to \mathbb{R}$ such that:*

$$(2.6) \qquad V \circ \varphi_{x^*} = x^* \circ T \quad \textit{for every } x^* \in X^*.$$

**Proof:** Let $h \in L_1(\mu)$ and $x^* \in X^*$, $x^* \neq 0$; by **Proposition 2.2**$(b)$ there is an $f \in L_1(\mu, X)$ such that $\varphi_{x^*} f = h$. then we put:

$$V(h) = x^* T f \tag{2.7}$$

The functional $V$ does not depend on the choice of $x^*$ but depends only on $T$. For if $V_{x^*}$ and $V_{y^*}$ are defined as in (2.7), with $x^*, y^* \neq 0$, then $V_{x^*}(h) = x^* T f$ if $h = \varphi_{x^*} f$ and $V_{y^*}(h) = y^* T g$ if $h = \varphi_{y^*} g$; but condition (2.4) on $T$ implies that $V_{x^*}(h) = V_{y^*}(h)$. It is easy to check that it is linear. We must show that $V$ is bounded. Since $\varphi_{x^*}$ is bounded and onto, by the open mapping principle there exists a constant $K = K_{x^*} > 0$ such that for every $h \in L_1(\mu)$, there is a solution $f \in L_1(\mu, X)$ of $\varphi_{x^*} f = h$, with $\|f\| \leq K.\|h\|$. From (2.7) we deduce that $\|V(h)\| \leq \|x^*\| \|T\| \|f\| \leq \|x^*\| \|T\| K \|h\|$, which proves that $V$ is bounded.
. It remains to prove (2.6). For $f \in L_1(\mu, X)$ and $x^* \in X^*$, we have $h = \varphi_{x^*} f \in L_1(\mu)$, and (2.7) gives $V(h) = V(\varphi_{x^*} f) = x^* T f$. Since $f$ and $x^*$ are arbitrary, (2.6) follows. Uniqueness is clear from (2.6) since $\varphi_{x^*}$ is onto.■

As a consequence of the preceding theorem let us note:

**2.8. Theorem:** *There is an isometric isomorphism between the Banach space* $\mathfrak{D}$ *and the topological dual* $L_1^*(\mu)$ *of* $L_1(\mu)$*, for each non trivial Banach space* $X$.

**Proof:** Define the operator $\Psi : \mathfrak{D} \to L_1^*(\mu)$ by: $T \in \mathfrak{D}$, $\Psi(T) = V$, where $V$ is the unique bounded functional on $L_1(\mu)$ attached to $T$ by **theorem 2.5**. It is not difficult to see that $\Psi$ is linear. We have to show that $\Psi$ is an isometry, that is, $\|V\| = \|T\|$ if $\Psi(T) = V$. First we prove the estimation

$$\|V\| = Sup\{\|V \circ \varphi_{x^*}\| : x^* \in X^*, \|x^*\| \leq 1\} \tag{2.9}$$

We have $\|V \circ \varphi_{x^*}\| \leq \|V\| \|\varphi_{x^*}\| = \|V\| \|x^*\|$, since $\|\varphi_{x^*}\| = \|x^*\|$ by $2.2(a)$. So we deduce $\|V \circ \varphi_{x^*}\| \leq \|V\|$, for all $x^* \in X^*$, with $\|x^*\| \leq 1$. Hence $Sup\{\|V \circ \varphi_{x^*}\| : x^* \in X^*, \|x^*\| \leq 1\} \leq \|V\|$. But $V \in L_1^*(\mu)$, consequently for each $\varepsilon > 0$ there is $h \in L_1(\mu)$ such that $\|h\|_1 \leq 1$ and $\|V\| - \varepsilon < |V(h)| \leq \|V\|$. Now let $y^* \in X^*$ as in $2.2(c)$ and choose $f \in L_1(\mu, X)$, such that $\|f\|_1 = \|h\|_1$ and $\varphi_{y^*} f = h$. Then $\|f\|_1 \leq 1$ and $|V \circ \varphi_{y^*}(f)| = |V(h)| \leq \|V \circ \varphi_{y^*}\| \|f\|_1$. Thus $|V(h)| \leq \|V \circ \varphi_{y^*}\| \leq Sup\{\|V \circ \varphi_{x^*}\| : x^* \in X^*, \|x^*\| \leq 1\}$. From the choice of $h$ we get $\|V\| - \varepsilon \leq Sup\{\|V \circ \varphi_{x^*}\| : x^* \in X^*, \|x^*\| \leq 1\}$. Letting $\varepsilon \downarrow 0$, we obtain $\|V\| \leq Sup\{\|V \circ \varphi_{x^*}\| : x^* \in X^*, \|x^*\| \leq 1\}$. So (2.9) is proved. To finish the norm equality $\|V\| = \|T\|$, we appeal to formula (2.6) and conclude: $\|V\| = Sup\{\|V \circ \varphi_{x^*}\| : x^* \in X^*, \|x^*\| \leq 1\} = Sup\{\|x^* \circ T\| : x^* \in X^*, \|x^*\| \leq 1\} = \|T\|$. To achieve the proof it remains to prove that $\Psi$ is onto. If $V \in L_1^*(\mu)$, then by the Riesz duality theorem, there is a unique $g \in L_\infty(\mu)$ such that $V(h) = \int_S hg\, d\mu$, for all $h \in L_1(\mu)$. Consider the operator $T_g$ on $L_1(\mu, X)$ given by formula (1.5). We have $T_g \in \mathfrak{D}$ and it is straightforward that $V$ and

$T_g$ are linked by equation (2.6). So from the definition of the operator $\Psi$ we deduce that $\Psi(T_g) = V$.■

Since $L_1^*(\mu)$ is isometrically isomorphic to $L_\infty(\mu)$, we deduce the following corollary:

**corollary:** *The class $\mathfrak{D}$ is isometrically isomorphic to $L_\infty(\mu)$. In other words, a bounded operator $T : L_1(\mu, X) \to X$ is in $\mathfrak{D}$ iff there is a unique $g \in L_\infty(\mu)$ such that $T = T_g$ and in this case $\|T\| = \|g\|_\infty$.*

Now we turn to another description of the class $\mathfrak{D}$, namely by a space of measures. This will be achieved via integrals with respect to operator valued measures.

