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SURVEY ARTICLE

Operator inequalities related to the arithmetic–geometric mean inequality and characterizations

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Abstract
In this survey, we shall present the characterizations of some distinguished classes of bounded linear operators acting on a complex separable Hilbert space in terms of operator inequalities related to the arithmetic–geometric mean inequality.

Keywords Unitary operator · Normal operator · Selfadjoint operator · Arithmetic–geometric mean inequality

Mathematics Subject Classification 47A30 · 47A05 · 47B15

1 Definitions and notations

Let $\mathfrak{B}(H)$ be the C*-algebra of all bounded linear operators acting on a complex separable Hilbert space H, and let $\mathcal{N}(H)$, $\mathbb{S}(H)$, and $\mathcal{U}(H)$ denote the class of all normal operators, the class of all selfadjoint operators, and the class of unitary operators in $\mathfrak{B}(H)$, respectively.

(1) We denote by

- $\mathfrak{I}(H)$, the group of all invertible elements in $\mathfrak{B}(H)$,
- $\mathcal{R}(H)$, the set of all operators with closed ranges in $\mathfrak{B}(H)$,
- $\mathbb{S}_0(H) = \mathbb{S}(H) \cap \mathfrak{I}(H)$, the set of all invertible selfadjoint operators in $\mathfrak{B}(H)$,
- $\mathbb{S}_{cr}(H) = \mathbb{S}(H) \cap \mathcal{R}(H)$, the set of all selfadjoint operators with closed ranges in $\mathfrak{B}(H)$,
- $\mathcal{N}_0(H) = \mathcal{N}(H) \cap \mathfrak{I}(H)$, the set of all invertible normal operators in $\mathfrak{B}(H)$,
- $\mathcal{N}_{cr}(H) = \mathcal{N}(H) \cap \mathcal{R}(H)$, the set of all normal operators with closed ranges in $\mathfrak{B}(H)$,

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- $\mathcal{U}_r(H) = \mathbb{S}_0(H) \cap \mathcal{U}(H)$, the set of all unitary reflection operators in $\mathfrak{B}(H)$,
- $u \otimes v$ (where $u,\ v \in H$), the operator of rank less or equal to one on H defined by $(u \otimes v)\, x = \langle x, v\rangle\, u$, for every $x \in H$,
- $\mathcal{F}_1(H) = \{x \otimes y : x,\ y \in H\}$, the set of all operators of rank less or equal to one on H,
- $|S|$, the positive square root of the positive operator S^*S (where $S \in \mathfrak{B}(H)$),
- $\{S\}' = \{X \in \mathfrak{B}(H) : SX = XS\}$, the commutant of S (where $S \in \mathfrak{B}(H)$),
- $(M)_1 = \{x \in M : \|x\| = 1\}$, for M be a subset of some normed space,
- $K \circ L = \left\{\sum_{i=1}^n \alpha_i \beta_i : (\alpha_1, \ldots, \alpha_n) \in K,\ (\beta_1, \ldots, \beta_n) \in L\right\}$, for $L,\ K \subset \mathbb{C}^n$, $n \geq 1$,
- $|\Gamma| = \sup_{\gamma\in\Gamma} |\gamma|$, where Γ is a bounded subset of the field of scalars,
- $\Gamma M = \{\lambda m : \lambda \in \Gamma,\ m \in M\}$, where M is a subspace of some vector space, and Γ is a subset of the field of scalars.
- E', the topological dual space of a normed space E.

(2) If $\mathcal{A}$ is a (real or complex) unital normed algebra, and $A \in \mathcal{A}$, then

- we denote by $\sigma(A)$ and $r(A)$, the spectrum and the spectral radius of A, respectively,
- we denote by $V(A)$ and $w(A)$, the algebraic numerical range and the numerical radius of A, respectively,
- A is called normaloid, if $w(A) = \|A\|$,
- A is called convexoid if $V(A) = co\sigma(A)$,
- if $\mathcal{A} = \mathfrak{B}(H)$, then $V(A) = \overline{W(A)}$ (where $\overline{W(A)}$ is the closure of the usual numerical range of A).

(3) For $S \in \mathfrak{B}(H)$, let $R(S)$ and $\ker S$ denote the range and the kernel of S, respectively.

(4) It is known that for $S \in \mathfrak{B}(H)$, then $S \in \mathcal{R}(H)$ if and only if there exits an operator $S^+ \in \mathcal{R}(H)$ satisfying the four following equations:

$$SS^+S = S,\ S^+SS^+ = S^+,\ (SS^+)^* = SS^+,\ (S^+S)^* = S^+S.$$

Then the operator S^+ if exists is unique, and it is called the Moore–Penrose inverse of S, and it satisfies that SS^+ and S^+S are orthogonal projections onto $R(S)$ and $R(S^*)$, respectively. It is clear that if $S \in \mathfrak{I}(H)$, then $S^+ = S^{-1}$, and if $S \in \mathfrak{B}(H)$ is a surjective operator (resp. injective with closed range), then $SS^+ = I$ (rep. $S^+S = I$).

(5) For every S in $\mathcal{R}(H)$,

- we associate the 2×2 matrix representation $S = \begin{bmatrix} S_1 & S_2 \\ 0 & 0 \end{bmatrix}$ with respect to the orthogonal direct sum $H = R(S) \oplus \ker S^*$,
- the operator S is called an EP operator if $R(S^*) = R(S)$, or equivalently $S_2 = 0$ and S_1 is invertible; in this case $S^+ = \begin{bmatrix} S_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$,
- if S is normal, then it is an EP operator.

(6) Let E be a (real or complex) normed space, and let $\mathfrak{B} = \mathfrak{B}(E)$ denote the normed algebra of all bounded linear operators acting on E.

