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# Weakened well-posedness of a hyperbolic characteristic boundary value problem

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**Abstract.** We deal with a hyperbolic characteristic boundary value problem for a Friedrichs symmetrizable system of first order with constant coefficients satisfying a weakened version of the so-called Uniform Kreiss–Lopatinskii (UKL) Condition on the boundary. The boundary value problem is weakly $L^2$ well-posed in the sense that it admits a unique solution satisfying an energy estimate where the failure of the UKL Condition yields a loss of regularity with respect to the data. The proof consists in splitting the original problem into two boundary value problems: a boundary value problem with a strictly dissipative boundary condition and another boundary value problem with a null source term in the interior equations. The $L^2$ solvability has been obtained thanks to a Fourier–Laplace analysis involving the weakened condition on the boundary.

We extend the analysis to an initial boundary value problem on a finite time interval $[0,T]$ by incorporating an arbitrary initial data. Assuming that the UKL Condition holds, we state a $L^2$ well-posedness result in the characteristic case.

**1. Introduction.** In this paper, we consider a boundary value problem (BVP) for a first order system of hyperbolic partial differential equations with constant coefficients. The aim is to study the $L^2$ well-posedness of the problem where the differential operator is assumed to be Friedrichs symmetrizable (Assumption 1.1) and the boundary to be characteristic (Assumption 1.2) and satisfies a weakened version of the so-called Uniform Kreiss–Lopatinskii Condition (Definition 1.1).

We consider the half-space

$$\Omega := \{x = (y, x_d) \in \mathbb{R}^d \colon x_d > 0\}, \quad \text{where} \quad y := (x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1},$$

whose boundary $\partial\Omega = \{x \in \mathbb{R}^d \colon x_d = 0\}$ is identified with $\mathbb{R}^{d-1}$.

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We set $Q = \mathbb{R} \times \Omega$ and $\Sigma = \mathbb{R} \times \partial\Omega$. We are interested in the following boundary value problem:

$$\begin{cases} Lu = \dfrac{\partial}{\partial t}u + \displaystyle\sum_{j=1}^{d} A_j \frac{\partial u}{\partial x_j} = F \quad \text{in } Q, \\ Bu = G \quad \text{on } \Sigma. \end{cases} \tag{1.1}$$

Here $A_1, \dots, A_d$ are real $N \times N$ matrices with constant coefficients defined on $Q$. The unknown $u = u(t,x)$ and the data $F = F(t,x)$ are vector valued functions defined on $Q$ with $N$ components.

$B$ is a given real $p \times N$ constant matrix and $G = G(t,x)$ is a vector valued function defined on $\Sigma$ with $p$ components.

We study the problem (1.1) under the following assumptions:

Assumption 1.1. The operator $L$ is *Friedrichs symmetrizable*, that is, there exists a symmetric positive definite matrix $S$ such that the matrices $SA_j$ for $j = 1, \dots, d$ are also symmetric.

Assumption 1.2. The boundary $\partial\Omega$ is characteristic for the BVP (1.1), which means that the matrix $A_d$ is singular. We assume that if $m$ stands for the dimension of its kernel, the matrix $A_d$ admits block form

$$\begin{pmatrix} 0_m & 0 \\ 0 & a_d \end{pmatrix}, \tag{1.2}$$

where the $(N-m) \times (N-m)$ block $a_d$ is invertible.

Accordingly, we decompose the unknown $u$ as $(u^I, u^{II})^T$, where $u^{II}$, valued in $\mathbb{C}^{N-m}$, stands for the non-characteristic component of $u$.

Assumption 1.3. For any $\xi = (\xi_1, \dots, \xi_{d-1}, \xi_d) =: (\eta, \xi_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ with $\xi \neq 0$, the symbol matrix $\sum_{j=1}^{d} \xi_j A_j := A(\eta) + \xi_d A_d$ admits $\lambda \equiv 0$ as an eigenvalue of multiplicity $m$, and the matrix $A(\eta) = \sum_{j=1}^{d-1} \xi_j A_j$ admits block form

$$A(\eta) = \begin{pmatrix} 0_m & a_{1,2}(\eta) \\ a_{2,1}(\eta) & a_2(\eta) \end{pmatrix}, \quad \eta \in \mathbb{R}^{d-1}. \tag{1.3}$$

Assumption 1.4. The matrix $B$ is a real $p \times N$ matrix of maximal rank $p$ equal to the number of positive eigenvalues of $A_d$ and satisfies $\ker A_d \subset \ker B$, which means that after a linear transformation $B$ has block form

$$B = \begin{pmatrix} 0_{p \times m} & B_2 \end{pmatrix} \tag{1.4}$$

with $B_2$ a $p \times (N-m)$ matrix.

The above boundary value problem is assumed to satisfy a fairly general boundary condition, namely a weakened version of the Kreiss–Lopatinskii

Condition (see e.g. Eller [4], Métivier [9], Benzoni-Gavage–Serre [1]). In order to define this condition, we introduce the frequency variables

$$\zeta = (\tau, \eta),\ \tau = \gamma + i\sigma \in \mathbb{C}_+, \quad \Re e(\tau) = \gamma > 0,\ \Im m(\tau) = \sigma \in \mathbb{R}, \quad \eta \in \mathbb{R}^{d-1},$$

and set

$$\Xi = \{(\tau, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1} \setminus (0,0) : |\tau|^2 + |\eta|^2 = 1,\ \gamma > 0\}.$$

We also introduce $\mathbb{E}_-(\tau, \eta)$, the stable invariant space associated to the singular system

$$(1.5)\qquad (\tau I_N + iA(\eta))\phi + A_d \frac{\partial \phi}{\partial x_d} = 0,$$

obtained after performing the Fourier–Laplace transform with respect to the tangential variables $(t, y)$ for the problem (1.1) with $F \equiv 0$.

The invariant space $\mathbb{E}_-(\tau, \eta)$ is obtained after decoupling the singular system into two systems with respect to the projector onto $\ker A_d$ along the range of $A_d$ (see further details in [1]).

Definition 1.1. We say that the BVP (1.1) satisfies the *s-weakened Kreiss–Lopatinskii Condition* (briefly $s$-WKL Condition) for some $s \geq 0$ if

$$(1.6)\qquad \exists C > 0,\ \forall (\tau, \eta) \in \Xi \quad \big(v \in \mathbb{E}_-(\tau, \eta)) \Rightarrow |A_d v| \leq C\gamma^{-s}|Bv|\big).$$

Assumption 1.5. The BVP (1.1) satisfies the $s$-WKL Condition for some $s \geq 0$.

Let us point out that for $s = 0$, the condition (1.6) is the standard formulation of the Uniform Kreiss–Lopatinskii (UKL) Condition.

Throughout this paper, the functions used may take either scalar, vector or matrix values. We warn the reader that we adopt, here and henceforth, with a slight abuse, the same notations for each type of values.

For a given $r \in \mathbb{R}$, we make use of the usual Sobolev space $H^r(\Sigma)$ of tempered distributions equipped with the family of norms

$$(1.7)\qquad \|u\|^2_{r,\gamma} := \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} (\gamma^2 + \sigma^2 + |\eta|^2)^r |\hat{u}(\sigma, \eta)|^2 \, d\eta \, d\sigma,$$

where $\hat{u}$ is the Fourier transform of $u$ and $\gamma > 0$ is a given parameter.