## 3 Operator valued measures representing the class $\mathfrak{D}$

**3.1.** The integration process we shall deal with in this section is performed by an operator valued additive set function $G : \mathcal{F} \to \mathcal{L}(X, E)$, where $\mathcal{L}(X, E)$ is the space of linear bounded operators of the Banach space $X$ into the Banach space $E$. The integral will be defined for measurable functions $f : S \to X$, under the assumption that $G$ is additive and with finite semivariarion. Let us recall that semivariation means the set function $\widetilde{G}$ on $\mathcal{F}$ given by $\widetilde{G}(B) = Sup \left\| \sum_i G(A_i).x_i \right\|$, where $B \in \mathcal{F}$, and the supremum taken over all finite partitions $\{A_i\}$ of $B$ in $\mathcal{F}$ and all finite systems of vectors $\{x_i\}$ in $X$, with $\|x_i\| \leq 1\ \forall i$. The function $G$ is said to be of finite semivariation if $\widetilde{G}(B)$ is finite for all $B \in \mathcal{F}$. A simple measurable function $s$ on $S$ with values in the Banach space $X$ is a function of the form $s(\bullet) = \sum_i 1_{A_i}(\bullet).x_i$, where $\{A_i\}$ is a finite partition of $S$ in $\mathcal{F}$, and $\{x_i\}$ is a finite system of vectors in $X$. The symbol $1_{A_i}$ means the characteristic function of the set $A_i$. A function $f : S \to X$ is said to be measurable if there is a sequence $s_n$ of measurable simple functions converging uniformly to $f$ on $S$. If we denote by $\mathcal{I}$ and $\mathcal{M}$ the sets of simple functions and measurable functions, respectively then $\mathcal{I}$ and $\mathcal{M}$ are subspaces of the Banach space of all bounded functions $f : S \to X$, with supremum norm. Moreover $\mathcal{I}$ is dense in $\mathcal{M}$.

We define the integral of the simple function $s(\bullet) = \sum_i 1_{A_i}(\bullet).x_i$ over the set $B \in \mathcal{F}$, with respect to $G$ by:

$$(3.2) \qquad \int_B s\ dG = \sum_i G(A_i \cap B).x_i$$

It is easy to check that the integral is well defined and satisfies:

$$(3.3) \qquad \left\| \int_B s dG \right\| \leq \|s\| .\widetilde{G}(B)$$

($\|s\|$ = supremum norm)
Let us observe that estimation (3.3) implies that the linear operator $U_B : \mathcal{I} \rightarrow E$, with $U_B(s) = \int_B s\, dG$ is bounded. So we can extend it in a unique manner to a bounded operator on the closure $\mathcal{M}$ of $\mathcal{I}$. This extension will be our integration process on the space $\mathcal{M}$ of measurable functions.We shall denote it also by $U_B$ with $U_B = U$ if $B = S$. Note that if $f \in \mathcal{M}$ and if $s_n$ is a sequence in $\mathcal{I}$ such that $\|f - s_n\| \rightarrow 0$ then the integral of $f$ is given by:

$$(3.4) \qquad U_B(f) = \int_B f\ dG = \lim_n \int_B s_n\ dG$$

By (3.3) the integral (3.4) does not depend on the sequence $s_n$ chosen converging to the function $f$. This simple integration process will be sufficient for our purpose. The outstanding facts are summarized in the following:

**3.5 Theorem:** *Let $G$ be an additive $\mathcal{L}(X, E)$-valued set function with finite semivariation on $\mathcal{F}$.Then:*
($a$) *The integral $\int_B f\ dG$ is linear in $f \in \mathcal{M}$ and satisfies:*

$$(3.6) \qquad \widetilde{G}(B) = Sup \left\{ \left\| \int_B f\ dG \right\|, \|f\| \leq 1, \quad f \in \mathcal{M} \right\}$$

*in other words the operator $U_B : \mathcal{M} \rightarrow E$ given by $U_B(f) = \int_B f\ dG$ is bounded with norm $\|U_B\| = \widetilde{G}(B)$, for each $B \in \mathcal{F}$. Conversely:*
($b$) *Let $U : \mathcal{M} \rightarrow E$ be a bounded operator. Then there is a unique additive set function $G : \mathcal{F} \rightarrow \mathcal{L}(X, E)$, with finite semivariation such that:*

$$(3.7) \qquad \forall\ f \in \mathcal{M}, \ \forall\ B \in \mathcal{F}, \quad U(f.1_B) = \int_B f\ dG$$

($c$) *Let $\Lambda : E \rightarrow Y$ be a bounded operator from $E$ into the Banach space $Y$. Let us define $\Lambda G : \mathcal{F} \rightarrow \mathcal{L}(X, Y)$ by $(\Lambda G)(B)x = \Lambda(G(B)x)$, $B \in \mathcal{F}$, $x \in X$. Then $\Lambda G$ is an additive $\mathcal{L}(X, Y)$-valued set function with finite semivariation and we have:*

$$(3.8) \qquad \forall\ f \in \mathcal{M}, \quad \int_S f\ d\Lambda G = \Lambda \left( \int_S f dG \right)$$