(i) Let $A=(A_1,\ldots,A_n),\ B=(B_1,\ldots,B_n)$ be two n-tuples of elements in $\mathfrak{B}$:
(.) we define the elementary operator (induced by $A,\ B$) $R_{A,B}$ on $\mathfrak{B}$ by:

$$\forall X\in\mathfrak{B},\ R_{A,B}(X)=\sum_{i=1}^{n}A_iXB_i,$$

(.) we denote by $\mathcal{R}(\mathfrak{B})$, the vector space of all elementary operators on $\mathfrak{B}$,
(.) we define the map $d(.):\mathcal{R}(\mathfrak{B})\longrightarrow\mathbb{R}$ by:

$$\forall R\in\mathcal{R}(\mathfrak{B}),\ d(R)=\sup_{\|X\|=1=\text{rank}X}\|R(X)\|,$$

(.) we consider the tensor product space

$$\mathfrak{B}\otimes\mathfrak{B}=\left\{\sum_{i=1}^{n}A_i\otimes B_i : n\geq 1,\ A_i,\ B_i\in\mathfrak{B},\ i=1,\ldots,n\right\},$$

(.) we denote by $\|.\|_\lambda$ the injective norm on $\mathfrak{B}\otimes\mathfrak{B}$ given by:

$$\left\|\sum_{i=1}^{n}A_i\otimes B_i\right\|_\lambda=\sup_{f,g\in(\mathfrak{B}')_1}\left|\sum_{i=1}^{n}f(A_i)g(B_i)\right|.$$

(ii) For $A,\ B\in\mathfrak{B}$, we define the particular elementary operator $U_{A,B}$ (called the Jordan algebra of symmetric operators) on $\mathfrak{B}$ by:

$$\forall X\in\mathfrak{B},\ U_{A,B}(X)=AXB+BXA.$$

(7) For two vectors x,y in a given (real or complex) inner product space, the relation $x/\!\!/y$ (that means x,y are linearly dependent) holds if and only if $\|x+\lambda y\|=\|x\|+\|y\|$, for some unit scalar λ. The two above conditions make sense in any normed space and the first condition implies the second, but the converse is false in general. So, we may introduce a new concept of the parallelism relation in the geometry of normed space as follows: for x,y in a given normed space, we say that x is norm-parallel to y $(x\parallel y)$, if $\|x+\lambda y\|=\|x\|+\|y\|$, for some unit scalar λ (this new concept of parallelism in the geometry of normed space was introduced in [16]). Note that this last relation is reflexive and symmetric, but not transitive.

2 Introduction

My main purpose of this survey paper is to present our characterizations (presented in several papers) of some distinguished classes of bounded linear operators acting on H, namely, the selfadjoint operators, the normal operators, and the unitary operators, in terms of operator inequalities.

Our first motivation was the so-called Corach–Porta–Recht inequality.

In [6], Corach et al. proved that for every invertible selfadjoint operator $S \in \mathfrak{B}(H)$, the following operator inequality holds

$$\forall X \in \mathfrak{B}(H), \quad \left\| SXS^{-1} + S^{-1}XS \right\| \geq 2 \left\| X \right\|. \tag{2.1}$$

So, it is interesting to describe the largest class of all operators $S \in \mathfrak{I}(H)$ satisfying (2.1). It is clear that this class contains $(\mathbb{C})_1 \, \mathbb{S}_0(H)$, and in [14], we had shown that it is exactly this class $(\mathbb{C})_1 \, \mathbb{S}_0(H)$ (the class all rotation of all selfadjoint operators $S \in \mathfrak{I}(H)$, or also the class of all operators $S \in \mathcal{N}_0(H)$ whose spectrum lies in a straight line through the origin).

Note that (2.1) is an immediate consequence of the known arithmetic–geometric mean inequality given as follows:

$$\forall A, B, X \in \mathfrak{B}(H), \quad \left\| A^*AX + XBB^* \right\| \geq 2 \left\| AXB \right\|. \tag{2.2}$$

We consider a second version of the arithmetic–geometric mean inequality which follows immediately from (2.2) given as follows:

$$\forall A, B, X \in \mathfrak{B}(H), \quad \left\| A^*AX \right\| + \left\| XBB^* \right\| \geq 2 \left\| AXB \right\|. \tag{2.3}$$

It is easy to see that for $S \in \mathfrak{B}(H)$, the three following properties are equivalent:

(i) S is normal,
(ii) $\forall X \in \mathfrak{B}(H), \ \|S^*X\| = \|SX\|$,
(iii) $\forall X \in \mathfrak{B}(H), \ \|XS^*\| = \|XS\|$.

From this fact and using (2.3), we may deduce that for every invertible normal operator $S \in \mathfrak{B}(H)$ the following operator inequality holds:

$$\forall X \in \mathfrak{B}(H), \quad \left\| SXS^{-1} \right\| + \left\| S^{-1}XS \right\| \geq 2 \left\| X \right\|. \tag{2.4}$$

Following the same problem of characterization cited above, so what is the largest class of all of all operators $S \in \mathfrak{I}(H)$ satisfying (2.4)? In [18], we had found that this class is exactly the class $\mathcal{N}_0(H)$ of all invertible normal operators in $\mathfrak{B}(H)$.

In [7], Fujii et al. had proved that the inequality (2.1) is equivalent to (2.2), and in [4], we found other operator inequalities that are also equivalent to (2.2), and here we cite three of them given by:

$$\forall X \in \mathfrak{B}(H), \quad \left\| SXS^{+} + S^{+}XS \right\| \geq 2 \left\| SS^{+}XS^{+}S \right\|, \tag{2.5}$$

for every $S \in \mathbb{S}_{cr}(H)$,

$$\forall X \in \mathfrak{B}(H), \quad \left\| S^2X + XS^2 \right\| \geq 2 \left\| SXS \right\|, \tag{2.6}$$

for every $S \in \mathbb{S}_{cr}(H)$,

$$\forall X \in \mathfrak{B}(H), \quad \left\| S^2 X + X S^2 \right\| \geq 2 \left\| SXS \right\|, \tag{2.7}$$

for every $S \in \mathbb{S}(H)$.

