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On the characterizations of some distinguished subclasses of Hilbert space operators

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Abstract. In this note, we present several characterizations for some distinguished classes of bounded Hilbert space operators (self-adjoint operators, normal operators, unitary operators, and isometry operators) in terms of operator inequalities.

1. Introduction and preliminaries

Let $\mathfrak{B}(H)$ be the C^*-algebra of all bounded linear operators acting on a complex Hilbert space H, and let $S(H)$, $\mathcal{N}(H)$, $\mathfrak{U}(H)$, and $\mathcal{V}(H)$ denote the class of all self-adjoint operators, the class of all normal operators, the class of all unitary operators, and the class of all isometry operators in $\mathfrak{B}(H)$, respectively.

We use the following notations:

- $\mathfrak{I}(H)$ is the group of all invertible elements in $\mathfrak{B}(H)$,
- $S_0(H) = S(H) \cap \mathfrak{I}(H)$ is the set of all invertible self-adjoint operators in $\mathfrak{B}(H)$,
- $\mathcal{N}_0(H) = \mathcal{N}(H) \cap \mathfrak{I}(H)$ is the set of all invertible normal operators in $\mathfrak{B}(H)$,
- $\mathcal{R}(H)$ is the set of all operators with closed ranges in $\mathfrak{B}(H)$,
- $S_{cr}(H) = S(H) \cap \mathcal{R}(H)$ is the set of all self-adjoint operators with closed ranges in $\mathfrak{B}(H)$,
- $\mathcal{N}_{cr}(H) = \mathcal{N}(H) \cap \mathcal{R}(H)$ is the set of all normal operators with closed range in $\mathfrak{B}(H)$,
- $(M)_1 = M \cap \{x \in X : \|x\| = 1\}$, where X is any normed space and M is a subset of X,

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- $x \otimes y$ (where $x, y \in H$) is the operator (of rank less than or equal to one) on H defined by $(x \otimes y)z = \langle z, y\rangle\, x$, for every $z \in H$,
- $|S|$ is the positive square root of the positive operator S^*S (where $S \in \mathfrak{B}(H)$),
- S_g is a generalized inverse of $S \in \mathfrak{B}(H)$ (is a solution in $\mathfrak{B}(H)$, if it exists, of the equations $SXS = S$, $XSX = X$; so SS_g and S_gS are projections in $\mathfrak{B}(H)$),
- $R(S)$ and $\ker S$ denote the range and the kernel of $S \in \mathfrak{B}(H)$, respectively,
- $r(S)$ is the spectral radius of an operator $S \in \mathfrak{B}(H)$,
- $\Gamma M = \{\lambda X \colon \lambda \in \Gamma,\ X \in M\}$, where Γ and M are subsets of $\mathbb{C}$ and $\mathfrak{B}(H)$, respectively.

Recall that an operator $S \in \mathfrak{B}(H)$ is said to be *paranormal* if $\left\|S^2x\right\| \geq \|Sx\|^2$, for every $x \in (H)_1$. It is known that

(i) for every $S \in \mathfrak{B}(H)$, S has a generalized inverse if and only if $S \in \mathcal{R}(H)$;
(ii) if $S \in \mathcal{R}(H)$, then there exists a unique generalized inverse denoted by S^+ (the Moore–Penrose inverse of S) such that SS^+ and S^+S are orthogonal projections onto $R(S)$ and $R(S^*)$, respectively;
(iii) if $S \in \mathcal{R}(H)$, then $R(S^+) = R(S^*)$ and $\ker S^+ = \ker S^*$;
(iv) if $S \in \mathcal{R}(H)$, then there exists a unique generalized inverse denoted by $S^\#$ (the group inverse of S) such that $SS^\# = S^\#S$ is a projection onto $R(S) = R(S^\#)$;
(v) if $S \in \mathfrak{I}(H)$, then $S^{-1} = S^+ = S^\#$.

An operator $S \in \mathcal{R}(H)$ is an EP operator, if $R(S) = R(S^*)$, or equivalently $S^+S = SS^+$. Note that any normal operator with a closed range is an EP operator, but the converse is not true even in a finite-dimensional space.

The ascent and descent of an operator $S \in \mathfrak{B}(H)$ are defined by

$$\operatorname{asc} S = \inf\{p \in \mathbb{N} \cup \{0\} : \ker(S^p) = \ker(S^{p+1})\},$$
$$\operatorname{dsc} S = \inf\{p \in \mathbb{N} \cup \{0\} : R(S^p) = R(S^{p+1})\},$$

where $\inf \emptyset = \infty$; if they are finite, they are equal and their common value is called the index of S and it is denoted by $\operatorname{ind} S$. An operator $S \in \mathfrak{B}(H)$ is group invertible (that means S has a group inverse) if and only if $\operatorname{ind} S \leq 1$. We denote by $\mathfrak{I}_1(H)$ the set of all operators $S \in \mathfrak{B}(H)$ such that $\operatorname{ind} S \leq 1$; it is clear that $\mathfrak{I}(H) \subset \mathfrak{I}_1(H)$.

Any nonzero operator $S \in \mathcal{R}(H)$ has the matrix representation $S = \begin{bmatrix} S_1 & S_2 \\ 0 & 0 \end{bmatrix}$ with respect to the orthogonal direct sum $H = R(S) \oplus \ker S^*$, and then $S^+ = \begin{bmatrix} S_1^*K^{-1} & 0 \\ S_2^*K^{-1} & 0 \end{bmatrix}$, where $K = S_1S_1^* + S_2S_2^*$ maps $R(S)$ onto itself and $K > 0$. Moreover, if $\operatorname{ind} S \leq 1$, then S_1 is invertible and $S^\# = \begin{bmatrix} S_1^{-1} & S_1^{-2}S_2 \\ 0 & 0 \end{bmatrix}$. It is easy to see that $S \in \mathcal{R}(H)$ is an EP operator if and only if $S_2 = 0$ and S_1 is invertible; in this case $S^+ = S^\# = \begin{bmatrix} S_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ (see [2]).

If $S \in \mathcal{R}(H)$ and $U|S|$ is the polar decomposition of S, then $S^+ = |S|^+ U^*$, where U is an isometry on $R(S^*) = R(|S|)$.

The purpose of this paper is to give several characterizations of some distinguished subclasses of $\mathfrak{B}(H)$ in terms of operator inequalities.

We start with the following elementary characterization of the distinguished subclass $\mathcal{N}(H)$:

$$\forall S \in \mathfrak{B}(H),\ (S \ \in \mathcal{N}(H)) \Leftrightarrow (\forall X \in \mathfrak{B}(H),\ \|S^* X\| = \|SX\|). \tag{N}$$

One of the most important operator inequalities in operator theory is the Arithmetic–Geometric Mean Inequality (see [7]):

$$\forall A, B, X \in \mathfrak{B}(H),\ \|A^* AX + XBB^*\| \geq 2\|AXB\|. \tag{AGMI1}$$

By the triangle inequality, it follows from this last inequality that

$$\forall A, B, X \in \mathfrak{B}(H),\ \|A^* AX\| + \|XBB^*\| \geq 2\|AXB\|. \tag{AGMI2}$$

Note that the operator inequality (AGMI1) is a particular form of the famous Heinz inequality (see [5]). In [4], it was proved that the operator inequality (AGMI1), the Heinz inequality, and four others inequalities hold (by using a direct proof) and follow from each other. In the next proposition, we state new operator inequalities that are equivalent to (AGMI1).