**Proof:** ($a$) To prove (3.6) start with $f$ simple and use (3.2) and the definition of $\widetilde{G}(B)$. For general $f$ use (3.4).
($b$) Define $G : \mathcal{F} \rightarrow \mathcal{L}(X, E)$ by $G(B).x = U(1_B.x)$, for $B \in \mathcal{F}$, and $x \in X$. Then $G$ is additive since $U$ is linear and $G$ is $\mathcal{L}(X, E)$-valued because $U$ is bounded. Now (3.7) is easily checked by (3.2) and (3.4).
($c$) To prove (3.8) start with $f$ simple and use the definition of $\Lambda G$, then apply (3.4), ( recall that the operator $\Lambda$ is bounded).■

Actually, part $(b)$ of this theorem is an integral representation of a bounded operator $U$ on the space $\mathcal{M}$ by means of an $\mathcal{L}(X,E)$-valued set function $G$ on $\mathcal{F}$.

The next step is to extend the preceding integration process from $\mathcal{M}$ to the space $L_1(\mu, X)$. The reader should observe that the space $\mathcal{M}$ is contained in $L_1(\mu, X)$, because functions in $\mathcal{M}$ are bounded and $\mu$ is a finite measure. The extension of the integral (3.4) from $\mathcal{M}$ to $L_1(\mu, X)$ will be achieved under the additional assumption that $\|G(A)\| \leq k.\mu(A)$ for some constant $k > 0$ and all $A \in \mathcal{F}$.

**3.9 Theorem:** *Let $G$ be an additive $\mathcal{L}(X,E)$-valued set function with finite semivariation on $\mathcal{F}$. Assume that:*

$$\text{(3.10)} \qquad \|G(A)\| \leq k.\mu(A)$$

*for some constant $k > 0$ and all $A \in \mathcal{F}$. Then we have:*
*$(a)$ The integral (3.4) is a linear operator from $\mathcal{M}$ to $E$ which is continuous with the $L_1(\mu, X)-$topology on $\mathcal{M}$ and satisfies:*

$$\text{(3.11)} \qquad \forall f \in \mathcal{M}, \quad \left\| \int_S f dG \right\| \leq k \int_S \|f\| \, d\mu$$

*$(b)$ The integral $\int_S f \, dG$, $f \in \mathcal{M}$, admits a unique extension to $L_1(\mu, X)$, still denoted by $\int_S f dG$, such that:*

$$\text{(3.12)} \qquad \forall f \in L_1(\mu, X), \quad \left\| \int_S f dG \right\| \leq k \int_S \|f\| \, d\mu$$

*$(c)$ The operator $f \to \int_S f dG$ is linear and bounded from $L_1(\mu, X)$ to $E$.*

**Proof:** $(a)$ Let $s(\bullet) = \sum_i 1_{A_i}(\bullet).x_i$ be a simple measurable function with values in $X$. From (3.10) we deduce $\left\| \int_S s \, dG \right\| = \left\| \sum_i G(A_i).x_i \right\| \leq \sum_i \|G(A_i)\| . \|x_i\| \leq \sum_i k\mu(A_i) . \|x_i\| = k \sum_i \mu(A_i) . \|x_i\| = k \int_S \|s\| \, d\mu$. So (3.11) is true for every $s \in \mathcal{I}$. Now if $f \in \mathcal{M}$, let $s_n \in \mathcal{I}$ be such that $s_n \to f$ uniformly on $S$. As $\mu$ is finite we deduce that $\int_S \|f - s_n\| \, d\mu \to 0$ and so $\int_S \|s_n\| \, d\mu \to \int_S \|f\| \, d\mu$. But $\left\| \int_S s_n dG \right\| \to \left\| \int_S f dG \right\|$ by (3.4). From the estimation above we know that $\left\| \int_S s_n \, dG \right\| \leq k \int_S \|s_n\| \, d\mu$, for all $n$. Letting $n \to \infty$ the validity of (3.11) follows. Hence the continuity of the operator $f \to \int_S f dG$ with the $L_1(\mu, X)-$topology on the space $\mathcal{M}$. Next to prove $(b)$, we shall construct an $E-$valued integration process on $L_1(\mu, X)$ with the set function $G$, that coincides with the integral (3.4) on $\mathcal{M}$. This will be the desired extension. Recall that the integral $\int_S s \, dG$, for $s$ simple, has been defined by formula (3.2). Now if $f \in L_1(\mu, X)$, there exist a sequence $s_n \in \mathcal{I}$ such that $\int_S \|f - s_n\| \, d\mu \to 0$.

By(3.11) the sequence $\int_S s_n dG$ is fundamental in the Banach space $E$, so the limit $lim_n \int_S s_n dG$ exists in $E$ and it is easy to check that this limit is independent of the choice of the sequence $s_n$ converging to $f$ in $L_1(\mu, X)$. So we can define:

$$(3.13) \qquad f \in L_1(\mu, X), \qquad \int_S f dG = lim_n \int_S s_n dG$$

where $s_n$ is any sequence in $\mathcal{I}$ converging to $f$ in the $L_1(\mu, X)$ sense.
Now if $f$ is a function in $\mathcal{M}$, every sequence $s_n \in \mathcal{I}$ which converges uniformly to $f$, converges also in the $L_1(\mu, X)$ sense. So the integrals (3.4) and (3.13) are the same for such $f$ and this proves that (3.13) is an extension of (3.4). To see the inequality (3.12), let $s_n \in \mathcal{I}$ converging in $L_1(\mu, X)$ to the function $f \in L_1(\mu, X)$. By (3.11) we have $\left\| \int_S s_n dG \right\| \leq k \int_S \|s_n\| d\mu$, for all $n$. Taking limits for both sides we get (3.12) from which uniqueness of the extension follows.Part $(c)$ is clear.■