Note that this family of operator inequalities is generated by a selfadjoint operator (invertible, with closed range, and any).

In [4], we have showed that (2.3) is equivalent to (2.4) and to the three following inequalities:

$$\forall X \in \mathfrak{B}(H), \ \left\| SXS^{+} \right\| + \left\| S^{+}XS \right\| \geq 2 \left\| SS^{+}XS^{+}S \right\|, \tag{2.8}$$

for every $S \in \mathcal{N}_{cr}(H)$,

$$\forall X \in \mathfrak{B}(H), \ \left\| S^2 X \right\| + \left\| X S^2 \right\| \geq 2 \left\| SXS \right\|, \tag{2.9}$$

for every $S \in \mathcal{N}_{cr}(H)$,

$$\forall X \in \mathfrak{B}(H), \ \left\| S^2 X \right\| + \left\| X S^2 \right\| \geq 2 \left\| SXS \right\|, \tag{2.10}$$

for every $S \in \mathcal{N}(H)$.

This second family of operator inequalities (2.4), (2.8), (2.9), and (2.10) that are equivalent to (2.3) is generated by a normal operator (invertible, with closed range, and any).

As we have done for the two characterizations cited above for the invertible case, it is interesting to describe (for the closed range case):

(i) the class of all operators $S \in \mathcal{R}(H)$ satisfying the operator inequality (2.5) or (2.6),
(ii) the class of all operators $S \in \mathcal{R}(H)$ satisfying the operator inequality (2.8) or (2.9).

In [21], using the two characterizations cited above for the invertible case, we had shown that the class (i) is exactly the class $(\mathbb{C})_1 \, \mathbb{S}_{cr}(H)$, and the class (ii) is $\mathcal{N}_{cr}(H)$.

But, unfortunately, after the publication of the paper, we have found a mistake in Lemma 1 of [21], and all results depend on it. So, in the corrigendum [22], we have presented a corrected proof of this lemma. Note that in the proof of this corrected lemma, we have used Theorem 6.3 of [14], where one of the conditions of this theorem is an equality between the spectrum of two positive operators. This condition is enough for the invertible case only, but does not suffice for non-invertible case and our lemma is for non-invertible case. But, to have a complete proof of the lemma, we need Theorem 6.3 with inclusion between spectrum instead of equality. We have mentioned in the proof of the corrected lemma that Theorem 6.3 remains true with inclusion between spectrum, but without argument. In this survey, we shall present this argument.

From the closed range case to the general situation of the two operator inequalities (2.7) and (2.10), we have proved in [23] that the class of all operators $S \in \mathfrak{B}(H)$ satisfying (2.7) (resp. (2.10)) is the class $(\mathbb{C})_1\, \mathbb{S}(H)$ (resp. $\mathcal{N}(H)$).

In this general situation, we have used a result of Halmos [9] that says the set $\mathfrak{D}(H) = \{S \in \mathfrak{B}(H) : S$ is left invertible or right invertible$\}$ is dense in $\mathfrak{B}(H)$ (where $\mathfrak{D}(H) \subset \mathcal{R}(H)$). In our proof, applying the characterizations cited above with the domain $\mathcal{R}(H)$ and the density of $\mathcal{R}(H)$ in $\mathfrak{B}(H)$, we conclude our characterizations in the general situation of the class $(\mathbb{C})_1\, \mathbb{S}(H)$ of all rotation of all selfadjoint operators in $\mathfrak{B}(H)$, and the class $\mathcal{N}(H)$ of all normal operators in $\mathfrak{B}(H)$.

Our idea in the above characterizations is to make connection between a family of operator inequalities related to the known arithmetic–geometric mean inequality (resp. the second version of the known arithmetic–geometric mean inequality) and some distinguished classes of operators.

In this survey, we adopt another and better strategy, we do not respect the chronological order of the publication of the original papers, but start with the invertible case, then the general situation, and then we deduce the closed range case; we also present the characterizations of the class of normal operators before the characterizations concerning the selfadjoint operators case. This new strategy gets rid of a heavy proof of one of the theorems concerning the closed range case.

For the third distinguished classes of operators, the class of all unitary operators $\mathcal{U}(H)$, we had proved in [19] that it is exactly the class of all operators $S \in (\mathfrak{I}(H))_1$ satisfying each of the following operator inequalities:

$$\forall X \in \mathfrak{B}(H), \quad \left\| SXS^{-1} + S^{-1}XS \right\| \leq 2\,\|X\|\,,$$
$$\forall X \in \mathfrak{B}(H), \quad \left\| SXS^{-1} \right\| + \left\| S^{-1}XS \right\| = 2\,\|X\|\,,$$
$$\forall X \in \mathfrak{B}(H), \quad \left\| SXS^{-1} \right\| + \left\| S^{-1}XS \right\| \leq 2\,\|X\|\,.$$

In Sect. 3 of this survey, we shall

(i) show that $d(.)$ is a norm on $\mathcal{R}\,(\mathfrak{B}(E))$,
(ii) prove that the two normed spaces $(\mathcal{R}\,(\mathfrak{B}(E))\,, d(.))$ and $\left(\mathfrak{B}(E) \otimes \mathfrak{B}(E), \|.\|_{\lambda}\right)$ are isometrically isomorphic,
(iii) introduce a new concept of the parallelism in the geometry of normed space (called the norm-parallelism),
(iv) give the concept of normaloid element in an abstract (real or complex) unital normed algebra, and in the C*-algebra $\mathfrak{B}(H)$ in terms of the norm-parallelism, precisely, we shall prove that for a (real or complex) unital normed algebra $\mathcal{A}$ with unit element I, for $A \in \mathcal{A}$, A is normaloid if and only if A is norm-parallel to the unit element I; and when $\mathcal{A} = \mathfrak{B}(H)$, A is normaloid if and only if A is norm-parallel to its adjoint A^*.