Proposition 1. *The following operator inequalities hold and follow from each other:*

(1) $\forall A, B, X \in \mathfrak{B}(H),\ \|A^* AX + XBB^*\| \geq 2\|AXB\|,$
(2) $\forall S, R \in S_{cr}(H),\ \forall X \in \mathfrak{B}(H),\ \|SXR^+ + S^+XR\| \geq 2\|SS^+XR^+R\|,$
(3) $\forall S, R \in S_{cr}(H),\ \forall X \in \mathfrak{B}(H),\ \left\|S^2X + XR^2\right\| \geq 2\|SXR\|,$
(4) $\forall S, R \in S_0(H),\ \forall X \in \mathfrak{B}(H),\ \left\|SXR^{-1} + S^{-1}XR\right\| \geq 2\|X\|,$
(5) $\forall S, R \in S(H),\ \forall X \in \mathfrak{B}(H),\ \left\|S^2X + XR^2\right\| \geq 2\|SXR\|,$
(1′) $\forall A, X \in \mathfrak{B}(H),\ \|A^* AX + XAA^*\| \geq 2\|AXA\|,$
(2′) $\forall S \in S_{cr}(H),\ \forall X \in \mathfrak{B}(H),\ \|SXS^+ + S^+XS\| \geq 2\|SS^+XS^+S\|,$
(3′) $\forall S \in S_{cr}(H),\ \forall X \in \mathfrak{B}(H),\ \left\|S^2X + XS^2\right\| \geq 2\|SXS\|,$
(4′) $\forall S \in S_0(H),\ \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| \geq 2\|X\|,$
(5′) $\forall S \in S(H),\ \forall X \in \mathfrak{B}(H),\ \left\|S^2X + XS^2\right\| \geq 2\|SXS\|.$

Proof. In [4], it was proved that the operator inequalities (1), (4), and (4′) hold and follow from each other.

It is easy to see that the equivalence (1) $\Leftrightarrow$ (5) holds (by using the polar decomposition of an operator).

(1) $\Rightarrow$ (2). Assume (1) holds. Let $S, R \in S_{cr}(H), X \in \mathfrak{B}(H)$. Since $S = S^*SS^+$ and $R = R^+RR^*$, then from (1) it follows that

$$\left\|SXR^+ + S^+XR\right\| = \left\|S^*S\left(S^+XR^+\right) + \left(S^+XR^+\right)RR^*\right\| \geq 2\left\|SS^+XR^+R\right\|.$$

Hence (2) holds.

The implications (2) $\Rightarrow$ (4), (1) $\Rightarrow$ (3), and (3) $\Rightarrow$ (4) are trivial. Hence the operator inequalities (1)–(5) are equivalent.

Going from pairs of operators to single operators, we deduce that the operator inequalities (1′)–(5′) are also equivalent.

Then the inequalities (1)–(5), and (1′)–(5′) given by the proposition hold and follow from each other. ∎

In the next proposition, we present a class of operator inequalities that are equivalent to (AGMI2).

Proposition 2. *The following operator inequalities hold and follow from each other:*

(1) $\forall A, B, X \in \mathfrak{B}(H),\ \|A^*AX\| + \|XBB^*\| \geq 2\|AXB\|,$
(2) $\forall S, R \in \mathcal{N}_{cr}(H),\ \forall X \in \mathfrak{B}(H),\ \|SXR^+\| + \|S^+XR\| \geq 2\|SS^+XR^+R\|,$
(3) $\forall S, R \in \mathcal{N}_{cr}(H),\ \forall X \in \mathfrak{B}(H),\ \left\|S^2X\right\| + \left\|XR^2\right\| \geq 2\|SXR\|,$
(4) $\forall S, R \in \mathcal{N}_0(H),\ \forall X \in \mathfrak{B}(H),\ \left\|SXR^{-1}\right\| + \left\|S^{-1}XR\right\| \geq 2\|X\|,$
(5) $\forall S, R \in \mathcal{N}(H),\ \forall X \in \mathfrak{B}(H), \left\|S^2X\right\| + \left\|XR^2\right\| \geq 2\|SXR\|,$
(1′) $\forall A, X \in \mathfrak{B}(H),\ \|A^*AX\| + \|XAA^*\| \geq 2\|AXA\|,$
(2′) $\forall S \in \mathcal{N}_{cr}(H),\ \forall X \in \mathfrak{B}(H),\ \|SXS^+\| + \|S^+XS\| \geq 2\|SS^+XS^+S\|,$
(3′) $\forall S \in \mathcal{N}_{cr}(H),\ \forall X \in \mathfrak{B}(H),\ \left\|S^2X\right\| + \left\|XS^2\right\| \geq 2\|SXS\|,$
(4′) $\forall S \in \mathcal{N}_0(H),\ \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geq 2\|X\|,$
(5′) $\forall S \in \mathcal{N}(H),\ \forall X \in \mathfrak{B}(H),\ \left\|S^2X\right\| + \left\|XS^2\right\| \geq 2\|SXS\|.$

Proof. (1) $\Rightarrow$ (2). Assume (1) holds. Let $S, R \in \mathcal{N}_{cr}(H), X \in \mathfrak{B}(H)$. Since $S^* = S^*SS^+$ and $R^* = R^+RR^*$, it follows from (1) and (N) that

$$\left\|SXR^+\right\| + \left\|S^+XR\right\| = \left\|S^*S\left(S^+XR^+\right)\right\| + \left\|\left(S^+XR^+\right)RR^*\right\| \geq 2\left\|SS^+XR^+R\right\|.$$

Hence (2) holds.

(2) $\Rightarrow$ (3). Assume (2) holds. Let $S, R \in \mathcal{N}_{cr}(H), X \in \mathfrak{B}(H)$. Then from (2) and since $SS^+S = S,\ RR^+R = R$, and $S^+S,\ RR^+$ are orthogonal projections, it follows that

$$\begin{aligned}\left\|S^2X\right\| + \left\|XR^2\right\| &\geq \left\|S\left(SXR\right)R^+\right\| + \left\|S^+\left(SXR\right)R\right\| \\ &\geq 2\left\|SS^+\left(SXR\right)R^+R\right\| = 2\|SXR\|.\end{aligned}$$

Thus (3) holds.

(3) $\Rightarrow$ (4). This implication is trivial.