We also consider the weighted Sobolev space $H^r_\gamma(\Sigma) = e^{\gamma t} H^r(\Sigma)$, the space of tempered distributions $u$ such that $e^{-\gamma t} u \in H^r(\Sigma)$.

We also use the space $L^2(\mathbb{R}_+, H^r(\Sigma))$ equipped with the obvious family of norms

$$(1.8)\qquad |||u|||^2_{r,\gamma} := \int\limits_{\mathbb{R}_+} \|u(\cdot, x_d)\|^2_{r,\gamma} \, dx_d.$$

If $r = 0$, we shall denote $H^0_\gamma(\Sigma) := L^2_\gamma(\Sigma)$ and $L^2\big(\mathbb{R}_+, H^0_\gamma(\Sigma)\big) := L^2_\gamma(Q)$.

The first main result of this paper is the following

Theorem 1.2. *Assume that the above assumptions are fulfilled for some $s \geq 0$. There exists a constant $C > 0$ such that for all $\gamma > 0$ and all $F \in L^2(\mathbb{R}_+, H^s_\gamma(\Sigma))$, $G \in H^s_\gamma(\Sigma)$, the problem* (1.1) *admits a unique solution $u \in L^2_\gamma(Q)$ such that $u^{II}_{|x_d=0} \in L^2_\gamma(\Sigma)$. Furthermore, $u$ satisfies the estimate*

$$\begin{aligned}(1.9)\qquad \gamma\|e^{-\gamma t}u\|^2_{L^2(Q)} + \|e^{-\gamma t}u^{II}_{|x_d=0}\|^2_{L^2(\Sigma)} \\ \leq C\left(\frac{1}{\gamma^{2s+1}}|||e^{-\gamma t}F|||^2_{s,\gamma} + \frac{1}{\gamma^{2s}}\|e^{-\gamma t}G\|^2_{s,\gamma}\right).\end{aligned}$$

In the non-characteristic boundary case, that is, when $\det A_d \neq 0$, assuming that the operator is strictly hyperbolic with $\mathcal{C}^\infty$ coefficients H. O. Kreiss [6] (see also Chazarain–Piriou [2]) was able to provide, under the UKL Condition, an a priori energy estimate by constructing an algebraic tool, known since as the Kreiss symmetrizer, leading to the $L^2$ solvability of the BVP. Majda and Osher [8] and Métivier [10] have extended this result to a class of operators larger than the strictly hyperbolic ones as soon as the symbol of the operator satisfies the so-called block structure condition.

Benzoni-Gavage and Serre [1] have constructed a Kreiss symmetrizer matrix under the UKL Condition for a symmetric hyperbolic system with constant coefficients and characteristic boundary, assuming that the symbol matrix $A(\eta)$ (1.3) has block form

$$A(\eta) = \begin{pmatrix} 0_m & a^T_{2,1}(\eta) \\ a_{2,1}(\eta) & a_2(\eta)\end{pmatrix}, \quad \eta \in \mathbb{R}^{d-1},$$

with $a_2(\eta) \equiv 0$.

These results have the important consequence that under the UKL Condition, one can provide an a priori energy estimate without loss of regularity of the solution with respect to the data.

However, in many examples of physical interest like linear elasticity or Euler equations from fluid dynamics, the UKL Condition breaks down. Recently, some weakened forms of the UKL Condition have been treated. Coulombel and Secchi [3] studied the linear stability of compressible vortex sheet. They were able to provide an a priori energy estimate similar to (1.9) in the case where $s = 1$. Eller [4] has proved under the $s$-WKL Condition a version of Theorem 1.2 in the non-characteristic case where the operator is constantly hyperbolic, and in the symmetric case.

In the present work, following a somewhat different approach, we give another proof by using ideas taken from [8], [3], and [5], and despite the general boundary conditions, we do not invoke specific algebraic tools like Kreiss symmetrizers [1, 2, 6] or other techniques to study the well-posednesss.

The strategy of the proof consists in splitting the original problem into two boundary value problems. In the first one, thanks to the assumption of

Friedrichs symmetrizability, the interior equations are coupled with a chosen strictly dissipative boundary condition with respect to the non-characteristic component of the solution. This type of problem is part of the framework of strictly dissipative hyperbolic BVPs for which we have well known results (see e.g. Lax–Phillips [7], Métivier [9], Benzoni-Gavage–Serre [1]).

In the second BVP, the interior equations have a null source term, but the boundary condition requires taking into account the $s$-WKL Condition. The constant coefficients case allows us to study the well-posedness by means of Fourier–Laplace analysis. The non-invertibility of the boundary matrix $A_d$ yields a singular system of ODEs. To overcome this difficulty, thanks to the block structure of the symbol matrix $A(\xi)$, we reduce the singular system to a non-characteristic system of ODEs with respect to the non-characteristic component after projecting the system onto $\ker A_d$ along the range of $A_d$.

We build an explicit solution of the non-characteristic part of the reduced system involving crucially the $s$-WKL Condition by means of the spectral projector onto the stable invariant subspace of the reduced resolvant matrix. We show finally that the original boundary value problem is weakly well-posed in the sense that it admits a unique $L^2$ solution, but only the trace of the non-characteristic component of the solution on the boundary is square integrable and an energy estimate is provided where the failure of the Uniform Kreiss–Lopatinskii Condition yields a loss of regularity with respect to the data.

Motivated by the conclusions of Theorem 1.2, it would be judicious to extend the analysis to a mixed problem on a finite time interval $[0, T]$ by incorporating an arbitrary initial data to the problem (1.1). We will do this in the particular case where Assumption 1.5 is satisfied for $s = 0$, i.e. in the case where the UKL Condition holds for the system $(L, B)$.

For this purpose, for a given $T > 0$, we set

$$Q_T := [0, T] \times \Omega \quad \text{with the boundary} \quad \Sigma_T := [0, T] \times \partial\Omega.$$

Consider the initial boundary value problem (IBVP)

$$\begin{cases} Lu = \dfrac{\partial}{\partial t}u + \displaystyle\sum_{j=1}^{d} A_j \frac{\partial u}{\partial x_j} = F & \text{in } Q_T, \\ Bu = G & \text{on } \Sigma_T, \\ u_{|t=0} = u_0 & \text{on } \Omega. \end{cases} \tag{1.10}$$

Consider the data $F \in L^2(Q_T)$, $G \in L^2(\Sigma_T)$ and $u_0 \in L^2(\Omega)$. Under the assumptions of Theorem 1.2 with $s = 0$, our second well-posedness result is the following.