In Sect. 4, we shall present

(i) some results concerning the injective norm of the two following elementary operators on $\mathfrak{B}(H)$, $X \to SXS^{-1} + S^{-1}XS$, and $X \to S^*XS^{-1} + S^{-1}XS^*$ (where S be an invertible operator in $\mathfrak{B}(H)$),

(ii) some characterizations of the class $\mathcal{U}(H)$ of all unitary operators in $\mathfrak{B}(H)$ in terms of operator inequalities.

In Sect. 5, we shall present

(i) Section 5.1: a family of operator inequalities that are equivalent to (2.3),
(ii) Section 5.2: the characterizations cited above of the classes $\mathcal{N}_0(H)$, $\mathcal{N}_{cr}(H)$, and $\mathcal{N}(H)$ in terms of operator inequalities related to (2.3),
(iii) Section 5.3: other characterizations of the class $\mathcal{N}_0(H)$, and we deduce some new general characterizations of the class $\mathcal{N}(H)$.

In Sect. 6, we shall present

(i) Section 6.1: a family of operator inequalities that are equivalent to (2.2),
(ii) Section 6.2: the characterizations cited above of the classes $(\mathbb{C})_1\, \mathbb{S}_0(H)$, $(\mathbb{C})_1\, \mathbb{S}_{cr}(H)$, and $(\mathbb{C})_1\, \mathbb{S}(H)$ in terms of operator inequalities related to (2.2).

3 Injective norm, norm-parallelism, and normaloid operator

In this section, we consider $B = \mathfrak{B}(E)$ the normed algebra of all bounded linear operators acting on a (real or complex) normed space E.

We shall show that $d(.)$ is a norm on the vector space $\mathcal{R}(\mathfrak{B})$, and the two normed spaces $(\mathcal{R}(\mathfrak{B}), d(.))$, $\left(\mathfrak{B} \otimes \mathfrak{B}, \|.\|_{\lambda}\right)$ are isometrically isomorphic; and we shall present the concept of normaloid element in any abstract (real or complex) unital normed algebra, and in the C*-algebra $\mathfrak{B}(H)$ in terms of the norm-parallelism.

We start with the following main theorem.

Proposition 3.1 [16] *Let $A = (A_1, \ldots, A_n)$, $B = (B_1, \ldots, B_n)$ be two n-tuples of elements in $\mathfrak{B}$. then the following equalities hold*:

$$\begin{aligned} d\left(R_{A,B}\right) &= \sup_{f,g \in (\mathfrak{B}^{'})_1} \left| \sum_{i=1}^{n} f(A_i) g\left(B_i\right) \right| \\ &= \sup_{f \in (\mathfrak{B}^{'})_1} \left\| \sum_{i=1}^{n} f(B_i) A_i \right\| \\ &= \sup_{f \in (\mathfrak{B}^{'})_1} \left\| \sum_{i=1}^{n} f(A_i) B_i \right\|. \end{aligned}$$

Proof We denote by k_1, k_2 and k_3 the supremum cited in the theorem in the same order. Let $x, y \in (E)_1$, $h \in \left(E^{'}\right)_1$, and let $f, g \in (\mathfrak{B}^{'})_1$. So, we have

$$\begin{aligned} d\left(R_{A,B}\right) &\geq \left\| \sum_{i=1}^{n} A_i\, (x \otimes h)\, B_i\, y \right\| \\ &= \left\| \left(\sum_{i=1}^{n} h\left(B_i y\right) A_i \right) x \right\|. \end{aligned}$$

By taking the supremum over $x \in (E)_1$, we have $d\left(R_{A,B}\right) \geq \left\| \sum_{i=1}^{n} h\left(B_i y\right) A_i \right\|$. Thus,

$$d\left(R_{A,B}\right) \geq \left| \sum_{i=1}^{n} f\left(A_i\right) h\left(B_i y\right) \right| = \left| h\left(\sum_{i=1}^{n} f\left(A_i\right) B_i y \right) \right|.$$

By taking the supremum over $h \in \left(E'\right)_1$ and over $y \in (E)_1$, we obtain

$$d\left(R_{A,B}\right) \geq \left\| \sum_{i=1}^{n} f\left(A_i\right) B_i \right\|.$$

Then,

$$d\left(R_{A,B}\right) \geq \left| \sum_{i=1}^{n} f\left(A_i\right) g\left(B_i\right) \right|.$$

So, we have $d\left(R_{A,B}\right) \geq k_1$. Since $k_1 \geq \left| g\left(\sum_{i=1}^{n} f\left(B_i\right) A_i\right) \right|$, then $k_1 \geq \left\| \sum_{i=1}^{n} f\left(B_i\right) A_i \right\|$. This gives us that $k_1 \geq k_2$. It is clear that $k_2 \geq \left| g\left(\sum_{i=1}^{n} f\left(A_i\right) B_i\right) \right|$, then $k_2 \geq k_3$. Since, $k_3 \geq \left| \sum_{i=1}^{n} f(A_i) h\left(B_i y\right) \right| = \left| f\left(\sum_{i=1}^{n} h\left(B_i y\right) A_i\right) \right|$, so we have

$$\begin{aligned} k_3 &\geq \left\| \sum_{i=1}^{n} h\left(B_i y\right) A_i \right\| \\ &\geq \left| \sum_{i=1}^{n} h\left(B_i y\right) A_i x \right| \\ &= \left\| \left(\sum_{i=1}^{n} A_i \left(x \otimes h\right) B_i \right) y \right\|. \end{aligned}$$

Thus, $k_3 \geq \left\| \sum_{i=1}^{n} A_i \left(x \otimes h\right) B_i \right\|$. Therefore, $k_3 \geq d\left(R_{A,B}\right)$. This completes the proof. □

Proposition 3.2 *The map* $d(.) : \mathcal{R}\left(\mathfrak{B}\right) \to \mathbb{R},\ R \mapsto d(R)$ *is a norm on* $\mathcal{R}\left(\mathfrak{B}\right)$.