(4) $\Rightarrow$ (1). Assume (4) holds. Then the following inequality holds:

$$\forall S, R \in \mathcal{N}_0(H),\ \forall X \in \mathfrak{B}(H),\ \left\|S^2X\right\| + \left\|XR^2\right\| \geq 2\left\|SXR\right\|.$$

Let $A, B, X \in \mathfrak{B}(H)$. Put $P = |A|$, $Q = |B^*|$. It is clear that the two operators $P + \epsilon I$ and $Q + \epsilon I$ are normal and invertible, for every $\epsilon > 0$. So, using the last inequality, we obtain

$$\forall \epsilon > 0,\ \|(P + \epsilon I)^2 X\| + \|X(Q + \epsilon I)^2\| \geq 2\|(P + \epsilon I)X(Q + \epsilon I)\|.$$

By letting $\epsilon \to 0$, we deduce (1).

(1) $\Rightarrow$ (5). This follows immediately by using (N).

(5) $\Rightarrow$ (3). This implication is trivial.

Therefore the operator inequalities (1)–(5) are equivalent.

Going from pairs of operators to single operators, we deduce that the operator inequalities $(1')$–$(5')$ are also equivalent.

(1) $\Rightarrow$ $(1')$. This implication is trivial.

$(1')$ $\Rightarrow$ (1). This follows immediately by using the Berberian technique.

Therefore the inequalities (1)–(5), and $(1')$–$(5')$ follow from each other. It remains to prove that one of them holds. It is clear that (1) is an immediate consequence of (AGMI1). But here, we shall give a direct proof of (1) independently of (AGMI1) by using the numerical arithmetic-geometric mean inequality. Let $A, B, X \in \mathfrak{B}(H)$. Then we have

$$\begin{aligned}\frac{1}{2}(\|A^*AX\| + \|XBB^*\|) &\geq \sqrt{\|A^*AX\|\|XBB^*\|} \geq \sqrt{\|BB^*X^*A^*AX\|} \\ &\geq \sqrt{r(BB^*X^*A^*AX)} = \sqrt{r(B^*X^*A^*AXB)} = \|AXB\|. \quad \blacksquare\end{aligned}$$

From Proposition 1 with the single operator case, we may introduce the following properties generated by operator inequalities:

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| \geq 2\left\|X\right\|, \quad (S \in \mathfrak{I}(H)), \tag{PS1}$$

$$\forall X \in \mathfrak{B}(H),\ \left\|S^2X + XS^2\right\| \geq 2\left\|SXS\right\|, \quad (S \in \mathcal{R}(H)), \tag{PS2}$$

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^+ + S^+XS\right\| \geq 2\left\|SS^+XS^+S\right\|, \quad (S \in \mathcal{R}(H)), \tag{PS3}$$

$$\forall X \in \mathfrak{B}(H),\ \left\|S^2X + XS^2\right\| \geq 2\left\|SXS\right\|, \quad (S \in \mathfrak{B}(H)). \tag{PS4}$$

Using Proposition 1, property (PS1) is satisfied for every $S \in \mathbb{C}^*S_0(H)$, each of the two properties (PS2), (PS3) is satisfied for every $S \in \mathbb{C}S_{cr}(H)$, and property

(PS4) is satisfied for every $S \in \mathbb{C}S(H)$. Note that Corach–Porta–Recht ([1]) proved with another motivation and independently of $(AGMI1)$ that property (PS1) is valid for every $S \in S_0(H)$.

From Proposition 2 with the single operator case, we may introduce the following properties generated by operator inequalities:

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geq 2\left\|X\right\|, \quad (S \in \mathfrak{I}(H)), \tag{PN1}$$

$$\forall X \in \mathfrak{B}(H),\ \left\|S^2X\right\| + \left\|XS^2\right\| \geq 2\left\|SXS\right\|, \quad (S \in \mathcal{R}(H)), \tag{PN2}$$

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^+\right\| + \left\|S^+XS\right\| \geq 2\left\|SS^+XS^+S\right\|, \quad (S \in \mathcal{R}(H)), \tag{PN3}$$

$$\forall X \in \mathfrak{B}(H),\ \left\|S^2X\right\| + \left\|XS^2\right\| \geq 2\left\|SXS\right\|, \quad (S \in \mathfrak{B}(H)). \tag{PN4}$$

Using Proposition 2, property (PN1) is satisfied for every $S \in \mathcal{N}_0(H)$, each of the two properties (PN2), (PN3) is satisfied for every $S \in \mathcal{N}_{cr}(H)$, and property (PN4) is satisfied for every $S \in \mathcal{N}(H)$.

So it is interesting to describe the subclasses characterized by the above properties that are generated by operator inequalities related to the known arithmetic-geometric mean inequality. This kind of problem was introduced in [9], by considering the Corach–Porta–Recht inequality ([1]). It was proved that property (PS1) characterizes exactly the subclass $\mathbb{C}^*S_0(H)$ (the subclass of all rotations of invertible self-adjoint operators in $\mathfrak{B}(H)$). Two other characteristic properties of the above subclass $\mathbb{C}^*S_0(H)$ were given in [10] by

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| = \left\|S^*XS^{-1} + S^{-1}XS^*\right\|,\ (S \in \mathfrak{I}(H)), \tag{PS5}$$

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| \geq \left\|S^*XS^{-1} + S^{-1}XS^*\right\|,\ (S \in \mathfrak{I}(H)). \tag{PS6}$$

Concerning the subclass $\mathcal{N}_0(H)$ of all invertible normal operators in $\mathfrak{B}(H)$, it was found that it is characterized by property (PN1) (see [10]), and also by three others (see [10], [12]) given by

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| = \left\|S^*XS^{-1}\right\| + \left\|S^{-1}XS^*\right\|, \tag{PN5}$$

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geq \left\|S^*XS^{-1}\right\| + \left\|S^{-1}XS^*\right\|, \tag{PN6}$$

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \leq \left\|S^*XS^{-1}\right\| + \left\|S^{-1}XS^*\right\|, \tag{PN7}$$

where $S \in \mathfrak{I}(H)$.

It is clear that the two properties (PS2), (PS3) (resp. (PN2), (PN3)) are extensions of property (PS1) (resp. (PN1)) from the domain $\mathfrak{I}(H)$ of invertible operators to the domain $\mathcal{R}(H)$ of operators with closed range.

Recently ([13]) it was showed that

(**i**) $\mathbb{C}S_{cr}(H)$ is characterized by each of the two properties (PS2), (PS3) and by each of the two following properties:

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{+} + S^{+}XS\right\| = \left\|S^{*}XS^{+} + S^{+}XS^{*}\right\|,\ (S \in \mathcal{R}(H)), \qquad \text{(PS7)}$$

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{+} + S^{+}XS\right\| \geq \left\|S^{*}XS^{+} + S^{+}XS^{*}\right\|,\ (S \in \mathcal{R}(H)), \qquad \text{(PS8)}$$

(**ii**) the class $\mathcal{N}_{cr}(H)$ is characterized by each of the two properties $(PN2)$, $(PN3)$ and by each of the following two properties:

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{+}\right\| + \left\|S^{+}XS\right\| = \left\|S^{*}XS^{+}\right\| + \left\|S^{+}XS^{*}\right\|, \qquad \text{(PN8)}$$

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{+}\right\| + \left\|S^{+}XS\right\| \geq \left\|S^{*}XS^{+}\right\| + \left\|S^{+}XS^{*}\right\|, \qquad \text{(PN9)}$$

where $S \in \mathcal{R}(H)$.