THEOREM 1.3. *Assume that the assumptions of Theorem* 1.2 *are fulfilled for* $s = 0$. *Then, for all* $F \in L^2(Q_T)$, $G \in L^2(\Sigma_T)$, $u_0 \in L^2(\Omega)$, *the problem*

(1.10) *admits a unique solution* $u \in L^2(Q_T)$ *such that* $u^{II}_{|x_d=0} \in L^2(\Sigma_T)$. *In addition,* $u \in \mathcal{C}([0,T], L^2(\Omega))$ *and the following estimate holds for all* $t \in [0,T]$ *and* $\gamma > 0$*:*

$$\begin{aligned}(1.11)\quad & \gamma \|e^{-\gamma t} u\|^2_{L^2(Q_t)} + \|e^{-\gamma t} u^{II}_{|x_d=0}\|^2_{L^2(\Sigma_t)} + e^{-2\gamma t}\|u(t)\|^2_{L^2(\Omega)} \\ & \qquad \le C\bigg(\frac{1}{\gamma}\|e^{-\gamma t}F\|^2_{L^2(Q_t)} + \|e^{-\gamma t}G\|^2_{L^2(\Sigma_t)} + \|u_0\|^2_{L^2(\Omega)}\bigg),\end{aligned}$$

The main novelty of this result is that it provides, in the framework of an operator with constant coefficients, semi-strong $L^2$ well-posedness for the IBVP (1.10), according to Lax–Phillips's terminology in [7], and in a certain way extends the statement of [5, Theorem 1.5] to the characteristic case.

The proof of Theorem 1.3 can be achieved by adopting the same strategy as in the proof of Theorem 1.2. It consists in splitting the original IBVP (1.10) into two initial boundary value problems. In the first one, we introduce an auxiliary problem with a strictly dissipative boundary condition, already considered in 2.1 by incorporating the nonzero initial data $u_0$, but with null data for the interior equations.

In the second IBVP, the initial datum has a null source term. The causality principle satisfied by the solution allows us to make use of the results of Theorem 1.2, leading to a well-posedness result for the IBVP (1.10).

The paper is organized as follows. In Section 2, we prove Theorem 1.2. An auxiliary strictly dissipative boundary value problem is studied in Subsection 2.1. Subsection 2.2 is devoted to developing a tangential Fourier–Laplace analysis for a boundary value problem with null source term. The results obtained in those two subsections allow us to conclude the proof of Theorem 1.2 by means of classical tools of harmonic analysis like the Paley–Wiener theorem and Plancherel's theorem. The proof of Theorem 1.3 is given in Section 3.

**2. Proof of Theorem 1.2.** For $\gamma > 0$, we introduce the new unknown $\tilde{u} = e^{-\gamma t}u$ and the new data $\tilde{F} := e^{-\gamma t}F$, $\tilde{G} := e^{-\gamma t}G$. Then the boundary value problem (1.1) becomes equivalent to

$$(2.1)\qquad \begin{cases} L_\gamma \tilde{u} := L\tilde{u} + \gamma\tilde{u} = \tilde{F} & \text{in } Q, \\ B\tilde{u} = \tilde{G} & \text{on } \Sigma. \end{cases}$$

**2.1. An auxiliary strictly dissipative boundary value problem.** Thanks to the assumption of Friedrichs symmetrizability, we define an auxiliary boundary value problem where the interior equations of (2.1) are coupled with a strictly dissipative boundary condition $M$ with respect to the non-characteristic component as follows.

In the context of a characteristic boundary case for a Friedrichs symmetrizable operator, we claim that a stricly dissipative boundary condition with respect to the non-characteristic component can always be chosen as noticed in [5, part 3, Remark 1.2].

Indeed, by Assumptions 1.1 and 1.2, let $S$ be a positive definite matrix such that $SA_d$ is also symmetric, where $A_d$ has the representation (1.2). It follows readily that there exists an $(N-m)\times(N-m)$ positive definite matrix $S_d$ such that

$$SA_d = \begin{pmatrix} 0_m & 0 \\ 0 & S_d a_d \end{pmatrix}, \tag{2.2}$$

where the matrix $S_d a_d$ is also symmetric. Therefore, knowing that the submatrix $a_d$ is invertible, in virtue of [5, part 3, Remark 1.2], there exists a matrix $M_d$ of size $p\times(N-m)$ such that the matrix $S_d a_d$ is negative definite on $\ker M_d$. Setting

$$M = \begin{pmatrix} 0_{p\times m} & M_d \end{pmatrix}, \tag{2.3}$$

the matrix $SA_d$ defined on $\ker M$ is obviously non-positive and vanishes only on $\ker A_d = \{v\in\mathbb{C}^N : v^{II}=0\}$. This property is characterized by the fact that there exist positive constants $c$ and $C$ such that for all $w\in\mathbb{C}^N$,

$$-\langle SA_d w, w\rangle_{\mathbb{C}^N} \geq c|w^{II}|^2 - C|Mw|^2 \tag{2.4}$$

(see [9, Chapter 2] for a proof in the non-characteristic case).

Consider the following BVP:

$$\begin{cases} L_\gamma \tilde{w} = \left(\dfrac{\partial}{\partial t}+\gamma\right)\tilde{w} + \displaystyle\sum_{j=1}^{d-1} A_j \dfrac{\partial \tilde{w}}{\partial x_j} + A_d \dfrac{\partial \tilde{w}}{\partial x_d} = \tilde{F} \quad \text{in } Q, \\ M\tilde{w} = 0 \quad \text{on } \Sigma. \end{cases} \tag{2.5}$$

This problem enters into the framework of hyperbolic strictly dissipative boundary value problems which has been treated by several authors (see e.g. Lax–Phillips [7], Métivier [9], Benzoni-Gavage–Serre [1]) and for which a well-posedness result has been established. Therefore, adapting for instance the proof of Métivier [9] in the characteristic boundary context for a Friedrichs symmetrizable operator (see [9, Proposition 2.2.13]), one gets

THEOREM 2.1. *Assume that the assumptions of Theorem* 1.2 *are fulfilled for some* $s\geq 0$. *Then there exists a constant* $C>0$ *such that, for all* $\gamma>0$, *and for a given* $F\in L^2(\mathbb{R}_+, H^s(\Sigma))$, *there exists a unique solution* $\tilde{w}\in L^2(\mathbb{R}_+, H^s(\Sigma))$ *of the problem* (2.5) *such that the trace* $\tilde{w}^{II}_{|x_d=0}$ *is in* $H^s(\Sigma)$ *and satisfies the following energy estimate:*

$$\gamma|||\tilde{w}|||^2_{s,\gamma} + \|\tilde{w}^{II}_{|x_d=0}\|^2_{s,\gamma} \leq \frac{C}{\gamma}|||\tilde{F}|||^2_{s,\gamma}. \tag{2.6}$$

**2.2. A Fourier–Laplace analysis for a boundary value problem.** Denote by $\tilde{u}$ a solution of the BVP (2.1) and by $\tilde{w}$ the solution of the BVP (2.5) and define

$$\tilde{v} = \tilde{u} - \tilde{w}. \tag{2.7}$$

We expect that $\tilde{v}$ must be a solution of the BVP

$$\begin{cases} L_\gamma \tilde{v} = A_d \dfrac{\partial \tilde{v}}{\partial x_d} + \left(\dfrac{\partial}{\partial t} + \gamma\right)\tilde{v} + \displaystyle\sum_{j=1}^{d-1} A_j \dfrac{\partial \tilde{v}}{\partial x_j} = 0 \quad \text{in } Q, \\ B_2 \tilde{v}^{II} = \tilde{G} - B_2 \tilde{w}^{II}_{|x_d=0} \quad \text{on } \Sigma. \end{cases} \tag{2.8}$$