Proof It is clear that

$$d(R) \geq 0,\ d(\lambda R) = |\lambda|\, d(R),\ d(R+S) \leq d(R) + d(S),$$

for every scalar λ, and for every $R, S \in \mathcal{R}\left(\mathfrak{B}\right)$. So, it remains to prove that if $d(R) = 0$, then $R = 0$, for every $R \in \mathcal{R}\left(\mathfrak{B}\right)$.

Now, let $A = (A_1, \ldots, A_n)$, $B = (B_1, \ldots, B_n)$ be two n-tuples of elements in $\mathfrak{B}$ such that $d\left(R_{A,B}\right) = 0$.

We may assume that $B_1, \ldots, B_m$ (where $m \leq n$) form a maximal linearly independent subset of $B_1, \ldots, B_n$. There exist m operators $C_1, \ldots, C_m \in sp\left\{A_1, \ldots, A_n\right\}$ such that $R_{A,B} = R_{C,D}$, where $C = \left(C_1, \ldots, C_m\right)$, $D = \left(B_1, \ldots, B_m\right)$. So, using the above proposition, we obtain $\sum_{i=1}^{m} f(C_i)B_i = 0$, for every $f \in \mathfrak{B}'$. Since $B_1, \ldots, B_n$ are linearly independent, $f(C_i) = 0$, for $i = 1, \ldots, m$, and for every $f \in \mathfrak{B}'$. This proves that $C_i = 0$, for $i = 1, \ldots, m$. Hence, $R_{A,B} = R_{C,D} = 0$. □

Corollary 3.3 *The two normed spaces $(\mathcal{R}(\mathfrak{B}), d(.))$ and $\left(\mathfrak{B} \otimes \mathfrak{B}, \|.\|_{\lambda}\right)$ are isometrically isomorphic.*

Proof Let the map

$$\begin{array}{ccc} \Gamma : \left(\mathfrak{B} \otimes \mathfrak{B}, \|.\|_{\lambda}\right) & \to & (\mathcal{R}(\mathfrak{B}), d(.)) \\ \sum_{i=1}^{n} A_i \otimes B_i & \longmapsto & R_{A,B}, \end{array}$$

where $A = (A_1, \ldots, A_n)$, $B = (B_1, \ldots, B_n) \in \mathfrak{B}^n$.

From Propositions 3.1 and 3.2, the map Γ is well defined and injective. It is clear that Γ is linear and surjective. Using again Proposition 3.1, we deduce that Γ is an isometry. □

Notation 3.4 (1) *According to the above identification, and for $R \in \mathcal{R}(\mathfrak{B})$, we use the notation $\|R\|_{\lambda}$ instead of $d(R)$, and we say it is the injective norm of R.*

(2) *Let $A = (A_1, \ldots, A_n)$, $B = (B_1, \ldots, B_n)$ be two n-tuples of elements in $\mathfrak{B}$; we put $D(R_{A,B}) = \sum_{i=1}^{n} \|A_i\| \|B_i\|$.*

Remark 3.5 For $R \in \mathcal{R}(\mathfrak{B})$, we have $\|R\|_{\lambda} \leq \|R\| \leq D(R)$.

In the next theorem, and for $A = (A_1, \ldots, A_n)$, $B = (B_1, \ldots, B_n)$ be two n-tuples of elements in $\mathfrak{B}$, then we shall characterize when the injective norm of the elementary operator $R_{A,B}$ gets its maximal value $D(R_{A,B})$, in terms of A_i, B_i. So, we need the following lemma, where the proof follows immediately from the Hahn–Banach Theorem.

Lemma 3.6 *Let $x_1, \ldots, x_n \in E$. Then the two following conditions are equivalent*:

(i) $\left\|\sum_{i=1}^{n} x_i\right\| = \sum_{i=1}^{n} \|x_i\|$,

(ii) $\exists f \in \left(E'\right)_1$, $f(x_i) = \|x_i\|$, $i = 1, \ldots, n$.

Proposition 3.7 [16] *Let $A = (A_1, \ldots, A_n)$, $B = (B_1, \ldots, B_n)$ be two n-tuples of elements in $\mathfrak{B}$. The following properties are equivalent*:

(i) $\left\|R_{A,B}\right\|_{\lambda} = D(R_{A,B})$,

(ii) *there exist two unit functional $f, g \in \mathfrak{B}'$, and n unit scalars $\lambda_1, \ldots, \lambda_n$ such that $f(A_i) = \lambda_i \|A_i\|$, $g(B_i) = \overline{\lambda_i} \|B_i\|$, for $i = 1, \ldots, n$.*

(iii) $\left\|\sum_{i=1}^{n} \lambda_i A_i\right\| = \sum_{i=1}^{n} \|A_i\|$, $\left\|\sum_{i=1}^{n} \overline{\lambda_i} B_i\right\| = \sum_{i=1}^{n} \|B_i\|$, *for some unit scalars $\lambda_1, \ldots, \lambda_n$.*

Proof We may assume that all A_i, B_i are nonzero.

The equivalence (ii) $\Longleftrightarrow$ (iii) follows from the above lemma.

(ii) $\Longrightarrow$ (i). Assume (ii) holds. Then from Proposition 3.1, $D(R_{A,B}) \geq \left\| R_{A,B} \right\|_{\lambda} \geq \left| \sum_{i=1}^{n} f(A_i) g\left(B_i\right) \right| = D(R_{A,B})$. This proves (i).