In [8], we found another characterization of the above class $\mathcal{N}_{cr}(H)$ given by

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{+}\right\| + \left\|S^{+}XS\right\| \leq \left\|S^{*}XS^{+}\right\| + \left\|S^{+}XS^{*}\right\|, \qquad \text{(PN10)}$$

where $S \in \mathcal{R}(H)$.

Note that property (PS4) (resp. (PN4)) is an extension of property (PS2) (resp. (PN2)) from the domain $\mathcal{R}(H)$ to the maximal domain $\mathfrak{B}(H)$.

Concerning the two subclasses characterized by the properties (PS4), (PN4) are still unknown; but the subclass characterized by (PS4) contains $\mathbb{C}S(H)$ and the subclass characterized by (PN4) contains $\mathcal{N}(H)$.

In the finite-dimensional case (here $\mathcal{R}(H) = \mathfrak{B}(H)$), (PS4) is exactly (PS2), (PN4) is exactly (PN2), then each of the properties (PS3), (PS4), (PS7), (PS8) (resp. (PN3), (PN4), (PN8), (PN9), (PN10)) characterizes $\mathbb{C}S(H)$ (resp. $\mathcal{N}(H)$). So, in the infinite-dimensional case, we may state the following two open problems:

Problem 1. Does property (PS4) characterize the class $\mathbb{C}S(H)$?

Problem 2. Does property (PN4) characterize the class $\mathcal{N}(H)$?

Remark 1. Let the following obvious inequality hold:

$$\forall X \in \mathfrak{B}(H), \quad \frac{\left\|S^2X\right\| + \left\|XS^2\right\|}{2} \geq \sqrt{\left\|S^2X\right\| \left\|XS^2\right\|}.$$

We may associate to it the following property:

$$\forall X \in \mathfrak{B}(H),\ \sqrt{\left\|S^2X\right\| \left\|XS^2\right\|} \geq \left\|SXS\right\|,\ \ (S \in \mathfrak{B}(H)). \qquad \text{(PN11)}$$

Note that from above, property (PN11) implies property (PN4).

From inequality (N) and using the same argument as used in the last part of the proof of Proposition 2, we find that the inequality (PN11) is satisfied for every normal operator in $\mathfrak{B}(H)$.

Conversely, let $S \in \mathfrak{B}(H)$ satisfy property (PN11). By taking $X = x \otimes y$ in (PN11), where $x, y \in (H)_1$, we find the following:

$$\forall x, y \in (H)_1,\ \left\|S^2 x\right\| \left\|S^{*2} y\right\| \geq \|Sx\|^2 \|S^* y\|^2 .$$

From this, it follows that $\left\|S^2\right\| = \|S\|^2$, and then $\left\|S^2 x\right\| \geq \|Sx\|^2$, $\left\|S^{*2} x\right\| \geq \|S^* x\|^2$, for every $x \in (H)_1$. Then S and S^* are paranormal. Using [14], we deduce that S is normal. Therefore, the property (PN11) characterizes the class $\mathcal{N}(H)$.

In this note, we will

(1) consider the following properties (extensions of the properties (PN5), (PN6), (PN7) from the domain $\mathfrak{I}(H)$ to the domain $\mathfrak{I}_1(H)$):

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{\#}\right\| + \left\|S^{\#}XS\right\| = \left\|S^*XS^{\#}\right\| + \left\|S^{\#}XS^*\right\|, \tag{PN12}$$

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{\#}\right\| + \left\|S^{\#}XS\right\| \geq \left\|S^*XS^{\#}\right\| + \left\|S^{\#}XS^*\right\|, \tag{PN13}$$

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{\#}\right\| + \left\|S^{\#}XS\right\| \leq \left\|S^*XS^{\#}\right\| + \left\|S^{\#}XS^*\right\|, \tag{PN14}$$

where $S \in \mathfrak{I}_1(H)$;

(2) give some new properties as follows:

$$\forall X \in \mathfrak{B}(H),\ \|S^*XS + SXS^*\| = 2\|SXS\|,\ \ (S \in \mathfrak{B}(H)), \tag{PS9}$$

$$\forall X \in \mathfrak{B}(H),\ \|S^*XS + SXS^*\| \geq 2\|SXS\|,\ \ (S \in \mathfrak{B}(H)), \tag{PS10}$$

$$\forall X \in \mathfrak{B}(H),\ \|S^*XS + SXS^*\| \leq 2\|SXS\|,\ \ (S \in \mathcal{R}(H)); \tag{PN15}$$

(3) give a characterization of a subclass of nonnormal operators, precisely the subclass $\mathcal{V}(H)$ of all isometry operators in $\mathfrak{B}(H)$.

We will prove that

(i) each of the three properties (PN12), (PN13), (PN14) characterizes $\mathcal{N}_{cr}(H)$,

(ii) each of the two properties (PS9), (PS10) characterizes $\mathbb{C}S(H)$, and property (PN15) characterizes $\mathcal{N}_{cr}(H)$,

(iii) the subclass $\mathcal{V}(H)$ is characterized by the following property:

$$\forall X \in \mathfrak{B}(H),\ \|X\| + \left\|S^{+}XS\right\| \leq 2\left\|SXS^{+}\right\|,\ \ (S \in (\mathcal{R}(H))_1). \tag{PI1}$$

2. Subclasses of normal operators and characterizations

In the next proposition, we shall extend the three properties (PN5), (PN6), (PN7) from the domain $\mathfrak{I}(H)$ to the domain $\mathfrak{I}_1(H)$ (where the usual inverse is replaced by the group inverse). We show that each of these extensions characterizes the class $\mathcal{N}_{cr}(H)$.

Proposition 3. *Let $S \in \mathfrak{I}_1(H)$. Then the following properties are equivalent:*

(i) $S \in \mathcal{N}_{cr}(H)$,

(ii) $\forall X \in \mathfrak{B}(H),\ \|SXS^{\#}\| + \|S^{\#}XS\| = \|S^*XS^{\#}\| + \|S^{\#}XS^*\|$,

(iii) $\forall X \in \mathfrak{B}(H),\ \|SXS^{\#}\| + \|S^{\#}XS\| \leq \|S^*XS^{\#}\| + \|S^{\#}XS^*\|$,

(iv) $\forall X \in \mathfrak{B}(H),\ \|SXS^{\#}\| + \|S^{\#}XS\| \geq \|S^*XS^{\#}\| + \|S^{\#}XS^*\|$.

Proof. (i)$\Longrightarrow$(ii). Assume (i) holds. Since $S^{\#} = S^+$, (ii) holds using (PN8).

(ii)$\Longrightarrow$(iii). The implication is trivial.