Performing the Fourier transform $\mathcal{F}$ with respect to the tangential variables $(t, y)$, the system (2.8) is rewritten as a singular ODE with respect to the normal variable $x_d$ parametrized by $\zeta = (\tau, \eta) = (\gamma + i\sigma, \eta) \in \mathbb{C}_+ \times \mathbb{R}^{d-1}$ of the form

$$\begin{cases} A_d \dfrac{\partial \phi}{\partial x_d}(\zeta, x_d) - \mathcal{A}(\zeta)\phi(\zeta, x_d) = 0, & \zeta \in \mathbb{C}_+ \times \mathbb{R}^{d-1},\ x_d > 0, \\ B_2 \phi^{II}(\zeta, 0) = H(\zeta), & \zeta \in \mathbb{C}_+ \times \mathbb{R}^{d-1},\ x_d = 0. \end{cases}, \tag{2.9}$$

where the unknown $\phi$ stands for $\mathcal{F}(\tilde{v})$, the Fourier transform with respect to the tangential variables $(t, y)$ of $\tilde{v}$, the expected $L^2$ solution of the problem (2.8), and

$$\mathcal{A}(\zeta) := -(\tau I_N + iA(\eta)), \quad A(\eta) := \sum_{j=1}^{d-1} \xi_j A_j, \tag{2.10}$$

$$H(\zeta) := \mathcal{F}(\tilde{G})(\sigma, \eta) - B_2 \mathcal{F}(\tilde{w}^{II})(\sigma, \eta, 0). \tag{2.11}$$

The strategy of the Fourier–Laplace analysis is to build a square integrable solution of the transformed system (2.9) that will satisfy energy estimates. This is necessary to get the well posedness of the original problem (2.8) through the inverse Laplace transform and the theorem of Paley-Wiener. The non-invertibility of the boundary matrix $A_d$ yields a singular system of ODEs. To overcome this difficulty, thanks to the block structure of the symbol matrix $A(\xi)$, we reduce the singular system to a non-characteristic system of ODEs with respect to the non-characteristic component after projecting the system onto $\ker A_d$ along the range of $A_d$.

Here and in what follows, with a slight abuse of notation, $\phi^{II}(\zeta, 0)$ denotes the trace of the solution $\phi^{II}(\zeta, \cdot)$ on $\Sigma$. Likewise, we denote by $\tilde{w}^{II}(\cdot, 0)$ the trace of $\tilde{w}^{II}$ which is well defined in $L^2(\Sigma)$.

According to Assumption 1.2, the block structure (1.2) of the boundary matrix $A_d$ permits one to consider the invertible matrix

$$A_d^- := \begin{pmatrix} I_m & 0 \\ 0 & a_d^{-1} \end{pmatrix}. \tag{2.12}$$

Multiplying on the left by $A_d^-$ the interior equation of the problem (2.9) we get

$$\mathbf{I}_{N-m}\frac{\partial\phi}{\partial x_d}(\zeta,x_d)=A_d^-\mathcal{A}(\zeta)\phi(\tau,\eta,x_d), \tag{2.13}$$

where $\mathbf{I}_{N-m}:=\operatorname{diag}(0_m,I_{N-m})$. Taking into account the block structure of the symbol matrix $A(\xi):=A(\eta)+\xi_d A_d$ (1.3) according to Assumption 1.3, the matrix $A_d^-\mathcal{A}(\zeta)$ takes the form

$$A_d^-\mathcal{A}(\zeta)=-\begin{pmatrix}\tau I_m & ia_{1,2}(\eta)\\ ia_d^{-1}a_{2,1}(\eta) & ia_d^{-1}a_2(\eta)+\tau a_d^{-1}\end{pmatrix}. \tag{2.14}$$

Applying the decomposition $\mathbb{C}^N=\mathbb{C}^m\oplus\mathbb{C}^{N-m}$, the system (2.9) with the unknown $\phi=(\phi^I,\phi^{II})^T$ can actually be rewritten in decoupled form

(2.15)

$$\begin{cases}\tau\phi^I(\zeta,x_d)+ia_{1,2}(\eta)\phi^{II}(\zeta,x_d)=0,\\ \dfrac{\partial\phi^{II}}{\partial x_d}(\zeta,x_d)=-ia_d^{-1}\big(a_{2,1}(\eta)\phi^I(\zeta,x_d)+(a_2(\eta)+i\tau I_{N-m})\phi^{II}(\zeta,x_d)\big).\end{cases}$$

We derive from the first equation of (2.15) an expression of $\phi^I$ as a function of $\phi^{II}$:

$$\phi^I(\zeta,x_d)=-\frac{i}{\tau}a_{1,2}(\eta)\phi^{II}(\zeta,x_d). \tag{2.16}$$

Plugging it into the second equation of (2.15), we get a reduced boundary value problem for an ODE parametrized by $\zeta\in\mathbb{C}_+\times\mathbb{R}^{d-1}$ with values in $\mathbb{C}^{N-m}$, with respect to the non-characteristic part $\phi^{II}$, of the form

$$\begin{cases}\dfrac{\partial\phi^{II}}{\partial x_d}(\zeta,x_d)-\mathcal{A}_2(\zeta)\phi^{II}(\zeta,x_d)=0,\\ B_2\phi^{II}(\zeta,0)=H(\zeta),\end{cases} \tag{2.17}$$

where for all $\zeta\in\mathbb{C}_+\times\mathbb{R}^{d-1}$,

$$\mathcal{A}_2(\zeta)=-a_d^{-1}\Big(\tau I_{N-m}+ia_2(\eta)+\frac{1}{\tau}a_{21}(\eta)a_{12}(\eta)\Big). \tag{2.18}$$

The problem (2.17) appears as a "generalized resolvent type equation" as pointed out by Guès, Métivier, Williams and Zumbrun [5].

In virtue of the deep analysis made by Benzoni-Gavage and Serre [1, Chapter 6], the matrix $\mathcal{A}_2(\zeta)$ is hyperbolic, in the sense that:

(1) For $\Re e(\tau)=\gamma>0$, the matrix $\mathcal{A}_2(\zeta)$ does not have purely imaginary eigenvalues.

(2) Let $\mathcal{E}_-(\zeta)$ (resp. $\mathcal{E}_+(\zeta)$) be the stable invariant subspace (resp. the unstable invariant subspace) of the matrix $\mathcal{A}_2(\zeta)$ with the decomposition

$$\mathbb{C}^{N-m} = \mathcal{E}_-(\zeta) \oplus \mathcal{E}_+(\zeta).$$

The stable subspace of $\mathcal{A}_2(\tau,\eta)$ has constant dimension when $\gamma > 0$. This dimension equals $p$.

Let $P_-(\zeta)$ be the spectral projector onto $\mathcal{E}_-(\zeta)$ along $\mathcal{E}_+(\zeta)$,

$$P_-(\zeta) = \frac{1}{2\pi i} \int_{\Gamma^-} (zI_{N-m} - \mathcal{A}_2(\zeta))^{-1} dz, \tag{2.19}$$

with $\Gamma^-$ a closed contour enclosing all eigenvalues of $\mathcal{A}_2(\zeta)$ of negative real part.