(i) $\Longrightarrow$ (ii). Assume (i) holds. The map $f \longmapsto \left\| \sum_{i=1}^{n} f(A_i) B_i \right\|$ is w*-continuous on $\mathfrak{B}'$, and $\left(\mathfrak{B}'\right)_1$ is w*-compact, so it follows that $\left\| R_{A,B} \right\|_{\lambda} = \left\| \sum_{i=1}^{n} f(A_i) B_i \right\|$, for some $f \in \left(\mathfrak{B}'\right)_1$. On the other hand, the Hahn–Banach guarantees also the existence of an element g in $\left(\mathfrak{B}'\right)_1$ such that $\sum_{i=1}^{n} \|A_i\| \|B_i\| = \left\| R_{A,B} \right\|_{\lambda} = \sum_{i=1}^{n} f(A_i) g\left(B_i\right)$. Since A_i, B_i are nonzero and $\left| f(A_i) \right| \leq \|A_i\|$, $\left| g\left(B_i\right) \right| \leq \|B_i\|$, for $i = 1, \ldots, n$, then $\left| f(A_i) \right| = \|A_i\|$, $\left| g\left(B_i\right) \right| = \|B_i\|$, and $f(A_i) g\left(B_i\right) = \|A_i\| \|B_i\|$, for $i = 1, \ldots, n$. Thus, $f(A_i) = \lambda_i \|A_i\|$, $g\left(B_i\right) = \overline{\lambda_i} \|B_i\|$, for $i = 1, \ldots, n$, and for some unit scalars $\lambda_1, \ldots, \lambda_n$. □

Corollary 3.8 [16] *Let A, $B \in \mathfrak{B}$. Then, the two following properties are equivalent*:

(i) *The injective norm of $U_{A,B}$ gets its maximal value* $2\|A\| \|B\|$,

(ii) $A \parallel B$.

Proof (i) $\Longrightarrow$ (ii). Assume (i) holds. Since $\left\| U_{A,B} \right\|_{\lambda} = 2\|A\| \|B\| = D(U_{A,B})$, and from the above proposition with the condition (iii), there exist two unit scalars α, β such that $\|\alpha A + \beta B\| = \|A\| + \|B\|$. Put $\lambda = \overline{\alpha}\beta$, then we have $\|A + \lambda B\| = \|A\| + \|B\|$, and where λ is a unit scalar. This proves (ii).

(ii) $\Longrightarrow$ (i). This implication follows immediately from the above proposition. □

In the end of this section, we shall present the concept of the normaloid element in an abstract unital algebra and in the C*-algebra $\mathfrak{B}(H)$ in terms of the norm-parallelism in the geometry of normed space.

Proposition 3.9 *Let $\mathcal{A}$ be a (real or complex) unital normed algebra with unit element I, and let $A \in \mathcal{A}$. The two following properties are equivalent*:

(i) *A is normaloid*,

(ii) $A \parallel I$.

Proof (i) $\Rightarrow$ (ii). Assume (i) holds. There exists a state f on $\mathcal{A}$ and a unit scalar λ such that $f(A) = \lambda \|A\|$. So, we obtain that $1 + \|A\| \geq \left\| I + \overline{\lambda} A \right\| \geq \left| f(I) + \overline{\lambda} f(A) \right| = 1 + \|A\|$. Hence, $\left\| I + \overline{\lambda} A \right\| = 1 + \|A\|$, and where $\left| \overline{\lambda} \right| = 1$. This proves (ii).

(ii) $\Rightarrow$ (i). Assume (ii) holds. Then, there exists a unit scalar λ such that $\|I + \lambda A\| = 1 + \|A\|$. Using Lemma 3.6, there exists $f \in \left(\mathcal{A}'\right)_1$ such that $f(I) = 1$, $f(\lambda A) = \|A\|$. Hence, f is a state on $\mathcal{A}$, $f(A) = \overline{\lambda} \|A\|$, and $\left| \overline{\lambda} \right| = 1$. Hence, A is normaloid. □

Corollary 3.10 *Let $A \in \mathfrak{B}$. Then, the two following properties are equivalent*:

(i) *The injective norm of the elementary operator $\mathfrak{B} \longrightarrow \mathfrak{B}$, $X \longmapsto AX + XA$ gets its maximal value* $2\|A\|$,

(ii) *A is normaloid.*

Proof This follows immediately from Corollary 3.8 and Proposition 3.9. □

Proposition 3.11 [17] *Let* $A \in \mathfrak{B}(H)$. *The two following properties are equivalent*:

(i) *A is normaloid*,
(ii) $A \parallel A^*$.

Proof (i) ⇒ (ii). Assume (i) holds. There exists a state f on $\mathfrak{B}(H)$ and a unit scalar λ such that $f(A) = \lambda \|A\|$. So, we obtain that $2\|A\| = f(\overline{\lambda}A + \lambda A^*) \le \left\|\overline{\lambda}A + \lambda A^*\right\| \le 2\|A\|$. Hence, $\left\|A + \lambda^2 A^*\right\| = \left\|\overline{\lambda}A + \lambda A^*\right\| = 2\|A\|$, and where $\left|\lambda^2\right| = 1$. Therefore, $A \parallel A^*$.

(ii) ⇒ (i). Assume (ii) holds. Then, there exists a unit scalar λ such that $\|A^* + \lambda A\| = 2\|A\|$. Since $A^* + \lambda A$ is normal, then there exists a state f on $\mathfrak{B}(H)$ such that $\|A^* + \lambda A\| = |f(A^* + \lambda A)|$. Hence, $2\|A\| = |f(A^* + \lambda A)| = \left|\overline{f(A)} + \lambda f(A)\right| \le 2|f(A)| \le 2\|A\|$. Thus, $|f(A)| = \|A\|$. This gives us that A is normaloid. □

4 On the injective norm of the two operators $X \longmapsto SXS^{-1} + S^{-1}XS$ and $X \longmapsto S^*XS^{-1} + S^{-1}XS^*$, unitary operators, and characterizations

In this section, we consider an invertible operator S in $\mathfrak{B}(H)$.