(iii)$\Longrightarrow$(i). Assume (iii) holds. Let $X = S(S^+)^2S - S^+S^2S^+$. Since $S^*SS^+ = S^+SS^* = S^*$ and $SS^+S^{\#} = S^{\#}S^+S = S^{\#}$, we obtain that

$$\begin{aligned}
S^*XS^{\#} &= S^*S(S^+)^2SS^{\#} - S^*S^+S^2S^+S^{\#} = S^*S^+SS^{\#} - S^*S^+SS^{\#} = 0,\\
S^{\#}XS^* &= S^{\#}S(S^+)^2SS^* - S^{\#}S^+S^2S^+S^* = S^{\#}SS^+S^* - S^{\#}SS^+S^* = 0,\\
S^{\#}XS &= S^{\#}S(S^+)^2SS - S^{\#}S^+S^2S^+S = S^{\#}S(S^+)^2S^2 - S^{\#}S,\\
SXS^{\#} &= SS(S^+)^2SS^{\#} - SS^+S^2S^+S^{\#} = S^2(S^+)^2SS^{\#} - SS^{\#}.
\end{aligned}$$

Applying (iii) for $X = S(S^+)^2S - S^+S^2S^+$, we get that $\|S^2(S^+)^2SS^{\#} - SS^{\#}\| + \|S^{\#}S(S^+)^2S^2 - S^{\#}S\| = 0$. Then $\|S^{\#}S(S^+)^2S^2 - S^{\#}S\| = 0$, so $S^{\#}SS^{+2}S^2 - S^{\#}S = 0$. Using the matrix representation with respect to the orthogonal direct sum $H = R(S) \oplus \ker S^*$ of the operators S, S^+, and $S^{\#}$ given in the introduction, we obtain that

$$\begin{aligned}
S^{\#}SS^{+2}S^2 &= \begin{bmatrix} S_1^*K^{-1}S_1^*K^{-1}S_1^2 + S_1^{-1}S_2S_2^*K^{-1}S_1^*K^{-1}S_1^2 & * \\ 0 & 0 \end{bmatrix}\\
&= \begin{bmatrix} I_1 & S_1^{-1}S_2 \\ 0 & 0 \end{bmatrix}\\
&= S^{\#}S
\end{aligned}$$

(where I_1 is the identity on $R(S)$). Hence $S_1^*K^{-1}S_1^*K^{-1}S_1^2 + S_1^{-1}S_2S_2^*K^{-1}S_1^*K^{-1}S_1^2 = I_1$. It follows that $S_1^{-1}(S_1S_1^* + S_2S_2^*)K^{-1}S_1^*K^{-1}S_1^2 = I_1$. Thus $S_1^{-1}S_1^*K^{-1}S_1^2 = I_1$, which implies that $S_1S_1^*K^{-1} = I_1$. So $K = S_1S_1^*$. Consequently $S_2 = 0$. Since S_1 is invertible, S is an EP operator.

Now applying again (iii) for $X = X_1 \oplus 0$ (where $X_1 \in B(R(S))$), we obtain

$$\forall X_1 \in B(R(S)), \left\|S_1 X_1 S_1^{-1}\right\| + \left\|S_1^{-1} X_1 S_1\right\| \leq \left\|S_1^* X_1 S_1^{-1}\right\| + \left\| S_1^{-1} X_1 S_1\right\|.$$

Using (PN7) with the Hilbert space $R(S)$, we find that S_1 is normal. Hence S is normal.

(i)$\Longrightarrow$(iv). This implication is trivial.

(iv)$\Longrightarrow$(i). Assume (iv) holds. If we put $X = x \otimes y$ (where $x, y \in H$) in (iv), then we obtain

$$\forall x, y \in H, \|Sx\| \left\|(S^\#)^* y\right\| + \left\|S^\# x\right\| \|S^* y\| \geq \|S^* x\| \left\|(S^\#)^* y\right\| + \left\|S^\# x\right\| \|Sy\|.$$

From this last inequality, it follows that $\ker S = \ker S^*$. Then S is an EP operator. Hence $S = S_1 \oplus 0$ with respect to the orthogonal direct sum $H = R(S) \oplus \ker S$, and where S_1 is an invertible operator in $\mathfrak{B}(R(S))$. Applying (iv) for $X = X_1 \oplus 0$ (where $X_1 \in B(R(S))$), we obtain the following inequality:

$$\forall X_1 \in \mathfrak{B}(R(S)),\ \left\|S_1 X S_1^{-1}\right\| + \left\|S_1^{-1} X S_1\right\| \geq \left\|S^* X S_1^{-1}\right\| + \left\|S_1^{-1} X S^*\right\|.$$

Then from this last inequality and using (PN6) with the Hilbert space $R(S)$, we obtain that S_1 is normal. Therefore S is normal. ∎

Remark 2. Note that the domain $\mathfrak{I}(H)$ (resp. $\mathfrak{I}_1(H)$, $\mathcal{R}(H)$) is for the existence of the usual inverse (resp. the group inverse, the Moore–Penrose inverse). The three properties (PN8), (PN9), (PN10) are extensions of (PN5), (PN6), (PN7) from the domain $\mathfrak{I}(H)$ to $\mathcal{R}(H)$, where the usual inverse is replaced by the Moore–Penrose inverse and each of them characterizes $\mathcal{N}_{cr}(H)$. In the above proposition, it was proved that each of the properties (PN12), (PN13), (PN14) characterizes also the class $\mathcal{N}_{cr}(H)$, and these properties are the extensions of the properties (PN5), (PN6), (PN7) from the domain $\mathfrak{I}(H)$ to $\mathfrak{I}_1(H)$, where the usual inverse is replaced by the group inverse.

Problem 3. Consider the following extension of property (PN1) from $\mathfrak{I}(H)$ to $\mathfrak{I}_1(H)$, and where the usual inverse is replaced by the group inverse:

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^\#\right\| + \left\|S^\# XS\right\| \geq 2\left\|SS^\# XS^\# S\right\|, \quad (S \in \mathfrak{I}_1(H)). \tag{P}$$

Note that the extension (PN3) of (PN1) from $\mathfrak{I}(H)$ to $\mathcal{R}(H)$ (where the usual inverse is replaced by the Moore–Penrose inverse) characterizes the class $\mathcal{N}_{cr}(H)$. But the class characterized by the extension (P) of (PN1) contains strictly $\mathcal{N}_{cr}(H)$ (if $\dim H \geq 2$): if $S \in \mathcal{N}_{cr}(H)$, then $S^\# = S^+$ and applying (PN3), we obtain that S satisfies property (P). It is easy to see that any projection of $\mathfrak{B}(H)$ satisfies property (P) (since, if $S \in \mathfrak{B}(H)$ is a projection, then $S^{\#} = S = S^2$). If $\dim H \geq 2$, $\mathfrak{B}(H)$ contains nonnormal projections (oblique projections).

So, it is interesting to describe the class characterized by property (P).

In the two next propositions, we introduce new forms of properties generated by operator inequalities.

