Acta Sci. Math. (Szeged)
70 (2004), 229–236

On the norm of elementary operators in standard operator algebras

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Communicated by L. Kérchy

Abstract. Let $\mathcal{A}$ be a complex normed algebra. For $A, B \in \mathcal{A}$, define a basic elementary operator $M_{A,B} : \mathcal{A} \to \mathcal{A}$ by $M_{A,B}(X) = AXB$.

Given a standard operator algebra $\mathcal{A}$ acting on a complex normed space and $A, B \in \mathcal{A}$ we have:

(i) The lower estimate $\|M_{A,B} + M_{B,A}\| \geq 2(\sqrt{2}-1)\|A\|\|B\|$ holds.

(ii) The lower estimate $\|M_{A,B} + M_{B,A}\| \geq \|A\|\|B\|$ holds if

$$\inf_{\lambda \in C} \|A + \lambda B\| = \|A\| \text{ or } \inf_{\lambda \in C} \|B + \lambda A\| = \|B\|.$$

(iii) The equality $\|M_{A,B} + M_{B,A}\| = 2\|A\|\|B\|$ holds if

$$\|A + \lambda B\| = \|A\| + \|B\| \text{ for some unit scalar } \lambda.$$

These results extend analogous estimates established earlier for standard operator subalgebras of Hilbert space operators.

1. Introduction

Let $\mathcal{A}$ and $B(H)$ be a complex normed algebra and the C^*-algebra of all bounded linear operators on a complex Hilbert space H, respectively. For $A, B \in \mathcal{A}$, define a basic elementary operator $M_{A,B}: \mathcal{A} \to \mathcal{A}$ by $M_{A,B}(X) = AXB$. An

Received August 21, 2003, and in revised form November 11, 2003.
AMS Subject Classification (2000): 46L35, 47L35, 47B47.

elementary operator is a finite sum $R_{A,B} = \sum_{i=1}^{n} M_{A_i,B_i}$, where $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ are two n-tuples of elements of $\mathcal{A}$.

Many facts about the relation between the norm of $M_{A,B} + M_{B,A}$ and the norms of A, B are known (e.g. [2], [3], [5] etc.). In a prime C^*-algebra (a prime C^*-algebra is a C^*-algebra where $M_{A,B} = 0$ implies $A = 0$ or $B = 0$), Mathieu [3] proved that $\|M_{A,B}\| = \|A\|\,\|B\|$ and $\|M_{A,B} + M_{B,A}\| \geq (2/3)\,\|A\|\,\|B\|$. The most obvious prime C^*-algebras are $B(H)$ and $\mathcal{C}_\infty(H)$ (the C^*-algebra of all compact operators on H). In [5], Stachó and Zalar investigated in a standard operator subalgebra of $B(H)$ (a standard operator subalgebra of $B(H)$ is a subalgebra of $B(H)$ containing all finite rank operators; it is not assumed that it is selfadjoint or closed with respect to any topology), where they proved that $\|M_{A,B} + M_{B,A}\| \geq 2(\sqrt{2} - 1)\,\|A\|\,\|B\|$ and they conjectured the following:

Conjecture 1. *Let $\mathcal{A}$ be a standard operator subalgebra of $B(H)$. The estimate $\|M_{A,B} + M_{B,A}\| \geq \|A\|\,\|B\|$ holds for every $A, B \in \mathcal{A}$.*

Note that this conjecture was verified in the following cases [6, 2]:

(i) in the Jordan algebra of symmetric operators of $B(H)$,

(ii) for $A, B \in B(H)$ such that $\inf_{\lambda\in\mathbb{C}} \|A + \lambda B\| = \|A\|$ or $\inf_{\lambda\in\mathbb{C}} \|B + \lambda A\| = \|B\|$.