Notation 4.1 *For* $A = (A_1, \ldots, A_n)$ *being an n-tuple of commuting operators in* $\mathfrak{B}(H)$, *we denote by*:

(1) Γ_A, *the set of all multiplicative functionals acting on the maximal commutative Banach algebra that contains the operators* $A_1, \ldots, A_n$,
(2) $\sigma(A) = \left\{(\varphi(A_1), \ldots, \varphi(A_n)) : \varphi \in \Gamma_A\right\}$, *the joint spectrum of* A.

We define the two particular elementary operators φ_S, ψ_S on $\mathfrak{B}(H)$ by

$$\begin{cases} \forall X \in \mathfrak{B}(H),\ \varphi_S(X) = SXS^{-1} + S^{-1}XS, \\ \forall X \in \mathfrak{B}(H),\ \psi_S(X) = S^*XS^{-1} + S^{-1}XS^*. \end{cases}$$

In this section, we shall present some results concerning the injective norm of φ_S and ψ_S, and we characterize the class of all invertible operators S for which $\|\varphi_S\|_\lambda$ (resp. $\|\psi_S\|_\lambda$) is minimal, and the class of all unitary operators in $\mathfrak{B}(H)$.

Lemma 4.2 [18] *Let* $A = (A_1, \ldots, A_n)$, $B = (B_1, \ldots, B_n)$ *be two n-tuples of commuting operators in* $\mathfrak{B}(H)$. *Then* $\left\|R_{A,B}\right\|_\lambda \ge |\sigma(A) \circ \sigma(B)|$; *and this inequality becomes an equality, if all* A_i *and* B_i *are normal operators.*

Proof Let (φ, ψ) be an arbitrary pair in $\Gamma_A \times \Gamma_B$. Using the Hahn–Banach theorem, we may extend φ and ψ to unit functional f and g on $\mathfrak{B}(H)$, respectively. So from Proposition 3.1, it follows that $\left\| R_{A,B} \right\|_\lambda \geq \left| \sum_{i=1}^n f(A_i) g(B_i) \right| = \left| \sum_{i=1}^n \varphi(A_i) \psi(B_i) \right|$. Therefore, $\left\| R_{A,B} \right\|_\lambda \geq |\sigma(A) \circ \sigma(B)|$.

Now, suppose all A_i and B_i are normal operators. It suffices to prove $\left\| R_{A,B} \right\|_\lambda \leq |\sigma(A) \circ \sigma(B)|$. Let f, g be two arbitrary unit functionals on $\mathfrak{B}(H)$, and let (φ, ψ) be an arbitrary pair in $\Gamma_A \times \Gamma_B$. Since $|\sigma(A) \circ \sigma(B)| \geq \left| \psi \left(\sum_{i=1}^n \varphi(A_i) B_i \right) \right|$, and $\sum_{i=1}^n \varphi(A_i) B_i$ is normal (from Putnam–Fuglede), then $|\sigma(A) \circ \sigma(B)| \geq \left\| \sum_{i=1}^n \varphi(A_i) B_i \right\|$. So that

$$|\sigma(A) \circ \sigma(B)| \geq \left\| \sum_{i=1}^n \varphi(A_i) g\left(B_i\right) \right\| = \left\| \varphi \left(\sum_{i=1}^n g\left(B_i\right) A_i \right) \right\|.$$

Using the same argument as used with B_i, we deduce that $|\sigma(A) \circ \sigma(B)| \geq \left\| \sum_{i=1}^n g\left(B_i\right) A_i \right\|$. From Proposition 3.1, it follows that $|\sigma(A) \circ \sigma(B)| \geq \left\| R_{A,B} \right\|_\lambda$. □

Lemma 4.3 *We have* $\left\| \psi_S \right\|_\lambda = \left\| \varphi_P \right\|_\lambda$, *where* $P = |S|$.

Proof Let $S = UP$ be the polar decomposition S. From the fact that

$$\{X \in \mathfrak{B}(H) : \|X\| = 1 = \operatorname{rank} X\} = \left\{U^* X : X \in \mathfrak{B}(H),\ \|X\| = 1 = \operatorname{rank} X\right\},$$

it follows that

$$\begin{aligned}
\left\| \psi_S \right\|_\lambda &= \sup_{\|X\|=1=\operatorname{rank} X} \left\| S^* X S^{-1} + S^{-1} X S^* \right\| \\
&= \sup_{\|X\|=1=\operatorname{rank} X} \left\| P U^* X P^{-1} U^* + P^{-1} U^* X P U^* \right\| \\
&= \sup_{\|X\|=1=\operatorname{rank} X} \left\| P \left(U^* X\right) P^{-1} + P^{-1} \left(U^* X\right) P \right\| \\
&= \left\| \varphi_P \right\|_\lambda .
\end{aligned}$$

□

Proposition 4.4 [18] *The following properties hold*:

(i) $\left\| \varphi_S \right\|_\lambda \geq \sup_{\lambda, \mu \in \sigma(S)} \left| \frac{\lambda}{\mu} + \frac{\mu}{\lambda} \right|$,

(ii) *if S is normal, the above inequality becomes an equality*,

(iii) *if S is normal, the following holds*:

$$\left\| \psi_S \right\|_\lambda = \sup_{\lambda, \mu \in \sigma(S)} \left(\left| \frac{\lambda}{\mu} \right| + \left| \frac{\mu}{\lambda} \right| \right).$$

Proof (i) and (ii) follow immediately from Lemma 4.2.