We will use the following two facts, which are easy to verify:

(**i**) if $S \in \mathfrak{I}(H)$, then S is unitary if and only if $\|S\| = 1 = \|S^{-1}\|$,

(**iii**) if $S \in \mathfrak{I}(H)$, then S is normal if and only if S^*S^{-1} is unitary.

Proposition 4. *Let $S \in \mathcal{R}(H)$. Then the following two properties are equivalent:*

(1) $S \in \mathcal{N}_{cr}(H)$,

(2) $\forall X \in \mathfrak{B}(H),\ \|S^*XS + SXS^*\| \leq 2\,\|SXS\|$.

Proof. The proof is trivial if $S = 0$. Assume now that $S \neq 0$.

(i)$\Longrightarrow$(ii). The implication follows immediately using the triangle inequality and property (N).

(ii)$\Longrightarrow$(i). Assume (ii) holds. We shall prove (i) in two steps.

Step 1. Suppose that S is invertible. Then applying (ii) for $X = S^{-1}(S^{-1})^*$, we obtain

$$1 + \left\|S^*S^{-1}\right\|^2 = \left\|S^*S^{-1}(S^*S^{-1})^* + I\right\| \leq 2\left\|S^*S^{-1}\right\|.$$

Hence $\left\|S^*S^{-1}\right\| = 1$. On the other hand, we remark that (ii) holds if we replace S by S^*. It follows that $\left\|S^*S^{-1}\right\| = 1 = \left\|(S^*S^{-1})^{-1}\right\|$. So, we have that S^*S^{-1} is unitary. Thus S is normal.

Step 2. The general case $S \in \mathcal{R}(H)$.

If we put $X = x \otimes y$ (where $x, y \in H$) in (ii), we obtain

$$\forall x, y \in H,\ \|S^*x \otimes S^*y + Sx \otimes Sy\| \leq 2\,\|Sx\|\,\|S^*y\|.$$

From this last inequality, it follows immediately that $\ker S^* = \ker S$. Then S is an EP operator. Hence $S = S_1 \oplus 0$ with respect to the orthogonal direct sum $H = R(S) \oplus \ker S^*$, and where S_1 is invertible. Applying (ii) for $X = X_1 \oplus 0$ (where $X_1 \in B(R(S))$, we obtain

$$\forall X_1 \in B(R(S)),\ \|S_1^*X_1S_1 + S_1X_1S_1^*\| \leq 2\,\|S_1X_1S_1\|.$$

From Step 1, S_1 is normal. Hence S is normal. ∎

Corollary 1. *The following property characterizes the class $\mathcal{N}_0(H)$:*

$$\forall X \in \mathfrak{B}(H),\ \left\|S^*S^{-1}X + XS^{-1}S^*\right\| \leq 2\left\|X\right\|, \quad (S \in \mathfrak{I}(H)). \tag{PN17}$$

Remark 3. From the above proposition, property $(PN14)$ characterizes the class $\mathcal{N}_{cr}(H)$.

Problem 4. Is it true that the class of all normal operators in $\mathfrak{B}(H)$ is characterized by

$$\forall X \in \mathfrak{B}(H),\ \|S^*XS + SXS^*\| \leq 2\|SXS\|, \quad (S \in \mathfrak{B}(H))? \tag{PN18}$$

Proposition 5. *Let $S \in \mathfrak{B}(H)$. Then the following three properties are equivalent:*

(i) $S \in \mathbb{C}S(H)$,
(ii) $\forall X \in \mathfrak{B}(H),\ \|S^*XS + SXS^*\| = 2\|SXS\|$,
(iii) $\forall X \in \mathfrak{B}(H),\ \|S^*XS + SXS^*\| \geq 2\|SXS\|$.

Proof. The proof is trivial if $S = 0$. Assume now that $S \neq 0$.

The two implications (i)$\Longrightarrow$(ii) and (ii)$\Longrightarrow$(iii) are trivial.

(iii)$\Longrightarrow$(i). Assume (iii) holds. By taking $X = x \otimes y$ (where $x, y \in (H)_1$) in (iii), we obtain

$$\forall x, y \in (H)_1,\ \|S^*x\|\|S^*y\| + \|Sx\|\|Sy\| \geq 2\|Sx\|\|S^*y\|. \tag{A}$$

From (A), we deduce that

$$\forall x, y \in (H)_1,\ \|S\|(\|S^*x\| + \|Sx\|) \geq 2\|Sx\|\|S^*y\|.$$

By taking the supremum in the above inequality over $y \in (H)_1$, we deduce that

$$\forall x \in H,\ \|S^*x\| \geq \|Sx\|. \tag{B}$$

Using the same argument as before, we deduce from (A) that

$$\forall x \in H,\ \|Sx\| \geq \|S^*x\|. \tag{C}$$

Them from (B) and (C), we obtain that S is normal. So from (iii) and (N), it follows that

$$\forall X \in \mathfrak{B}(H),\ \|S^*XS + SXS^*\| = \|S^*XS\| + \|SXS^*\|. \tag{D}$$

Let now $X \in \mathfrak{B}(H)$ and put $A = SXS^*$, $B = S^*XS$. Then from (D), we deduce that

$$\begin{aligned}\|A\|^2 + \|B\|^2 + 2\|A\|\|B\| &= \|(A+B)^*(A+B)\| = \|A^*A + B^*B + 2\,\mathrm{Re}(B^*A)\| \\ &\leq \|A\|^2 + \|B\|^2 + 2\|B^*A\| \leq \|A\|^2 + \|B\|^2 + 2\|A\|\|B\|.\end{aligned}$$

Hence, $\|B^*A\| = \|A\|\,\|B\|$. So, we have

$$\forall X \in \mathfrak{B}(H),\ \|S^*XS\|\,\|SXS^*\| = \left\|(S^*XS)^*\,(SXS^*)\right\|.$$

Since $S \neq 0$ and S is normal, we may choose a vector $y \in H$ such that $Sy \neq 0$ and $S^*y \neq 0$. By taking $X = x \otimes y$ (where x arbitrary in H) in the last inequality, we obtain that $\|S^*x\|\,\|Sx\| = |\langle S^*x, Sx\rangle|$ for all $x \in H$.

Thus S^*x and Sx are linearly dependent, for every $x \in H$. Since S is normal, then for every $x \in H$ such that $Sx \neq 0$, there exists a complex number $\lambda(x)$ (depending on x) of modulus one such that $S^*x = \lambda(x)Sx$. By taking $X = x \otimes y$ in (D), for $x, y \in H$ such that $Sx \neq 0$, $Sy \neq 0$, we deduce that $|1 + \lambda(x)\overline{\lambda(y)}| = 2$ and so $\lambda(x) = \lambda(y)$. Hence there exists a constant real number θ such that $S^*x = e^{i\theta}Sx$, for every $x \in H$. Thus $S^* = e^{i\theta}S$. Put $M = e^{i\frac{\theta}{2}}S$. Then M is self-adjoint in $\mathfrak{B}(H)$ and $S = e^{-i\frac{\theta}{2}}M$. Therefore (i) holds. ■

Corollary 2. *The class $\mathbb{C}^*S_0(H)$ is characterized by each of the following two properties:*

$$\forall X \in \mathfrak{B}(H),\ \left\|S^*S^{-1}X + XS^{-1}S^*\right\| = 2\,\|X\|\,,\quad (S \in \mathfrak{I}(H)), \tag{PS11}$$

$$\forall X \in \mathfrak{B}(H),\ \left\|S^*S^{-1}X + XS^{-1}S^*\right\| \geq 2\,\|X\|\,,\quad (S \in \mathfrak{I}(H)). \tag{PS12}$$

Remark 4. From the above proposition, we deduce that each of the two propositions (PS9) and (PS10) characterizes exactly the class $\mathbb{C}S(H)$.