The $s$-WKL Condition (2.20) can then be rewritten in terms of the subspace $\mathcal{E}_-(\zeta)$ and the representation (1.4) of the matrix $B$:

$$\exists C > 0,\ \forall \zeta \in \Xi \quad \big(v \in \mathcal{E}_-(\zeta) \Rightarrow |v| \le C\gamma^{-s}|B_2 v|\big). \tag{2.20}$$

Let us introduce the operator

$$\mathbb{J}(\zeta) := B_2 P_-(\zeta).$$

As a consequence of the $s$-WKL Condition (2.20), for any $\zeta \in \mathbb{C}_+ \times \mathbb{R}^{d-1}$, the operator $\mathbb{J}(\zeta)$ is an isomorphism between $\mathcal{E}_-(\zeta)$ and $\mathbb{C}^p$, so $\mathbb{J}^{-1}H(\zeta) \in \mathcal{E}_-(\zeta)$.

Therefore, in virtue of the definition of the stable subspace, one can first state the following crucial result.

PROPOSITION 2.2. *Assume that the assumptions of Theorem 1.2 are fulfilled for some $s \ge 0$. Then there exists a constant $C > 0$ such that, for all $\gamma > 0$, and for any $\tilde{F} \in L^2(\mathbb{R}_+, H^s(\Sigma))$, $\tilde{G} \in H^s(\Sigma)$, and any $\zeta \in \mathbb{C}_+ \times \mathbb{R}^{d-1}$, the problem (2.17) admits a unique solution $\phi^{II}(\zeta,\cdot) \in L^2(\mathbb{R}_+)$, and it satisfies the following estimate:*

$$\int_{\mathbb{R}^d} |\phi^{II}_{|x_d=0}(\gamma + i\sigma, \eta, 0)|^2 \, d\sigma \, d\eta \le C\left(\frac{1}{\gamma^{2s}}\|\tilde{G}\|^2_{s,\gamma} + \frac{1}{\gamma^{1+2s}}|||\tilde{F}|||^2_{s,\gamma}\right). \tag{2.21}$$

*Proof.* The problem (2.17) admits a unique $L^2(\mathbb{R}_+)$ solution of the form

$$\phi^{II}(\zeta, x_d) = \exp(x_d \mathcal{A}_2(\zeta))\mathbb{J}^{-1}(H(\zeta)). \tag{2.22}$$

The trace $\phi^{II}(\zeta,0) = \mathbb{J}^{-1}H(\zeta)$ belongs to $\mathcal{E}_-(\zeta)$. Therefore, applying the $s$-WKL Condition (2.20), and invoking homogeneity, for any $\zeta \in \mathbb{C}_+ \times \mathbb{R}^{d-1}$ one has

$$|\phi^{II}(\zeta,0)|^2 \le C\frac{|\zeta|^{2s}}{\gamma^{2s}}|B_2\phi^{II}(\zeta,0)|^2 = C\frac{|\zeta|^{2s}}{\gamma^{2s}}|H(\zeta)|^2.$$

Thus, from the identity (2.11), we get

$$|\phi^{II}(\zeta,0)|^2 \le C\frac{|\zeta|^{2s}}{\gamma^{2s}}\big(|\mathcal{F}(\tilde{G})(\sigma,\eta)|^2 + |\mathcal{F}\tilde{w}^{II}(\sigma,\eta,0)|^2\big). \tag{2.23}$$

Integrating the estimate (2.23) over $\mathbb{R}\times\mathbb{R}^{d-1}$, we get

$$\begin{aligned}\int_{\mathbb{R}^d} |\phi^{II}_{|x_d=0}(\gamma+i\sigma,\eta,0)|^2\,d\sigma\,d\eta \\ \le \frac{C}{\gamma^{2s}}\Big(\int_{\mathbb{R}^d}(\gamma^2+\sigma^2+|\eta|^2)^s|\mathcal{F}(\tilde{G})(\sigma,\eta)|^2\,d\sigma\,d\eta \\ + \int_{\mathbb{R}^d}(\gamma^2+\sigma^2+|\eta|^2)^s|\mathcal{F}(\tilde{w}^{II}_{|x_d=0})(\sigma,\eta)|^2\,d\eta\,d\sigma\Big).\end{aligned} \tag{2.24}$$

The right hand side of the estimate (2.24) is none other than

$$\frac{C}{\gamma^{2s}}\Big(\int_{\mathbb{R}^d}(\gamma^2+\sigma^2+|\eta|^2)^s|\mathcal{F}\big(e^{-\gamma\cdot}G\big)(\sigma,\eta)|^2\,d\eta\,d\sigma + \int_{\mathbb{R}^d}(\gamma^2+\sigma^2+|\eta|^2)^s|\mathcal{F}\big(e^{-\gamma\cdot}w^{II}_{|x_d=0}\big)(\sigma,\eta)|^2\,d\eta\,d\sigma\Big).$$

In other words,

$$\int_{\mathbb{R}^d} |\phi^{II}_{|x_d=0}(\gamma+i\sigma,\eta,0)|^2\,d\sigma\,d\eta \le C\frac{1}{\gamma^{2s}}(\|\tilde{G}\|^2_{s,\gamma} + \|\tilde{w}^{II}_{|x_d=0}\|^2_{s,\gamma}). \tag{2.25}$$

Now, using the estimate (2.6) satisfied by the solution $\tilde{w}$ of the auxiliary problem (2.5):

$$\|\tilde{w}^{II}_{|x_d=0}\|^2_{s,\gamma} \le \frac{C}{\gamma}\|\tilde{F}\|^2_{s,\gamma}, \tag{2.26}$$

one has

$$\int_{\mathbb{R}^d} |\phi^{II}_{|x_d=0}(\gamma+i\sigma,\eta,0)|^2\,d\sigma\,d\eta \le C\bigg(\frac{1}{\gamma^{2s}}\|\tilde{G}\|^2_{s,\gamma} + \frac{1}{\gamma^{1+2s}}|||\tilde{F}|||^2_{s,\gamma}\bigg),$$

which ends the proof of the proposition. ■

Likewise, the equality (2.16) determines the characteristic component $\phi^I$ which is also in $L^2(\mathbb{R}_+)$. We deduce readily that for any $\zeta\in\mathbb{C}_+\times\mathbb{R}^{d-1}$, the function $\phi(\zeta,\cdot) = (\phi^I(\zeta,\cdot),\phi^{II}(\zeta,\cdot))^T$ is the unique solution of the problem (2.9) which belongs to $L^2(\mathbb{R}_+)$.