Here, we are interested in the more general case where $\mathcal{A}$ is a standard operator algebra acting on a complex normed space. We shall prove that $\|R_{A,B}\| \geq \sup\{|\sum_{i=1}^{n} f(A_i)g(B_i)| : f, g \in (\mathcal{A}^*)_1\}$, for any two n-tuples $A = (A_1, \ldots, A_n)$, $B = (B_1, \ldots, B_n)$ of elements of $\mathcal{A}$ (where $(\mathcal{A}^*)_1$ is the unit sphere of $\mathcal{A}^*$). As a consequence of this main result (in our general situation), we show that the Stachó–Zalar lower bound remains true, and the estimate $\|M_{A,B} + M_{B,A}\| \geq \|A\|\,\|B\|$ holds if one of the following conditions is satisfied:

(1) $\inf_{\lambda\in\mathbb{C}} \|A + \lambda B\| = \|A\|$,

(2) $\inf_{\lambda\in\mathbb{C}} \|B + \lambda A\| = \|B\|$,

(3) $\inf_{\lambda\in\mathbb{C}} \|A + \lambda B\| \leq (1/2)\,\|A\|$,

(4) $\inf_{\lambda\in\mathbb{C}} \|B + \lambda A\| \leq (1/2)\,\|B\|$.

So the conjecture of Stachó–Zalar (in our general situation) remains unknown only in the case:

(5) $(1/2)\,\|A\| < \inf_{\lambda\in\mathbb{C}} \|A + \lambda B\| < \|A\|$ and $(1/2)\,\|B\| < \inf_{\lambda\in\mathbb{C}} \|B + \lambda A\| < \|B\|$.

On the other hand, we are interested in the following problem:

Problem 1. Let $\mathcal{A}$ be a standard operator algebra acting on a complex normed space. For which n-tuples A, B of elements of $\mathcal{A}$ does the equality $\|R_{A,B}\| = \sum_{i=1}^{n} \|A_i\| \|B_i\|$ hold? In particular, for which $A, B \in \mathcal{A}$ does the equality $\|M_{A,B} + M_{B,A}\| = 2\|A\| \|B\|$ hold?

2. Preliminaries

If Ω is a complex Banach algebra with unit I, then the algebraic numerical range of an element A in Ω is by definition $W_0(A) = \{f(A) : f \in P(\Omega)\}$ (where $P(\Omega) = \{f \in \Omega^* : f(I) = 1 = \|f\|\}$), the numerical radius of an element A in Ω is by definition $w(A) = \sup\{|\lambda| : \lambda \in W_0(A)\}$, and the joint algebraic numerical range of an n-tuple $A = (A_1, \dots, A_n)$ of elements of Ω is by definition $W_0(A) = \{(f(A_1), \dots, f(A_n)) : f \in P(\Omega)\}$.

The numerical range of an element A in $B(H)$ is by definition $W(A) = \{\langle Ax, x\rangle : x \in H, \|x\| = 1\}$. It is known [7] that if $A \in B(H)$, then $W_0(A) = W(A)^-$ (where $W(A)^-$ denotes the closure of $W(A)$).

Definition 1. Let E be a complex normed space and let $\mathcal{A}$ denote a subalgebra of $B(E)$. $\mathcal{A}$ is called a standard operator subalgebra of $B(H)$ if it contains all finite rank operators.

Notation 1.

(1) For $(x, f) \in E \times E^*$ and $A, B \in \mathcal{A}$, we denote by
  - (i) $x \otimes f$ the operator defined on E by $(x \otimes f)y = f(y)x$,
  - (ii) $U_{A,B}$ the operator defined on $\mathcal{A}$ by $U_{A,B}(X) = AXB + BXA$,
  - (iii) $V_{A,B}$ the operator defined on $\mathcal{A}$ by $V_{A,B}(X) = AXB - BXA$.

(2) We denote by $(E)_1$ the unit sphere of E.

(3) Let K be a bounded subset of $\mathbb{C}$ and let $M, N \subset \mathbb{C}^n$; we denote by
  - (i) $|K|$ the non-negative number $\sup\{|\lambda| : \lambda \in K\}$,
  - (ii) by $M \circ N$ the subset $\{\sum_{i=1}^{n} \alpha_i \beta_i : (\alpha_1, \dots, \alpha_n) \in M,\ (\beta_1, \dots, \beta_n) \in N\}$ of $\mathbb{C}$.