3. Isometry operators and characterizations

Concerning the third distinguished subclass $\mathfrak{U}(H)$ of all unitary operators in $\mathfrak{B}(H)$, it was characterized by each of the following three properties (see [11]):

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| \leq 2\,\|X\|\,,\quad (S \in (\mathfrak{I}(H))_1)\,, \tag{PU1}$$

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| = 2\,\|X\|\,,\quad (S \in (\mathfrak{I}(H))_1)\,, \tag{PU2}$$

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \leq 2\,\|X\|\,,\quad (S \in (\mathfrak{I}(H))_1)\,. \tag{PU3}$$

In this section, we shall characterize the class $\mathcal{V}(H)$ (that contains $\mathfrak{U}(H)$) of all isometry operators in $\mathfrak{B}(H)$, and we deduce another characterization of $\mathfrak{U}(H)$.

An operator $S \in \mathfrak{B}(H)$ is called quasinormal if S and S^*S commute.

It is known that if $S \in \mathcal{R}(H)$ is a partial isometry, then $S^* = S^+$.

Proposition 6. *Let $S \in (\mathcal{R}(H))_1$. Then the following two properties are equivalent:*

(i) $S \in \mathcal{V}(H)$,

(ii) $\forall X \in \mathfrak{B}(H), \|X\| + \|S^+XS\| \leq 2\,\|SXS^+\|$.

Proof. The implication (i)$\Longrightarrow$(ii) is trivial.

(ii)$\Longrightarrow$(i). Assume (ii) holds. We prove (i) in three steps:

Step 1. We prove that S is injective.

If we replace X by $x \otimes y$ (for $x, y \in H$) in (ii), we obtain

$$\forall x, y \in H, \quad \|x\| \, \|y\| + \left\|S^+ x\right\| \|S^* y\| \le 2 \, \|Sx\| \left\|(S^+)^* y\right\|.$$

By taking $x \in \ker S$ and choosing $y \neq 0$ in the above inequality, we obtain that $x = 0$. Hence S is injective.

Step 2. We prove that $(S^2)^+ S = S^+$.

Since S is injective with a closed range, S^2 is also injective with a closed range. Thus $S^+ S = (S^2)^+ S^2 = I$.

It is known that S^+ is the unique solution of the following four equations: $SXS = S$, $XSX = X$, $(XS)^* = XS$, $(SX)^* = SX$. It is easy to see that $(S^2)^+ S$ satisfies the first three equations. Now, we prove that $(S^2)^+ S$ satisfies the last equation. Since the operator $S(S^2)^+ S$ is a projection, it suffices to prove that its norm is less than or equal to one. By taking $X = S(S^2)^+ S$ in (ii), we obtain

$$\left\|S(S^2)^+ S\right\| + \left\|S^+ S(S^2)^+ SS\right\| \le 2 \left\|SS(S^2)^+ SS^+\right\| \le 2 \left\|S^2 (S^2)^+\right\| \left\|SS^+\right\| \le 2.$$

Thus $\left\|S(S^2)^+ S\right\| \le 1$. Hence $S((S^2)^+ S)$ is a projection of norm less than or equal to one. Thus $S(S^2)^+ S$ is a self-adjoint projection. So $(S^2)^+ S$ satisfies the above four equations. Therefore $(S^2)^+ S = S^+$.

Step 3. We prove that $S \in \mathcal{V}(H)$.

Since $S^2 (S^2)^+ = SS^+ S^2 (S^2)^+$ and SS^+, $S^2 (S^2)^+$ are self-adjoint, $S^2 (S^2)^+ = S^2 (S^2)^+ SS^+$. So from Step 2, we obtain $S^2 (S^2)^+ = S^2 (S^+)^2$. Since S^2 is injective, we have $(S^+)^2 = (S^2)^+$.

Returning to (ii) and replacing X by SXS^+, we obtain

$$\forall X \in B(H), \left\|SXS^+\right\| + \|X\| \le 2 \left\|S^2 X (S^2)^+\right\|.$$

Put $P = |S|$, $R = \left|S^2\right|$. Since S and S^2 are both injective with closed range, P and R are invertible. By taking the polar decomposition of each of the two operators S and S^2 in the last inequality, we obtain

$$\forall X \in B(H), \left\|PXP^{-1}\right\| + \|X\| \le 2 \left\|RXR^{-1}\right\|. \qquad (*)$$

If we replace X by $R^{-1} X R$ in the above inequality, we obtain

$$\forall X \in B(H), \left\|PR^{-1} XRP^{-1}\right\| + \left\|R^{-1} XR\right\| \le 2 \, \|X\|.$$

So from (PN1), we get that

$$\forall X \in B(H), \ \left\|PR^{-1}XRP^{-1}\right\| + \left\|R^{-1}XR\right\| \leq \left\|RXR^{-1}\right\| + \left\|R^{-1}XR\right\|.$$

Thus for all $X \in B(H)$, $\left\|PR^{-1}XRP^{-1}\right\| \leq \left\|RXR^{-1}\right\|$. Hence

$$\forall X \in B(H), \ \left\|PR^{-2}X\left(PR^{-2}\right)^{-1}\right\| \leq \|X\|.$$

So, we obtain that for all $X \in B(H)$, $\left\|PR^{-2}XR^2P^{-1}\right\| = \|X\|$. Using again (∗) and replacing X by $R^{-2}XR^2$, we find

$$\forall X \in B(H), \quad \left\|PR^{-2}XR^2P^{-1}\right\| + \left\|R^{-2}XR^2\right\| \leq 2\left\|R^{-1}XR\right\|.$$

So, we have $\|X\| + \left\|R^{-2}XR^2\right\| \leq 2\left\|R^{-1}XR\right\|$ for all $X \in B(H)$. Thus,

$$\forall X \in B(H), \ \left\|RXR^{-1}\right\| + \left\|R^{-1}XR\right\| \leq 2\|X\|.$$

Using (PU3) and since R is positive, we obtain $R = \|R\| I$. Then (∗) becomes

$$\forall X \in B(H), \ \left\|PXP^{-1}\right\| \leq \|X\|.$$

Thus $1 \leq \|P\| \left\|P^{-1}\right\| = \sup_{\|X\|=1} \left\|PXP^{-1}\right\| \leq 1$. Hence $\|P\| \left\|P^{-1}\right\| = 1$. So we have $P = \|P\| I = I$. Hence $S^*S = P^2 = I$. Therefore $S \in \mathcal{V}(H)$. ■

Remark 5. From the above proposition, property (PI1) characterizes $\mathcal{V}(H)$.