As a converse let us point out the following

**3.10 Theorem:** *Let $T : L_1(\mu, X) \to E$ be a bounded operator from $L_1(\mu, X)$ to $E$. Then there exists a unique set function $G : \mathcal{F} \to \mathcal{L}(X, E)$ with finite semivariation satisfying* (3.10), *with the constant $k = \|T\|$ and such that*:

$$(3.14) \qquad f \in L_1(\mu, X), \qquad Tf = \int_S f dG$$

*Moreover $G$ is $\sigma-$additive in the uniform topology of $\mathcal{L}(X, E)$.*

**Proof:** Define $G$ on $\mathcal{F}$ by the formula:

$$(3.15) \qquad A \in \mathcal{F},\ x \in X \qquad G(A).x = T(1_A(\bullet).x)$$

It is clear that $G(A)$ is linear on $X$ for each $A \in \mathcal{F}$ and we have $\|G(A).x\| = \|T(1_A(\bullet).x)\| \leq \|T\|.\mu(A).\|x\|$. So we deduce that the function $G$ sends $\mathcal{F}$ to $\mathcal{L}(X, E)$ and satisfies $\|G(A)\| \leq \|T\|.\mu(A)$, hence the validity of (3.10) with $k = \|T\|$. On the other hand (3.14) is easily checked from (3.15) for simple functions by linearity, and then extended to arbitrary $f \in L_1(\mu, X)$, by the apropriate limiting process. Finally to get the $\sigma-$additivity of $G$, let $A_n$ be a sequence in $\mathcal{F}$ with $A_n \searrow \phi$, then $\mu(A_n) \to 0$ and since $\|G(A_n)\| \leq \|T\|.\mu(A_n)$ for all $n$, we obtain $G(A_n) \to 0$ in the uniform topology of $\mathcal{L}(X, E)$, whence the $\sigma-$additivity of $G$.■

Now we consider operators $T$ in the class $\mathfrak{D}$. We prove that the operator valued function $G$ attached to an operator $T \in \mathfrak{D}$, according to (3.15), allows an interesting characterization of such operators.

**3.16 Theorem:** *Let $T : L_1(\mu, X) \to X$ be a bounded operator on $L_1(\mu, X)$ into $X$. Then $T$ is in the class $\mathfrak{D}$ if and only if the operator valued function attached to it according to* (3.15) *is of the form:*

$$(3.17) \qquad A \in \mathcal{F},\ x \in X \qquad G(A).(\bullet) = \lambda(A).I(\bullet)$$

*where $\lambda$ is a bounded measure absolutely continuous with respect to $\mu$, and $I$ is identity operator of $X$.*

**Proof:** If $T$ is in the class $\mathfrak{D}$, then by the corollary of theorem (2.8) there is a unique $g \in L_\infty(\mu)$ such that $T = T_g$, that is for all $f \in L_1(\mu, X)$, $Tf = \int_S fg\ d\mu$. On the other hand we have from (3.14), $Tf = \int_S fdG$ with $G$ given by (3.15). So taking $f = 1_A(\bullet).x$, for $A \in \mathcal{F}$, $x \in X$, in the two preceding expressions of $Tf$, we get $G(A).x = \left(\int_A g.x\ d\mu\right) = \left(\int_A g\ d\mu\right).x$. Hence the validity of (3.17) with $\lambda(A) = \int_A g\ d\mu$. Since $\mu$ is finite, the function $g$ is in $L_1(\mu)$ and then it is clear that $\lambda$ is a bounded measure absolutely continuous with respect to $\mu$. Now suppose that the operator valued function attached to $T$ according to (3.15) is of the form: $G(A).x = \lambda(A).x$, with $\lambda$ a bounded measure absolutely continuous with respect to $\mu$. So we can write $\lambda(A) = \int_A g\ d\mu$, $A \in \mathcal{F}$, for some unique $g \in L_1(\mu)$. Actually the function $g$ belongs to $L_\infty(\mu)$. Indeed by (3.15), $G(A).x = T(1_A(\bullet).x)$ and we deduce that $\|G(A).x\| = \left|\left(\int_A gd\mu\right)\right|.\|x\| \leq \|T\|\mu(A)\|x\|$, which implies $\left|\left(\int_A gd\mu\right)\right| \leq \|T\|\mu(A)$, for all $A \in \mathcal{F}$. Consequently $\|g\|_\infty \leq \|T\|$, that is $g \in L_\infty(\mu)$. Now let us write the formula $G(A).x = \lambda(A).x$ as $\int_S 1_A.xdG = \int_S g.1_A.xd\mu$, and extend it by linearity to$\int_S sdG = \int_S g.sd\mu$, for $s$ simple in $L_1(\mu, X)$. If $f \in L_1(\mu, X)$, let $s_n$ be a sequence of simple functions converging to $f$ in $L_1(\mu, X)$. Then $g.s_n$ converges to $g.f$ in $L_1(\mu, X)$, since $g \in L_\infty(\mu)$, so we deduce that $\int_S g.s_n\ d\mu$ goes to $\int_S g.f\ d\mu$. But $\int_S s_n\ dG = \int_S g.s_n\ d\mu$, for all $n$ and by (3.13), $\int_S fdG = lim_n \int_S s_n dG$, consequently $\int_S g.f\ d\mu = \int_S fdG$, for all $f \in L_1(\mu, X)$. But from (3.14) we have, $Tf = \int_S fdG$ for $f \in L_1(\mu, X)$, thus $Tf = \int_S g.f\ d\mu = T_g f$, that is $T \in \mathfrak{D}$.■

