Tutorial series 1

Mathematical Analysis I

Exercise 01:

1. Prove that the number $\frac{\ln 3}{\ln 2}$ is irrational.
2. Let x be the number: $0;234234\cdots234\cdots$. By comparing $1000x$ and x, write x in the form of a rational fraction.
3. Let $x, y \in \mathbb{R}$, prove that

$$\begin{cases} \max(x,y) = \frac{x+y+|x-y|}{2} \\ \min(x,\ y) = \frac{x+y-|x-y|}{2} \end{cases}$$

Exercise 02:

Assume $x = \sqrt[3]{20+14\sqrt{2}} + \sqrt[3]{20-14\sqrt{2}}$

1. Show that x is a solution of an equation of the form $x^3 = ax + b$, with $a, b \in \mathbb{R}$.
2. Solve this last one and determine x.

Exercise 03:

Let $x, y, z \in \mathbb{R}$, prove the following inequalities:

1. $||x| - |y|| \leq |x+y| \leq |x| + |y|$.
2. $|x| + |y| \leq |x+y| + |x-y|$.
3. $\sqrt{x^2+y^2} \leq |x| + |y|$.
4. $xy \leq \frac{1}{2}(x^2+y^2)$.

Exercise 04:

For all $x, y \in \mathbb{R}$, Show that:

1. $f(x) = E(x)$ is an increasing function.
2. $E(x) + E(y) \leq E(x+y) \leq E(x) + E(y) + 1$.
3. $\forall n \in \mathbb{N}^*,\ E\left(\frac{E(nx)}{n}\right) = E(x)$.

Exercise 05:

Let A and B be two non-empty and bounded parts of $\mathbb{R}$, show that:

1. If $A \subset B$, then

$$\begin{cases} \sup(A) \leq \sup(B) \\ \inf(B) \leq \inf(A). \end{cases}$$

2. $\sup(A \cup B) = \max(\sup(A), \sup(B))$.
3. $\inf(A \cup B) = \min(\inf(A), \inf(B))$.

Exercise 06:

Put the following sets in the form of an interval of $\mathbb{R}$ or a union of intervals.

1. $A_1 = \{x \in \mathbb{R},\ x^2 \leq 1\}$.
2. $A_2 = \{x \in \mathbb{R},\ x^3 \leq 1\}$.
3. $A_3 = \left\{x \in \mathbb{R},\ -1 \leq \frac{2x}{x^2+1} < 1\right\}$.
4. $A_4 = \left\{x \in \mathbb{R},\ \frac{1}{|x|} > 1\right\}$.

Exercise 07:

Determine (if exist) the lower bound, upper bound, The greatest lower bound, and the least upper bound of the following sets:

1. $A_1 = [1,2] \cap \mathbb{Q}$.
2. $A_2 = \left\{u_n = (-1)^n + \frac{1}{n+1} /\ n \in \mathbb{N}\right\}$.
3. $A_3 = \left\{\frac{m}{n} / m,\ n \in \mathbb{N}^* \text{ and } m < 2n\right\}$.
4. $A_4 = \left\{u_n = \sin\left(\frac{2n\pi}{7}\right) /\ n \in \mathbb{Z}\right\}$.
5. $A_5 = \left\{x \in \mathbb{R}, 1 < \frac{2x}{x^2+1} < 1\right\}$.