The purpose of this paper is to extend the following theorems in more general forms.

Theorem 1. [5] *Let $\mathcal{A}$ be a standard operator subalgebra of $B(H)$. Then $\|U_{A,B}\| \geq 2(\sqrt{2} - 1)\|A\| \|B\|$, for any $A, B \in \mathcal{A}$.*

Theorem 2. [2] *Let* $A, B \in B(H)$ *such that* $\inf_{\lambda\in\mathbb{C}} \|A+\lambda B\| = \|A\|$ *or* $\inf_{\lambda\in\mathbb{C}} \|B+\lambda A\| = \|B\|$. *Then* $\|U_{A,B}\| \geq \|A\| \, \|B\|$.

Theorem 3. [4] *Let* $\mathcal{A}$ *be a standard operator subalgebra of* $B(H)$ *and* $A, B \in \mathcal{A}$ *such that* $w(A^*B) = \|A\| \, \|B\|$. *Then* $\|U_{A,B}\| = 2\,\|A\| \, \|B\|$.

Remark 1. It is known [1] that if $A, B \in B(H)$, then $\|A+\lambda B\| = \|A\| + \|B\|$ for some unit scalar λ if and only if $w(A^*B) = \|A\| \, \|B\|$. So Theorem 3 may be reformulated as follows:

Let $\mathcal{A}$ be a standard operator subalgebra of $B(H)$ and $A, B \in \mathcal{A}$ such that $\|A+\lambda B\| = \|A\| + \|B\|$ for some unit scalar λ. Then $\|U_{A,B}\| = 2\,\|A\| \, \|B\|$.

3. A lower bound of the norm of $R_{A,B}$

In this section, we consider a standard operator algebra $\mathcal{A}$ acting on a complex normed space E.

Theorem 4. *Let* $A = (A_1, \dots, A_n)$ *and* $B = (B_1, \dots, B_n)$ *be two* n*-tuples of elements of* $\mathcal{A}$. *Then*

$$\|R_{A,B}\| \geq \sup \left\{ \left| \sum_{i=1}^{n} f(A_i) g(B_i) \right| : f, g \in (\mathcal{A}^*)_1 \right\}.$$

Proof. Let $x, y \in (E)_1$, $f, g \in (\mathcal{A}^*)_1$ and $h \in (E^*)_1$. Then we have:

$$\begin{aligned} \|R_{A,B}\| &\geq \left\| \sum_{i=1}^{n} A_i (x \otimes h) B_i \right\| \\ &\geq \left\| \sum_{i=1}^{n} A_i (x \otimes h) B_i y \right\| = \left\| \sum_{i=1}^{n} h(B_i y) A_i x \right\|. \end{aligned}$$

Thus $\|R_{A,B}\| \geq \sup_{\|x\|=1} \|\sum_{i=1}^{n} h(B_i y) A_i x\| = \|\sum_{i=1}^{n} h(B_i y) A_i\|$. So that $\|R_{A,B}\| \geq |\sum_{i=1}^{n} h(B_i y) f(A_i)| = |h\left(\sum_{i=1}^{n} f(A_i) B_i y\right)|$.

Then $\|R_{A,B}\| \geq \sup\left\{ |h\left(\sum_{i=1}^{n} f(A_i) B_i y\right)| : h \in (E^*)_1 \right\} = \|\sum_{i=1}^{n} f(A_i) B_i y\|$. Hence $\|R_{A,B}\| \geq \sup_{\|y\|=1} \|\sum_{i=1}^{n} f(A_i) B_i y\| = \|\sum_{i=1}^{n} f(A_i) B_i\|$. Therefore $\|R_{A,B}\| \geq |\sum_{i=1}^{n} f(A_i) g(B_i)|$.