# Part 2.

## Bochner integral in locally convex spaces

Let $X$ be a locally convex Hausdorff space, whose topology is generated by a family $\{p_\alpha\}$ of continuous seminorms. We assume that $\{p_\alpha\}$ is separating, this means that for each nonzero $x \in X$ there is a $p_\alpha$ such that $p_\alpha(x) \neq 0$. Moreover we assume that $X$ is sequentially complete, that is, every Cauchy sequence in $X$ is convergent. For all details on such spaces, the reader is referred to [13], especially the sections $1.25, 1.36, 1.37$ there. The construction of the Bochner integral we give in this context is, as far as we know, new. (for other approachs see [1, 5, 14]). On the space $L_1(\mu, X)$ of Bochner integrable functions we define a family of separating seminorms that make this space locally convex. Finally we introduce a special class of bounded operators from $L_1(\mu, X)$ into $X$ whose structure is , in many respects, similar to some well known operators from $L_1(\mu)$ into $\mathbb{R}$.

For the needs of measurability and integration, we fix an abstract measure space $(S, \mathcal{F}, \mu)$, where $\mathcal{F}$ is a $\sigma-$field on the set $S$ and $\mu$ a finite positive measure on $\mathcal{F}$.

## 1. Measurability

**1.1. Definition:** A function $f : S \longrightarrow X$ is called elementary if its range $f(S)$ is finite.
If we put $f(S) = \{x_1, x_2, ..., x_n\}$ and $A_j = \{s : f(s) = x_j\}$ then the sets $A_j$ form a partition of $S$ and we can write $f$ in the consolidated form
$f(\bullet) = \sum\limits_{j=1}^{n} x_j\, 1_{A_j}(\bullet)$, where $1_{A_j}$ is the characteristic function of the set $A_j$.

**1.2. Definition:** An elementary function $f(\bullet) = \sum\limits_{j=1}^{n} x_j\, 1_{A_j}(\bullet)$ is measurable if we have $A_j \in \mathcal{F}$ for every $j$. We denote by $\mathcal{E}(X)$ the set of all elementary measurable functions $f : S \longrightarrow X$. Then we have:

**1.3 Proposition:** $\mathcal{E}(X)$ is a vector space on $\mathbb{R}$.

**Proof:** Let $f, g$ be in $\mathcal{E}(X)$ and $\lambda \in \mathbb{R}$. Put $f(\bullet) = \sum\limits_{n} x_n\, 1_{A_n}(\bullet)$
$g(\bullet) = \sum\limits_{m} y_m\, 1_{B_m}(\bullet)$, then $(f+g)(\bullet) = \sum\limits_{n,m} (x_n + y_m)\ 1_{A_n \cap B_m}(\bullet)$ and $(\lambda f)(\bullet) = \sum\limits_{n} \lambda x_n\, 1_{A_n}(\bullet)$.■

**1.4. Remark:** Let $T$ be any mapping from $X$ into $Y$.
If $f(\bullet) = \sum\limits_{n} x_n\, 1_{A_n}(\bullet)$ then $(T \circ f)(\bullet) = \sum\limits_{n} T(x_n)\ 1_{A_n}(\bullet)$.

**1.5. Definition:** A function $f : S \longrightarrow X$ is measurable if there is a sequence $(f_n)$ of elementary measurable functions such that:

$$\underset{n}{Lim}\, p_\alpha (f_n - f) = 0$$

for each $p_\alpha$.
This means that for each $s \in S$, each $\epsilon > 0$, and each $p_\alpha$, there is $N = N_{s,\epsilon,p_\alpha} \geq 1$ such that $\forall n \geq N$, $p_\alpha (f_n(s) - f(s)) < \epsilon$.

**1.6. Proposition:** The set $M(X)$ of all measurable functions $f : S \longrightarrow X$ is a vector space on $\mathbb{R}$.

**Proof:** Let $f, g$ be in $M(X)$ and let $f_n, g_n$ be sequences of elementary functions such that $p_\alpha (f_n - f) \longrightarrow 0$ and $p_\alpha (g_n - g) \longrightarrow 0$, for each $p_\alpha$. Then we have $p_\alpha ((f_n + g_n) - (f + g)) \leq p_\alpha (f_n - f) + p_\alpha (g_n - g)$, so the sequence of elementary functions $f_n + g_n$ gives the measurability of $f + g$.
Likewise for $\lambda \in \mathbb{R}$,we have $p_\alpha (\lambda f_n - \lambda f) = |\lambda|\, p_\alpha (f_n - f) \longrightarrow 0$, which gives $\lambda f \in M(X)$.■

## 2. Bochner integration

**2.1. Definition:** Let $f(\bullet) = \sum_{j=1}^{n} x_j \, 1_{A_j}(\bullet)$ be an elementary measurable function. We define the integral of $f$ by the vector $\int_S f\, d\mu \in X$ :

$$\int_S f\, d\mu = \sum_{j=1}^{n} \mu(A_j) .x_j$$

Since $\mu$ is finite this integral is well defined.

**2.2. Proposition:** $(a)$The integral is linear from $\mathcal{E}(X)$ into $X$.

$(b)$. For every $f \in \mathcal{E}(X)$ and every $p_\alpha$ we have

$$p_\alpha\left(\int_S f\, d\mu\right) \leq \int_S p_\alpha(f)\ d\mu$$

where $p_\alpha(f)$ is the positive elementary function given by

$p_\alpha(f)(\bullet) = \sum_{j=1}^{n} p_\alpha(x_j)\ 1_{A_j}(\bullet)$ whose integral is $\int_S p_\alpha(f)\ d\mu = \sum_{j=1}^{n} p_\alpha(x_j)\ \mu(A_j)$.