On the other hand, one can consider $\phi(\zeta,\cdot)$ as a solution of the singular ODE defined in (2.9), but coupled with the strictly dissipative boundary condition $M_d$ defined in (2.3),

$$\begin{cases} A_d \dfrac{\partial \phi}{\partial x_d}(\zeta, x_d) - \mathcal{A}(\zeta)\phi(\zeta, x_d) = 0, & \zeta \in \mathbb{C}_+ \times \mathbb{R}^{d-1},\ x_d > 0, \\ M_d \phi^{II}(\zeta, 0) := K(\zeta). \end{cases} \tag{2.27}$$

The following proposition holds:

Proposition 2.3. *There exists $C > 0$ such that for any $\gamma > 0$ and any $\zeta \in \mathbb{C}_+ \times \mathbb{R}^{d-1}$, one has*

$$\gamma \|\phi(\zeta, \cdot)\|^2_{L^2(\mathbb{R}_+)} + |\phi^{II}(\zeta, 0)|^2 \leq C|K(\zeta)|^2 = C|M_d \phi^{II}(\zeta, 0)|^2. \tag{2.28}$$

*Proof.* Thanks to the Friedrichs symmetrizable assumption 1.1, for all $j = 1, \dots, d$, the matrices $SA_j$ are symmetric. Thus, for any $\eta \in \mathbb{R}^{d-1}$, the matrix $SA(\eta)$ is also symmetric, where $A(\eta)$ has the block structure (1.3),

$$A(\eta) = \begin{pmatrix} 0_m & a_{1,2}(\eta) \\ a_{2,1}(\eta) & a_2(\eta) \end{pmatrix},$$

according to Assumption 1.3.

In particular, we have already seen that the submatrix $S_d$ of the matrix $S$, of size $(N - m) \times (N - m)$, defined in (2.2) is symmetric and positive definite.

Returning to the singular ODE (2.9) and multiplying on the left by the matrix $S$, in virtue of the expression (2.10) of the matrix $\mathcal{A}(\zeta)$, one has

$$SA_d \frac{\partial \phi}{\partial x_d}(\zeta, \cdot) + \tau S\phi(\zeta, \cdot) + iSA(\eta)\phi(\zeta, \cdot) = 0. \tag{2.29}$$

Now, taking the real part of the scalar product with $\phi$ in $\mathbb{C}^N$ and using the fact that $SA(\eta)$ is symmetric, this implies that

$$\Re e \left\langle SA_d \frac{\partial \phi}{\partial x_d}(\zeta, \cdot), \phi(\zeta, \cdot) \right\rangle_{\mathbb{C}^N} = -\gamma \langle S\phi(\zeta, \cdot), \phi(\zeta, \cdot) \rangle_{\mathbb{C}^N}. \tag{2.30}$$

Integration by parts with respect to $x_d$ on $\mathbb{R}_+$ yields

$$\begin{aligned} 2\gamma (S\phi(\zeta, \cdot), \phi(\zeta, \cdot))_{L^2(\mathbb{R}_+)} &= \langle SA_d \phi(\zeta, 0), \phi(\zeta, 0) \rangle_{\mathbb{C}^N} \\ &= \langle S_d a_d \phi^{II}(\zeta, 0), \phi^{II}(\zeta, 0) \rangle_{\mathbb{C}^{N-m}}. \end{aligned} \tag{2.31}$$

At this stage, we use on the one hand the fact that $S$ is positive, and on the other hand the fact that the inequality (2.4) holds since the matrix $S_d a_d$ defined on $\ker M_d$ is negative definite. It follows that there exists a constant $C > 0$ such that

$$\gamma \|\phi(\zeta, \cdot)\|^2_{L^2(\mathbb{R}_+)} \leq C|M_d \phi^{II}(\zeta, 0)|^2 - |\phi^{II}(\zeta, 0)|^2,$$

and the proof of Proposition 2.3 is complete. ∎

Integrating once again over $\mathbb{R}\times\mathbb{R}^{d-1}$ the estimate (2.28) and adding the right hand side of (2.21), we get

$$\text{(2.32)}\qquad \gamma\int_{\mathbb{R}^{d+1}_+}|\phi(\gamma+i\sigma,\eta,x_d)|^2\,d\sigma\,d\eta\,dx_d+\int_{\mathbb{R}^d}|\phi^{II}_{|x_d=0}(\gamma+i\sigma,\eta,0)|^2\,d\sigma\,d\eta \le C|M_d|^2\left(\frac{1}{\gamma^{2s}}\|\tilde{G}\|^2_{s,\gamma}+\frac{1}{\gamma^{1+2s}}|\!|\!|\tilde{F}|\!|\!|^2_{s,\gamma}\right).$$

Note that the function $\tau\mapsto\phi(\tau,\cdot,\cdot)$ is holomorphic in $\mathbb{C}_+$ with values in $L^2(\mathbb{R}^{d-1}\times\mathbb{R}_+)$. The estimate (2.32) shows that

$$\text{(2.33)}\qquad \sup_{\gamma\geq\gamma_0}\int_{\mathbb{R}}\|\phi(\gamma+i\sigma,\cdot,\cdot)\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}_+)}d\sigma<\infty$$

for all $\gamma_0>0$. Thanks to the theorem of Paley–Wiener, the above properties show that there exists a function $v\colon (t,y,x_d)\mapsto\tilde{v}(t,y,x_d)$ such that $e^{-\gamma t}v:=\tilde{v}\in L^2(\mathbb{R}\times\mathbb{R}^{d-1}\times\mathbb{R}_+)$ for $\gamma>0$, and for which the function $\phi$ is exactly its Fourier transform with respect to the $(t,y)$ variables.

Applying the inverse Fourier operator to the problem (2.9), we deduce that $\tilde{v}$ is the unique solution of $L_\gamma\tilde{v}=0$. Moreover, the trace $\tilde{v}^{II}_{|x_d=0}$ is well defined in $L^2(\varSigma)$ and satisfies the boundary condition $B\tilde{v}^{II}_{|x_d=0}=\tilde{G}-B\tilde{w}^{II}_{|x_d=0}$ on $\varSigma$.

Eventually, applying Plancherel's theorem to (2.32), we see that the solution $v$ satisfies the estimate

$$\text{(2.34)}\qquad \gamma\|e^{-\gamma t}v\|^2_{L^2(Q)}+\|e^{-\gamma t}v^{II}_{|x_d=0}\|^2_{L^2(\varSigma)} \le C\left(\frac{1}{\gamma^{2s+1}}|\!|\!|e^{-\gamma t}F|\!|\!|^2_{s,\gamma}+\frac{1}{\gamma^{2s}}\|e^{-\gamma t}G\|^2_{s,\gamma}\right).$$

Recall that the unique solution $\tilde{w}$ of the auxiliary problem (2.5) belongs to $L^2(Q)$, with $\tilde{w}^{II}_{|x_d=0}\in L^2(\varSigma)$ and satisfies the estimate (2.6).

Finally, setting $\tilde{u}=\tilde{v}-\tilde{w}\in L^2(Q)$, we see that $u:=e^{\gamma t}\tilde{u}$ is the unique solution of the problem (1.1) such that $\tilde{u}^{II}_{|x_d=0}\in L^2(\varSigma)$ and satisfies the energy estimate

$$\gamma\|e^{-\gamma t}u\|^2_{L^2(Q)}+\|e^{-\gamma t}u^{II}_{|x_d=0}\|^2_{L^2(\varSigma)} \le C\left(\frac{1}{\gamma^{2s+1}}|\!|\!|e^{-\gamma t}F|\!|\!|^2_{s,\gamma}+\frac{1}{\gamma^{2s}}\|e^{-\gamma t}G\|^2_{s,\gamma}\right),$$

which completes the proof of Theorem 1.2.