From the preceding proposition, we can obtain the following corollary:

Corollary 3. *The class $\mathfrak{U}(H)$ is characterized by each of the following two properties:*

$$\forall X \in \mathfrak{B}(H), \ \|X\| + \left\|SXS^{-1}\right\| \leq 2\left\|S^{-1}XS\right\|, \quad (S \in (\mathfrak{I}(H))_1), \tag{PU4}$$

$$\forall X \in \mathfrak{B}(H), \ \|X\| + \left\|S^{-1}XS\right\| \leq 2\left\|SXS^{-1}\right\|, \quad (S \in (\mathfrak{I}(H))_1). \tag{PU5}$$

Proposition 7. *Let $S \in (\mathcal{R}(H))_1$. Then the following properties are equivalent:*

(i) *S is a direct sum of an isometry and zero,*

(ii) *S is a partial isometry and quasinormal,*

(iii) $\forall X \in B(H), \|X\| + \|SXS^+\| \geq 2\|S^+XS\|$.

Proof. (i)⇔(ii). This equivalence follows immediately from the definitions and elementary properties of a partial isometry operator and a quasinormal operator.

(ii)⟹(iii). Assume (ii) holds. Then $S = S^*S^2$, so we have $R(S) \subset R(S^*)$. Using Douglas's Theorem [3], we obtain $\|Sx\| \geq \|S^*x\|$, for every $x \in H$. Hence, $\|SX\| \geq \|S^*X\|$, for every X in $B(H)$.

Now let $X \in B(H)$. Then we have $\|X\| \geq \|S^*XS\| = \|S^+XS\|$ (since $\|S\|=1$ and $S^* = S^+$), and

$$\left\|SXS^+\right\| = \|SXS^*\| \geq \|S^*XS^*\| = \|SX^*S\| \geq \|S^*X^*S\| = \|S^*XS\| = \left\|S^+XS\right\|.$$

Hence $\|X\| + \|SXS^+\| \geq 2\,\|S^+XS\|$. This proves (iii).

(iii)$\Longrightarrow$(ii). Assume (iii) holds. It follows immediately that $1 + \|S^+\|\,\|S\| \geq 2\,\|S^+\|\,\|S\|$. Hence $\|S^+\|\,\|S\| \leq 1$. Thus $\|S^+\|\,\|S\| = 1$, so $\|S^+\| = \|S\| = 1$. Using [6, Theorem 3.1], S is a partial isometry.

It remains to prove that S is quasinormal. By taking $X = Sx \otimes Sx$ (where $x \in H$) in (iii), we obtain

$$\forall x \in H, \quad \|Sx\|^2 + \left\|S^2x\right\|^2 \geq 2\,\|S^*Sx\|^2. \tag{$\dagger$}$$

Since S^* is also a partial isometry,

$$\forall x \in H, \quad \|S^*Sx\| = \|Sx\|. \tag{$\ddagger$}$$

From ($\dagger$) and ($\ddagger$), we obtain the inequality $\left\|S^2x\right\| \geq \|Sx\|$ for all $x \in H$. Since $\|S\| = 1$, we have $\left\|S^2x\right\| = \|Sx\|$ for all $x \in H$.

Hence $S^*(I - S^*S)S = 0$ (where $I - S^*S \geq 0$). Then $(I - S^*S)S = 0$. So $S = (S^*S)\,S = S\,(S^*S)$. Therefore S is quasinormal. ■

As an immediate consequence of the above proposition, we have the following corollary:

Corollary 4. *The class $\mathfrak{U}(H)$ is characterized by each of the following two properties:*

$$\forall X \in \mathfrak{B}(H),\ \|X\| + \left\|SXS^{-1}\right\| \geq 2\left\|S^{-1}XS\right\|, \quad (S \in (\mathfrak{I}(H))_1), \tag{PU6}$$

$$\forall X \in \mathfrak{B}(H),\ \|X\| + \left\|S^{-1}XS\right\| \geq 2\left\|SXS^{-1}\right\|, \quad (S \in (\mathfrak{I}(H))_1). \tag{PU7}$$

As an immediate consequence of both Corollary 3 and Corollary 4, we deduce the following characterization:

Corollary 5. *The class $\mathfrak{U}(H)$ is characterized by the following property:*

$$\forall X \in \mathfrak{B}(H),\ \|X\| + \left\|SXS^{-1}\right\| = 2\left\|S^{-1}XS\right\|, \quad (S \in (\mathfrak{I}(H))_1). \tag{PU8}$$

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References

[1] G. Corach, R. Porta and L. Recht, An operator inequality, *Linear Algebra Appl.*, **142** (1990), 153–158.

[2] D.S. Djordjević and J.J. Koliha, Characterizing Hermitian, normal and EP operators, *Filomat*, **21** (2007), 39–54.

[3] R.G. Douglas, On majorization, factorization, and range inclusion of operators on Hilbert space, *Proc. Amer. Math. Soc.*, **17** (1966), 413–415.

[4] J. Fujii, M. Fujii, T. Furuta and R. Nakamoto, Norm inequalities equivalent to Heinz inequality, *Proc. Amer. Math. Soc.*, **118** (1993), 827–830.

[5] E. Heinz, Beiträge zur Störungstheorie der Spektralzerlegung, *Math. Ann.*, **123** (1951), 415–438 (in German).

[6] M. Mbekhta, Partial isometry and generalized inverses, *Acta Sci. Math (Szeged)*, **70** (2004), 767–781.

[7] A. McIntosh, *Heinz Inequalities and Perturbation of Spectral Families*, Macquarie Mathematical Reports, Macquarie Univ., 1979.

[8] S. Menkad, Partial isometries and norm equalities for operators, *Matematicki Vesnik*, **67** (2015), 269–276.

[9] A. Seddik, Some results related to the Corach–Porta–Recht inequality, *Proc. Amer. Math. Soc.*, **129** (2001), 3009–3015.

[10] A. Seddik, On the injective norm and characterization of some subclasses of normal operators by inequalities or equalities, *J. Math. Anal. Appl.*, **351** (2009), 277–284.

[11] A. Seddik, Characterization of the class of unitary operators by operator inequalities, *Linear Multilinear Algebra*, **59** (2011), 1069–1074.

[12] A. Seddik, Closed operator inequalities and open problems, *Math. Inequal. Appl.*, **14** (2011), 147–157.

[13] A. Seddik, Moore–Penrose inverse and operator inequalities, *Extracta Math.*, **30** (2015), 29–39.

[14] T. Yamazaka and M. Yanagida, Relations between two operator inequalities and their applications to paranormal operators, *Acta Sci. Math. (Szeged)*, **69** (2003), 377–389.

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