■

Corollary 1. *Let $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ be two n-tuples of elements of $\mathcal{A}$ such that $\|\sum_{i=1}^n A_i\| = \sum_{i=1}^n \|A_i\|$ and $\|\sum_{i=1}^n B_i\| = \sum_{i=1}^n \|B_i\|$. Then $\|R_{A,B}\| = \sum_{i=1}^n \|A_i\| \|B_i\|$.*

Proof. From the hypothesis $\|\sum_{i=1}^n A_i\| = \sum_{i=1}^n \|A_i\|$ and $\|\sum_{i=1}^n B_i\| = \sum_{i=1}^n \|B_i\|$ and using the Hahn–Banach theorem, we may choose f_0, g_0 in $(\mathcal{A}^*)_1$ such that $f_0(A_i) = \|A_i\|$ and $g_0(B_i) = \|B_i\|$, for $i = 1, \ldots, n$.

So from Theorem 4, we obtain:

$$\|R_{A,B}\| \geq \left|\sum_{i=1}^n f_0(A_i) g_0(B_i)\right| = \sum_{i=1}^n \|A_i\| \|B_i\| .$$

Then the result follows immediately.

∎

The next corollary is a particular case of the above result.

Corollary 2. *Let $A, B \in \mathcal{A}$ such that $\|A + \lambda B\| = \|A\| + \|B\|$ for some unit scalar λ. Then $\|U_{A,B}\| = 2 \|A\| \|B\|$.*

Remark 2. The above result gives a general form of Theorem 3.

Corollary 3. *Assume E is a Banach space and $\mathcal{A} = B(E)$. Let $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ be two n-tuples of elements of $\mathcal{A}$. Then*

$$\|R_{A,B}\| \geq |W_0(A) \circ W_0(B)| .$$

Proof. The proof follows immediately from the above theorem and since $P(\mathcal{A}) \subset (\mathcal{A}^*)_1$.

∎

Corollary 4. *Let $A, B \in \mathcal{A}$. Then $\|U_{A,B}\| \geq 2(\sqrt{2} - 1) \|A\| \|B\|$.*

Proof. We may assume without loss of the generality, that $\|A\| = \|B\| = 1$.

For every $f, g \in (\mathcal{A}^*)_1$, we obtain from Theorem 4:

$$\|U_{A,B}\| \geq |f(A)g(B) + f(B)g(A)| \tag{1}$$

Applying inequality (1) for $g = f$, we obtain:

$$\|U_{A,B}\| \geq 2 |f(A)f(B)| . \tag{2}$$

By the Hahn–Banach theorem, we may choose f_0 and g_0 in $(\mathcal{A}^*)_1$ such that $f_0(B) = g_0(A) = 1$.

Put $f_0(A) = \alpha$ and $g_0(B) = \beta$.

For $f = f_0$ and $g = g_0$ inequality (1) yields :

$$\|U_{A,B}\| \geq |1 + \alpha\beta| \geq 1 - |\alpha\beta| \,. \tag{3}$$

Applying inequality (2) twice for $f = f_0$ and for $f = g_0$, we obtain:

$$\begin{cases} \|U_{A,B}\| \geq 2\,|\alpha| & (4) \\ \|U_{A,B}\| \geq 2\,|\beta| \,. & (5) \end{cases}$$

From (3), (4) and (5), we obtain $\|U_{A,B}\|^2 + 4\,\|U_{A,B}\| \geq 4\,|\alpha\beta| + 4(1 - |\alpha\beta|) = 4$. Therefore $\|U_{A,B}\| \geq 2(\sqrt{2} - 1)$.

■

Remark 3. Note that the estimate given in the above corollary is obtained by Stachó–Zalar in the particular case of a standard operator algebra acting on a Hilbert space, see [5]; but here we have obtained it, using another way, in a more general situation.

Corollary 5. *Let* $A, B \in \mathcal{A}$ *such that* $\inf_{\lambda\in\mathbb{C}} \|A + \lambda B\| = \|A\|$ *or* $\inf_{\lambda\in\mathbb{C}} \|B + \lambda A\| = \|B\|$. *Then*:

(i) $\|U_{A,B}\| \geq \|A\|\,\|B\|$,

(ii) $\|V_{A,B}\| \geq \|A\|\,\|B\|$.