## 3. Proof of Theorem 1.3

**3.1. Solutions of the boundary value problem localized in time.** As a consequence of Theorem 1.2, we find that the solution of the BVP (1.1) satisfies the causality principle when the data vanish in $\{t<0\}$.

Proposition 3.1. *In the framework of Theorem 1.2 in the case $s = 0$, for any $\gamma > 0$, if both $F \in L^2_\gamma(Q)$ and $G \in L^2_\gamma(\Sigma)$ vanish for $t < 0$, then the solution $u \in L_\gamma(Q)$ of the BVP (1.1) vanishes for $t < 0$.*

*Proof.* We refer for instance to [1, 2, 9] for a proof which remains true in the context of Theorem 1.2. ∎

For a given $T > 0$, we introduce the sets

$$\Omega_T := ]-\infty, T] \times \Omega \quad \text{with boundary} \quad \omega_T := ]-\infty, T] \times \partial\Omega.$$

Consider now the BVP (1.1) but defined in the time interval $]-\infty, T]$:

$$\begin{cases} Lu = F & \text{in } \Omega_T := ]-\infty, T] \times \mathbb{R}^d_+, \\ Bu_{|x_d=0} = G & \text{on } \omega_T := ]-\infty, T] \times \mathbb{R}^{d-1}. \end{cases} \tag{3.1}$$

The causality principle stated in Proposition 3.1 implies the following important result for the boundary value problem (3.1).

Theorem 3.2. *Assume that the assumptions of Theorem 1.2 are satisfied for $s = 0$. Let $F \in L^2(\Omega_T)$ and $G \in L^2(\omega_T)$ be such that $F \equiv 0$ and $G \equiv 0$ for $t < 0$. Then there exists a unique solution $u \in L^2(\Omega_T)$ of the problem (3.1) such that $u^{II}_{|x_d=0} \in L^2(\omega_T)$ and $u$ vanishes for $t < 0$. In addition $u \in \mathcal{C}([0,T], L^2(\Omega))$ and for any $\gamma > 0$, it satisfies the estimate*

$$\begin{aligned} \gamma\|e^{-\gamma t}u\|^2_{L^2(\Omega_t)} + \|e^{-\gamma t}u^{II}_{|x_d=0}\|^2_{L^2(\omega_t)} + e^{-2\gamma t}\|u(t)\|^2_{L^2(\Omega)} \\ \le C\left(\frac{1}{\gamma}\|e^{-\gamma t}F\|^2_{L^2(\Omega_t)} + \|e^{-\gamma t}G\|^2_{L^2(\omega_t)}\right) \end{aligned} \tag{3.2}$$

*for all $t \in [0,T]$.*

*Proof.* Extend $F$ and $G$ by zero for $t > T$ and get $\pi(F) \in L^2(Q)$ and $\pi(G) \in L^2(\Sigma)$. One deduce readily from the assumptions about $F$ and $G$ that $\pi(F) \in L^2_\gamma(Q)$ and $\pi(G) \in L^2_\gamma(\Sigma)$ for all $\gamma > 0$. Consequently, by Theorem 1.2, there exists a unique $\breve{u} \in L^2_\gamma(Q)$ with $\breve{u}^{II} \in L^2_\gamma(\Sigma)$ which is a solution of the BVP

$$L\breve{u} = \pi(F) \quad \text{in } Q, \qquad B\breve{u}_{|x_d=0} = \pi(G) \quad \text{in } \Sigma. \tag{3.3}$$

Moreover $\breve{u}$ satisfies the estimate (1.9) in the case $s = 0$. Applying Proposition 3.1 one deduces from the properties of $F$ and $G$ that $\breve{u}$ vanishes for $t < 0$. Set $u := \breve{u}_{|\Omega_T}$. Then $u \in L^2(\Omega_T)$ and vanishes for $\{t < 0\}$. Furthermore, $u^{II}_{|x_d=0} \in L^2(\omega_T)$ and one can readily verify that $u$ is a solution of the BVP (3.1) in the sense of distributions. Therefore, in virtue of (1.9),

$$\begin{aligned} \gamma\|e^{-\gamma t}u\|^2_{L^2(\Omega_t)} + \|e^{-\gamma t}u^{II}_{|x_d=0}\|^2_{L^2(\omega_t)} \\ \le C\left(\frac{1}{\gamma}\|e^{-\gamma t}F\|^2_{L^2(\Omega_t)} + \|e^{-\gamma t}G\|^2_{L^2(\Sigma_t)}\right) \end{aligned} \tag{3.4}$$

for all $t \in [0,T]$. As a consequence of the causality principle, the uniqueness of the solution $u$ is obtained easily by using a truncation procedure as for instance in [1, 9].

The solution $u$ can be seen as defined in $Q_T = [0,T] \times \mathbb{R}^d_+$. The last step of the proof is to show that $u$ is continuous in the time variable. Consider the solution $\breve{u} \in L^2_\gamma(Q)$ of the problem (3.3). Using a standard mollifier $j_\varepsilon$ in tangential variables supported in $\{t > 0\}$ and the associated convolution operator $J_\varepsilon$, one can consider a suitable regularized sequence $(w^\varepsilon)$ defined by

$$w^\varepsilon = e^{\gamma t} J_\varepsilon e^{-\gamma t} \breve{u} \in L^2(\mathbb{R}_+, H^\infty_\gamma(\Sigma)).$$

This sequence satisfies $(w^\varepsilon)^{II}_{|x_d=0} \in H^{+\infty}_\gamma(\Sigma)$. As $\breve{u}$ vanishes for $t < 0$ and the mollifier $j_\varepsilon$ is supported in $\{t > 0\}$, $w^\varepsilon$ also vanishes for $t < 0$. In addition, the operator $L$ having constant coefficients, $w^\varepsilon$ is a solution of the BVP

$$Lw^\varepsilon = f^\varepsilon \quad \text{in } Q, \qquad Bw^\varepsilon_{|x_d=0} = g^\varepsilon \quad \text{on } \Sigma, \tag{3.5}$$

where $f^\varepsilon := e^{\gamma t} J_\varepsilon(e^{-\gamma t}\pi(F)) \in L^2(\mathbb{R}_+, H^{+\infty}_\gamma(\Sigma))$ and $g^\varepsilon := e^{\gamma t} J_\varepsilon(e^{-\gamma t}\pi(G)) \in H^{+\infty}_\gamma(\Sigma))$. Furthermore, the following convergences hold:

$$\begin{aligned} & w^\varepsilon \xrightarrow[\varepsilon\to 0]{} \breve{u} \quad \text{in } L^2_\gamma(Q), \\ & (w^\varepsilon_{|x_d=0})^{II} \xrightarrow[\varepsilon\to 0]{} (\breve{u}_{|x_d=0})^{II} \quad \text{in } L^2_\gamma(\Sigma), \\ & Lw^\varepsilon \xrightarrow[\varepsilon\to 0]{} \pi(F) \quad \text{in } L^2_\gamma(Q), \\ & Bw^\varepsilon_{|x_d=0} \xrightarrow[\varepsilon\to 0]{} \pi(G) \quad \text{in } L^2_\gamma(\Sigma). \end{aligned} \tag{3.6}$$