**Proof:** $(a)$ Put $f(\bullet) = \sum_{j=1}^{n} x_j \, 1_{A_j}(\bullet)$, $g(\bullet) = \sum_{k=1}^{m} y_k \, 1_{B_k}(\bullet)$

then $(f+g)(\bullet) = \sum_{j,k} (x_j + y_k)\ 1_{A_j \cap B_k}(\bullet)$ and $(\lambda f)(\bullet) = \sum_{j} \lambda x_j \, 1_{A_j}(\bullet)$.

This yields $\int_S (f+g)\ d\mu \sum_{1\leq k\leq m} \sum_{1\leq j\leq n} (x_j + y_k)\ \mu(A_j \cap B_k) = \sum_{j=1}^{n} \mu(A_j) .x_j + \sum_{k=1}^{m} \mu(B_k) .y_k = \int_S f\, d\mu + \int_S g\, d\mu$

Likewise we can prove that $\int_S \lambda.f\, d\mu = \lambda. \int_S f\, d\mu$ for $\lambda \in \mathbb{R}$.

$(b)$ We have $p_\alpha\left(\int_S f\, d\mu\right) = p_\alpha\left(\sum_{j=1}^{n} \mu(A_j) .x_j\right) \leq \sum_{j=1}^{n} \mu(A_j) .p_\alpha(x_j) = \int_S p_\alpha(f)\ d\mu.\blacksquare$.

**2.3. Proposition:** Let $T : X \longrightarrow Y$ be a linear operator from $X$ into a locally convex space $Y$.

Let $f \in \mathcal{E}(X)$, then $T \circ f \in \mathcal{E}(Y)$ and we have:

$$T\left(\int_S f\, d\mu\right) = \int_S T \circ f\, d\mu.$$

**Proof:** Let $f(\bullet) = \sum_{j=1}^{n} x_j \, 1_{A_j}(\bullet)$, with $\int_S f\, d\mu = \sum_{j=1} \mu(A_j) .x_j$, then $(T \circ f)(\bullet) = \sum_{j=1}^{n} T(x_j)\ 1_{A_j}(\bullet)$ and

$\int_S T \circ f\, d\mu = \sum_{j=1}^{n} \mu(A_j) .T(x_j) = T\left(\sum_{j=1}^{n} \mu(A_j) .x_j\right)$, by the linearity of $T$, so we deduce that $T\left(\int_S f\, d\mu\right) = \int_S T \circ f\, d\mu.\blacksquare$.

**2.4. Definition:** A measurable function $f : S \longrightarrow X$ is Bochner integrable if there is a sequence $f_n$ of elementary measurable functions such that for each $p_\alpha$, $\underset{n}{Lim}\; p_\alpha(f_n - f) = 0$ uniformly on $S$. Since the measure $\mu$ is assumed finite, this implies that $\underset{n}{Lim}\; \int_S p_\alpha(f_n - f)\, d\mu = 0$, for each $p_\alpha$.
To define the Bochner integral of $f$ let us observe that if $f_n$ is such a sequence of elementary functions we have:
$\int_S p_\alpha(f_n - f_m)\, d\mu \leq \int_S p_\alpha(f_n - f)\, d\mu + \int_S p_\alpha(f_m - f)\, d\mu$.
So $\underset{n,m}{Lim}\; \int_S p_\alpha(f_n - f_m)\, d\mu = 0$. But $p_\alpha \int_S (f_n - f_m)\, d\mu \leq \int_S p_\alpha(f_n - f_m)\, d\mu$ by **Proposition 2.2**$(b)$, this implies that the sequence of integrals $\int_S f_n\, d\mu$ is Cauchy. As the space $X$ is assumed sequentially complete, $\int_S f_n\, d\mu$ converges. This allows to define the Bochner integral of $f$ by the vector:

$$\int_S f\, d\mu = \underset{n}{Lim}\; \int_S f_n\, d\mu.$$

If $g_n$ is another sequence of elementary functions such that $p_\alpha(g_n - f) \longrightarrow 0$ uniformly on $S$, it is easy to check, from the continuity of $p_\alpha$ that $\underset{n}{Lim}\; \int_S f_n\, d\mu = \underset{n}{Lim}\; \int_S g_n\, d\mu$, so the Bochner integral $\int_S f\, d\mu$ is well defined.

In the sequel we will denote by $L_1(\mu, X)$ the set of all Bochner integrable functions $f : S \longrightarrow X$, where as usual two integrable functions are considered as identical if they are equal $\mu-$almost everywhere.

**2.5. Proposition:** $L_1(\mu, X)$ is a vector space on $\mathbb{R}$ and we have:
$(a)$. The integral as defined is linear from $L_1(\mu, X)$ into $X$.
$(b)$. For every $f \in L_1(\mu, X)$ and every $p_\alpha$ we have
$p_\alpha\left(\int_S f\, d\mu\right) \leq \int_S p_\alpha(f)\; d\mu$

**Proof:**
$(a)$ Let $f, g$ be in $L_1(\mu, X)$ and let $f_n, g_n$ be in $\mathcal{E}(X)$ such that $p_\alpha(f_n - f) \longrightarrow 0$ and $p_\alpha(g_n - f) \longrightarrow 0$, uniformly on $S$. Since we have $p_\alpha((f+g) - (f_n + g_n)) \leq p_\alpha(f_n - f) + p_\alpha(g_n - f) \longrightarrow 0$, it follows that $p_\alpha((f+g) - (f_n + g_n)) \longrightarrow 0$ uniformly on $S$. This yields
$\int_S (f+g)\; d\mu = \underset{n}{Lim}\; \int_S (f_n + g_n)\; d\mu = \underset{n}{Lim}\; \int_S f_n\; d\mu + \underset{n}{Lim}\; \int_S g_n\; d\mu = \int_S f\, d\mu + \int_S g\, d\mu$. Likewise we have $\int_S \lambda.f\, d\mu = \lambda. \int_S f\, d\mu$.