Proof. If $\inf_{\lambda\in\mathbb{C}} \|A + \lambda B\| = \|A\|$, then by using the Hahn–Banach theorem, there exists f_0 in $(\mathcal{A}^*)_1$ such that $f_0(B) = 0$ and $f_0(A) = \|A\|$. So by Theorem 4, we obtain:

$$\begin{cases} \|U_{A,B}\| \geq \sup\{|f_0(A)g(B) + f_0(B)g(A)| : g \in (\mathcal{A}^*)_1\} \\ \|V_{A,B}\| \geq \sup\{|f_0(A)g(B) - f_0(B)g(A)| : g \in (\mathcal{A}^*)_1\} \,. \end{cases}$$

Thus

$$\begin{cases} \|U_{A,B}\| \geq \|A\| \sup\{|g(B)| : g \in (\mathcal{A}^*)_1\} = \|A\|\,\|B\| \\ \|V_{A,B}\| \geq \|A\| \sup\{|g(B)| : g \in (\mathcal{A}^*)_1\} = \|A\|\,\|B\| \,. \end{cases}$$

■

Remark 4. Corollary 5 (i) gives a general form of Theorem 2 and it is obtained by a direct proof.

Theorem 5. *Let $A, B \in \mathcal{A}$ such that* $\inf_{\lambda\in\mathbb{C}} \|A+\lambda B\| \le (1/2)\|A\|$ *or* $\inf_{\lambda\in\mathbb{C}} \|B+\lambda A\| \le (1/2)\|B\|$. *Then* $\|U_{A,B}\| \ge \|A\|\,\|B\|$.

Proof. Suppose $\inf_{\lambda\in\mathbb{C}} \|A+\lambda B\| \le (1/2)\|A\|$.

By a simple computation, we obtain that $V_{A,B} = V_{A+\lambda B,B}$, for all complex λ. Then $\|V_{A,B}\| \le 2\inf_{\lambda\in\mathcal{C}} \|A+\lambda B\|\,\|B\|$.

Thus $\|V_{A,B}\| \le \|A\|\,\|B\|$. Since $\|U_{A,B}\| + \|V_{A,B}\| \ge \|U_{A,B}+V_{A,B}\| = 2\|A\|\,\|B\|$, so $\|U_{A,B}\| \ge \|A\|\,\|B\|$.

By the same argument, we obtain the theorem with the second condition. ■

Remark 5. From Corollary 5 (i) and Theorem 5, we have obtained that the conjecture of Stachó–Zalar (in our general situation) is verified in the following cases:

(1) $\inf_{\lambda\in\mathbb{C}} \|A+\lambda B\| = \|A\|$,
(2) $\inf_{\lambda\in\mathbb{C}} \|B+\lambda A\| = \|B\|$,
(3) $\inf_{\lambda\in\mathbb{C}} \|A+\lambda B\| \le (1/2)\|A\|$,
(4) $\inf_{\lambda\in\mathbb{C}} \|B+\lambda A\| \le (1/2)\|B\|$.

So, it remains unknown only in the case where $(1/2)\|A\| < \inf_{\lambda\in\mathbb{C}} \|A+\lambda B\| < \|A\|$ and $(1/2)\|B\| < \inf_{\lambda\in\mathbb{C}} \|B+\lambda A\| < \|B\|$.

Theorem 6. *Let $A, B \in \mathcal{A}$. Then* $\|U_{A,B}\| \ge (1/2)\|V_{A,B}\|$.

Proof. We may assume without loss of the generality, that $\|B\| = 1$.

By the Hahn–Banach theorem, there exists f_0 in $(\mathcal{A}^*)_1$ such that $f_0(B) = 1$. Put $f_0(A) = \alpha$.

It follows from Theorem 4 that

$$\|U_{A,B}\| \ge \sup\{|f_0(A)g(B) + f_0(B)g(A)| : g \in (\mathcal{A}^*)_1\} = \|A+\alpha B\|\,.$$

Since $\|V_{A,B}\| = \|V_{A+\alpha B,B}\| \le 2\|A+\alpha B\|$, then $\|U_{A,B}\| \ge (1/2)\|V_{A,B}\|$. ■

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