The assumptions of Theorem 1.2 are fulfilled by the BVP (3.5) for $s = 0$. Moreover the source terms $f^\varepsilon$ and $g^\varepsilon$ vanish in $\{t < 0\}$, as also does the solution $w^\varepsilon$, so for any $\gamma > 0$ we have an estimate similar to (3.4),

$$\begin{aligned} & \gamma \|e^{-\gamma t} w^\varepsilon\|^2_{L^2(\Omega_t)} + \|e^{-\gamma t}(w^\varepsilon)^{II}_{|x_d=0}\|^2_{L^2(\omega_t)} \\ & \qquad \le C\left(\frac{1}{\gamma}\|e^{-\gamma t} f^\varepsilon\|^2_{L^2(Q)} + \|e^{-\gamma t} g^\varepsilon\|^2_{L^2(\Sigma)}\right) \end{aligned} \tag{3.7}$$

for all $0 \le t \le T$. Using Assumption 1.1 of Friedrichs symmetrizability of $L$, we derive by integration by parts the following estimate for all $\varepsilon > 0$:

$$\begin{aligned} & e^{-2\gamma t}\|w^\varepsilon(t)\|^2_{L^2(\Omega)} \\ & \le C\left(\frac{1}{\gamma}\|e^{-\gamma t} Lw^\varepsilon\|^2_{L^2(\Omega_t)} + \gamma\|e^{-\gamma t} w^\varepsilon\|^2_{L^2(\Omega_t)} + \|e^{-\gamma t}(w^\varepsilon)^{II}_{|x_d=0}\|^2_{L^2(\omega_t)}\right). \end{aligned} \tag{3.8}$$

We refer to [1] for the derivation of this estimate. Combining estimates (3.7) and (3.8) we get

$$\text{(3.9)}\qquad \gamma\|e^{-\gamma t}w^\varepsilon\|^2_{L^2(\Omega_t)} + \|e^{-\gamma t}(w^\varepsilon)^{II}_{|x_d=0}\|^2_{L^2(\omega_t)} + e^{-2\gamma t}\|w^\varepsilon(t)\|^2_{L^2(\Omega)} \le C\bigg(\frac{1}{\gamma}\|e^{-\gamma t}f^\varepsilon\|^2_{L^2(Q)} + \|e^{-\gamma t}g^\varepsilon\|^2_{L^2(\Sigma)}\bigg).$$

Applying (3.9) to the difference $(w^\varepsilon - w^{\varepsilon'})_{|Q_T}$, we deduce from (3.6) that $(w^\varepsilon_{|Q_T})$ is a Cauchy sequence in the Banach space $\mathcal{C}([0,T],L^2(\Omega))$, so convergent in $\mathcal{C}([0,T],L^2(\Omega))$. We deduce thanks to (3.6) that the limit can only be $u$ the solution of the BVP (3.1), by invoking the uniqueness of the solution. Moreover, passing to the limit in (3.9), we obtain the estimate (3.2). ■

**3.2. End of the proof of Theorem 1.3.** We introduce an auxiliary problem with a strictly disspative boundary condition, already considered in Section 2.1 by incorporating the nonzero initial data $u_0$. Consider the matrix $M$ defined in (2.3). The matrix $SA_d$ defined on $\ker M$ is obviously nonpositive and vanishes only on $\ker A_d$. Consider then the following auxiliary IBVP:

$$\text{(3.10)}\qquad \begin{cases} Lw = F & \text{in } Q_T,\\ Mw = 0 & \text{on } \Sigma_T,\\ w_{|t=0} = u_0 & \text{in } \Omega. \end{cases}$$

The problem (3.10) is a hyperbolic strictly dissipative initial boundary value problems for which a well-posedness result has been established by several authors [7, 8, 9, 1].

Theorem 3.3. *There is a constant $C>0$ such that for all $F\in L^2(Q_T)$ and $u_0\in L^2(\Omega)$, there exists a unique solution $w\in L^2(Q_T)$ of the problem (3.10) for which $w^{II}_{|x_d=0}\in L^2(\Sigma_T)$. Furthermore $w\in\mathcal{C}([0,T],L^2(\Omega))$ and the following estimate holds for all $t\in[0,T]$ and $\gamma>0$:*

$$\text{(3.11)}\qquad \gamma\|e^{-\gamma t}w\|^2_{L^2(Q_t)} + \|e^{-\gamma t}w^{II}_{|x_d=0}\|^2_{L^2(\Sigma_t)} + e^{-2\gamma t}\|w(t)\|^2_{L^2(\mathbb{R}^d_+)} \le C\bigg(\frac{1}{\gamma}\|e^{-\gamma t}F\|^2_{L^2(Q_t)} + \|u_0\|^2_{L^2(\Omega)}\bigg).$$

Consider now the solution $w$ of the problem (3.10) for which $w^{II}_{|x_d=0}\in L^2(\Sigma_T)$, and the following homogeneous IBVP:

$$\text{(3.12)}\qquad \begin{cases} Lv = 0 \quad \text{in } Q_T,\\ Bv_{|x_d=0} = G - Bw_{|x_d=0} \quad \text{on } \Sigma_T,\\ v_{|t=0} = 0 \quad \text{in } \Omega. \end{cases}$$

The data $(G-Bw_{|x_d=0}) = \big(G-B_2 w^{II}_{|x_d=0}\big)$ of the boundary condition belongs to $L^2(\Sigma_T)$. Extend it by 0 from $\Sigma_T$ to $\omega_T$ and get a function in $L^2(\omega_T)$. The assumptions of Theorem 3.2 are then fulfilled in the case $s = 0$ for the problem (3.12). Thus, there exists a solution $v \in L^2(\Omega_T)$ of the IBVP (3.12) which vanishes on $\{t < 0\}$ such that $v^{II}_{|x_d=0} \in L^2(\omega_T)$. Furthermore $v \in \mathcal{C}([0,T], L^2(\Omega))$ and it satisfies for all $t \in [0,T]$ the estimate

$$\begin{aligned}(3.13)\qquad \gamma\|e^{-\gamma t}v\|^2_{L^2(Q_t)} + \|e^{-\gamma t}v^{II}_{|x_d=0}\|^2_{L^2(\Sigma_t)} + e^{-2\gamma t}\|v(t)\|^2_{L^2(\Omega)}\\ \le C(\|e^{-\gamma t}G\|^2_{L^2(\Sigma_t)} + \|e^{-\gamma t}w^{II}\|^2_{L^2(\Sigma_t)}).\end{aligned}$$

Then $u := v + w$ is obviously a solution of the IBVP (1.10) in $L^2(Q_T)$ such that $u^{II}_{|x_d=0} \in L^2(\Sigma_T)$. In addition, $u \in \mathcal{C}([0,T], L^2(\Omega))$ and, by combining estimates (3.11), (3.13), it satisfies the estimate (1.11). The uniqueness of the solution $u$ follows readily from the uniqueness of solution of the problem (3.1). This completes the proof of Theorem 1.3.

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