$(b)$ Let $f_n$ be in $\mathcal{E}(X)$ defining $\int_S f\, d\mu$. By proposition **2.2**$(b)$ $p_\alpha\left(\int_S f_n\, d\mu\right) \leq \int_S p_\alpha(f_n)\; d\mu$ for all $n$. This implies $p_\alpha\left(\int_S f\, d\mu\right) = p_\alpha\left(\underset{n}{Lim} \int_S f_n\, d\mu\right) = \left(\underset{n}{Lim}\; p_\alpha\left(\int_S f_n\, d\mu\right)\right) \leq \liminf_n \; \int_S p_\alpha(f_n)\; d\mu \leq \liminf_n \left(\int_S p_\alpha(f_n - f)\, d\mu + \int_S p_\alpha(f)\; d\mu\right) = \int_S p_\alpha(f)\; d\mu.\blacksquare$.

**2.6. Proposition:** Let $T : X \longrightarrow Y$ be a linear continuous operator from $X$ into a locally convex space $Y$.
Let $f \in L_1(\mu, X)$, then $T \circ f \in L_1(\mu, Y)$ and we have:

$$T\left(\int_S f\, d\mu\right) = \int_S T \circ f\, d\mu.$$

**Proof:** Let $f_n$ be in $\mathcal{E}(X)$ defining $\int_S f\,d\mu$, i.e $\underset{n}{Lim}\ p_\alpha(f_n - f) \longrightarrow 0$ uniformly on $S$. By the continuity of $T$, if $q$ is a seminorm on $Y$ there is a seminorm $p_\alpha$ on $X$ such that $q(Tx) \leq p_\alpha(x)$, for every $x \in X$. It follows that $q(Tf_n - Tf) = qT(f_n - f)\ d\mu \leq p_\alpha(f_n - f) \to 0$ uniformly on $S$. We deduce that $q(Tf_n - Tf) \longrightarrow 0$ uniformly on $S$ for each $q$. So the sequence $Tf_n$, which is in $\mathcal{E}(Y)$ by Proposition **2.3,** is defining the integral of $Tf$ by $\int_S Tf\,d\mu = \underset{n}{Lim} \int_S Tf_n d\mu$. By **Proposition 2.3** once more we have
$\int_S Tf_n d\mu = T\int_S f_n d\mu$ for all $n$.
Since $\underset{n}{Lim}\int_S f_n d\mu = \int_S f\,d\mu$, we get $\underset{n}{Lim}\int_S Tf_n d\mu = T\left(\int_S f\,d\mu\right)$, by the continuity of $T$. this gives $T\left(\int_S f\,d\mu\right) = \int_S T\circ f\,d\mu$.■.

## 3. Bounded operators on $\mathbf{L}_1(\mu, X)$

First we start by defining on $L_1(\mu, X)$ a family $\{\widetilde{p_\alpha}\}$ of continuous seminorms which will make $L_1(\mu, X)$ a locally convex space.

Let us observe that for each $p_\alpha$, we have $p_\alpha(f)$ bounded on $S$ if $f \in L_1(\mu, X)$.To see this let $f_n$ be in $\mathcal{E}(X)$ defining $\int_S f\,d\mu$, i.e $Lim p_\alpha(f_n - f) = 0$ uniformly on $S$, (**Definition 2.4**), so if $\epsilon > 0$, there is $N \geq 1$ such that $|p_\alpha(f) - p_\alpha(f_N)| \leq p_\alpha(f_N - f) < \epsilon$ uniformly on $S$. We deduce that $p_\alpha(f) < \epsilon + p_\alpha(f_N)$ on $S$ and $p_\alpha(f_N)$ is bounded on $S$ since $f_N \in \mathcal{E}(X)$.

Now define $\widetilde{p_\alpha}$ on $L_1(\mu, X)$ by:

$$(3.1) \qquad f \in L_1(\mu, X) \qquad \widetilde{p_\alpha}(f) = \underset{t\in S}{Sup}\ p_\alpha(f(t))$$

Then $\widetilde{p_\alpha}$ is a seminorm on $L_1(\mu, X)$ and the family $\{\widetilde{p_\alpha}\}$ is separating. To see this, let $f$ be in $L_1(\mu, X)$ with $f \neq 0$, that is $f(t) \neq 0$ for some $t \in S$. Since the family $\{p_\alpha\}$ is assumed separating on $X$, there is a $p_\alpha$ such that $p_\alpha(f(t)) > 0$, so that $\widetilde{p_\alpha}(f) > 0$.
Since the family of seminorms $\{\widetilde{p_\alpha}\}$ is separating, it makes $L_1(\mu, X)$ a locally convex space such that each $\widetilde{p_\alpha}$ is continuous ([13], section 1.37).

In what follows we define a special class of bounded operators from $L_1(\mu, X)$ into $X$ which are, in many respects, similar to some well known operators from $L_1(\mu)$ into $\mathbb{R}$. First let us observe:

**3.2. Lemma:** Let $g \in L_\infty(\mu)$, then for every $f \in L_1(\mu, X)$
$g.f \in L_1(\mu, X)$.

**Proof:** Since $g \in L_\infty(\mu)$, there is a sequence $(g_n)$ of simple measurable functions $g_n : S \longrightarrow \mathbb{R}$ converging uniformly to $g$ on $S$. Since $f \in L_1(\mu, X)$, there is a sequence $f_n$ of elementary measurable functions such that for each $p_\alpha$, $\underset{n}{Lim}\ p_\alpha(f_n - f) = 0$ uniformly on $S$. But $g_n.f_n$ is elementary measurable, and we have:

$p_\alpha(g_n.f_n - g.f) = p_\alpha[(g_n.f_n - g_n.f) + (g_n.f - g.f)]$
$\leq |g_n - g|.p_\alpha(f) + |g_n|.p_\alpha(f_n - f)$
$\leq |g_n - g|.\widetilde{p_\alpha}(f) + |g_n|.p_\alpha(f_n - f) \longrightarrow 0,\ n \longrightarrow \infty,$ uniformly on $S$.

Consequently we have $f.g \in L_1(\mu, X)$ .■.

Now we define a class $\{T_g\ ,g \in L_\infty(\mu)\}$ of operators $T_g$, by the following recipe:

**3.3. Definition:** For each fixed $g \in L_\infty(\mu)$, $T_g$ sends $L_1(\mu, X)$ into $X$ by the formula:

$$f \in L_1(\mu, X), \qquad T_g(f) = \int_S fg\, d\mu$$

**3.4. Theorem:.** Let $L_1(\mu, X)$ be endowed with the seminorms $\{\widetilde{p_\alpha}\}$ given by (3.1), and let $X$ be equipped with the seminorms $\{p_\alpha\}$, then the operators $T_g$ are linear and bounded.

**Proof:** The linearity is clear from **2.5** $(a)$. To see boundedness, let $p_\alpha$ be a seminorm on $X$, by **2.5** $(b)$ we have:

$p_\alpha(T_g(f)) = p_\alpha\left(\int_S fg\, d\mu\right) \leq \int_S p_\alpha(fg)\, d\mu$. Since $p_\alpha(fg) = |g|\ p_\alpha(f)$, we deduce that $p_\alpha(T_g(f)) \leq \int_S |g|\ p_\alpha(f)\, d\mu \leq \|g\|_\infty . \widetilde{p_\alpha}(f) . \mu(X)$, which proves that $T_g$ is bounded.■.

In what follows, we quote some properties of the operators $T_g$, whose proof comes from facts about Bochner integral (**2.5-2.6**).We denote by $E^{'}$ the strong dual of the space $E$ :

**3.5. Proposition:** $(a)$ If $\theta \in X^{'}$, then $\theta \circ T_g \in L_1^{'}(\mu, X)$ .

$(b)$ If $\theta \in X^{'}$, then $\theta \circ T_g(f) = \int_S g\, \theta f\, d\mu$, for every $f \in L_1(\mu, X)$ .

$(c)$ If $\theta, \sigma$ are in $X^{'}$, and $\varphi, \psi$ in $L_1(\mu, X)$, then:

$$\theta \circ \varphi = \sigma \circ \psi \Longrightarrow \theta \circ T_g(\varphi) = \sigma \circ T_g(\psi)$$

These properties, especially property $(c)$, lead to the following:

**Open problem:** Let $T : L_1(\mu, X) \longrightarrow X$ be a linear bounded operator from $L_1(\mu, X)$ into $X$ satisfying condition **3.5**$(c)$, that is:

If $\theta, \sigma$ are in $X^{'}$, and $\varphi, \psi$ in $L_1(\mu, X)$, then:

$$\theta \circ \varphi = \sigma \circ \psi \Longrightarrow \theta \circ T(\varphi) = \sigma \circ T(\psi)$$

Does there exist a $g \in L_\infty(\mu)$ such that:

$$T(f) = \int_S fg\, d\mu, \text{ for all } f \in L_1(\mu, X) .$$

**References**

[1] **R. Beckmann and A. Deitmar,** *arXiv 403.320, [math F.A]*, 2014**.**

[2] **J.Diestel, Uhl, J.J** *Vector Measures*,
AMS Math.Surveys 15 1977.

[3] **N.Dinculeanu :** *Vector Measures*, New York USA Pergamon Press 1967

[4] **N.Dunford, J.T.Schwartz :** *Linear operators*, Part 1: General Theory

Wiley classics library.1988

[5] **R. K. Goodrich:** *A Riesz representation theorem,* Proc. Amer. Math. Soc. 1970 24 ,629-636

[6] **E.Hille, R.S.Phillips :***Functional Analysis and Semigroups*, AMS Colloquium.1957

[7] **L.Meziani:***Integral Representation for a class of Vector Valued Operators*,Proc.Amer.Math.Soc.2002, 130, 2067-2077.

[8] **L.Meziani :** *A theorem of Riesz type with Pettis integrals in topological vector spaces,* J. Math. Anal. Appl.,2008. 340/2 817-824,

[9] **L.Meziani, S. M Alsulami:***Weak Integrals and Bounded Operators in TVS* Advances in Pure Mathematics,2013. 3 475-478,

[10] **L.Meziani, Maha Noor Waly and Saleh Almezel**, *Characterizing Linear Bounded Operators via Integral,* International Mathematical Forum,2009 4 , 233-240,

[11] **Laurent Vanderputten :** *Representation of Operators Defined on the Space of Bochner Integrable Functions,* Extracta Mathematicae 2001.16**,** 383-391

[12] **W.Rudin:** *Real and Complex Analysis,* McGraw-Hill. 1970

[13] **W.Rudin:** *Functional Analysis,* McGraw-Hill. 1991

[14].**V.I. Rybakov:** *A Generalization of The Bochner Integral to Locally Convex Spaces,* Tula State Pedagogical Institute. Translated from Matematicheskie Zametki, Vol. 18 No.4 1975.