(iii) Assume S is normal, and let UP be its polar decomposition.

Since S is invertible and normal, then $\sigma(P) = \{|\lambda| : \lambda \in \sigma(S)\}$. So from the above lemma and (ii), we obtain $\|\psi_S\|_\lambda = \|\varphi_P\|_\lambda = \sup_{\lambda,\mu\in\sigma(S)} \left(\left|\frac{\lambda}{\mu}\right| + \left|\frac{\mu}{\lambda}\right|\right)$. □

Corollary 4.5 (i) *We have* $\|\varphi_S\|_\lambda \geq 2$.

(ii) *If S is normal, then the injective norm of φ_S gets its minimal value 2, if and only if the following spectral condition holds*:

$$\forall \lambda, \mu \in \sigma(S), \; \left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right| \leq 2.$$

(iii) *If* $\|\varphi_S\|_\lambda = 2$, *then the interior of $\sigma(S)$ is empty.*

Proof (i) and (ii) follow immediately from the above proposition.

(iii) Assume $\|\varphi_S\|_\lambda = 2$. Thus, $\left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right| \leq 2$, for every $\lambda, \mu \in \sigma(S)$. Hence, every straight line passing through the origin intercept $\sigma(S)$ in at most two points. This proves that the interior of $\sigma(S)$ is empty. □

Proposition 4.6 [18] *Let P be a positive and invertible operator in $\mathfrak{B}(H)$. Then we have*

$$\|\varphi_P\|_\lambda = \|P\| \left\|P^{-1}\right\| + \frac{1}{\|P\| \left\|P^{-1}\right\|}.$$

Proof Let the operator M_P defined on $\mathfrak{B}(H)$ by

$$\forall X \in \mathfrak{B}(H), \; M_P(X) = PXP^{-1}.$$

Since $\sigma(M_P) = \sigma(P)\sigma(P^{-1})$, $\sigma(\varphi_P) = \left\{f(M_P) + \frac{1}{f(M_P)} : f \in \Gamma\right\}$ (where Γ is the set of all multiplicative functionals on the maximal commutative Banach algebra in $\mathfrak{B}(\mathfrak{B}(H))$ that contains M_P), and from the above proposition, it is easy to see that $\|\varphi_P\|_\lambda = \sup_{\lambda,\mu\in\sigma(P)} \left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right| = \sup_{z\in\sigma(M_P)} \left|z + \frac{1}{z}\right|$. So, using the fact that $\min \sigma(P) = \frac{1}{\|P^{-1}\|}$ and $\max \sigma(P) = \|P\|$, then $\min \sigma(M_P) = \frac{1}{\|P\|\|P^{-1}\|} = p$, and $\max \sigma(M_P) = \|P\| \left\|P^{-1}\right\| = \frac{1}{p}$. On the other hand, since $\max_{p\leq t\leq \frac{1}{p}} \left(t + \frac{1}{t}\right) = p + \frac{1}{p}$, this maximum is attainable at p and $\frac{1}{p}$. Thus, the result follows immediately from the fact that $p \in \sigma(M_P)$. □

Proposition 4.7 [18] *The following properties hold*:

(i) $\|\psi_S\|_\lambda = \|S\| \left\|S^{-1}\right\| + \frac{1}{\|S\|\left\|S^{-1}\right\|}$,

(ii) *if S is selfadjoint, then* $\|\varphi_S\|_\lambda = \|S\| \left\|S^{-1}\right\| + \frac{1}{\|S\|\left\|S^{-1}\right\|}$,

(iii) *if S is normal, then* $\|\varphi_S\|_\lambda \leq \|S\| \left\|S^{-1}\right\| + \frac{1}{\|S\|\left\|S^{-1}\right\|}$.

Proof Let $S = UP$ be the polar decomposition of S.

(i) From Lemma 4.3 and Proposition 4.6, and since $\|S\| = \|P\|$, $\left\|S^{-1}\right\| = \left\|P^{-1}\right\|$, it follows that

$$\begin{aligned}\left\|\psi_S\right\|_\lambda &= \left\|\varphi_P\right\|_\lambda \\ &= \|P\| \left\|P^{-1}\right\| + \frac{1}{\|P\| \left\|P^{-1}\right\|} \\ &= \|S\| \left\|S^{-1}\right\| + \frac{1}{\|S\| \left\|S^{-1}\right\|}.\end{aligned}$$

(ii) This implication follows immediately from (i).
(iii) Assume S normal. Then, using Proposition 4.4, and (i), we obtain

$$\begin{aligned}\left\|\varphi_S\right\|_\lambda &= \sup_{\lambda,\mu\in\sigma(S)} \left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right| \\ &\leq \sup_{\lambda,\mu\in\sigma(S)} \left(\left|\frac{\lambda}{\mu}\right| + \left|\frac{\mu}{\lambda}\right|\right) \\ &= \left\|\psi_S\right\|_\lambda \\ &= \|S\| \left\|S^{-1}\right\| + \frac{1}{\|S\| \left\|S^{-1}\right\|}.\end{aligned}$$

□

From above, it is clear that $\left\|\psi_S\right\|_\lambda \geq 2$. In the next corollary, we shall deduce three necessary and sufficient conditions for which $\left\|\psi_S\right\|_\lambda$ gets its minimal value 2. Note that the condition $\|S\| \left\|S^{-1}\right\| = 1$ is equivalent to $\frac{1}{\|S\|} S$ is unitary.

Corollary 4.8 [18] *The following properties are equivalent*:

(i) $\forall X \in \mathfrak{B}(H)$, $\left\|S^* X S^{-1}\right\| + \left\|S^{-1} X S^*\right\| = 2\|X\|$,
(ii) $\forall X \in \mathfrak{B}(H)$, $\left\|S^* X S^{-1} + S^{-1} X S^*\right\| = 2\|X\|$,
(iii) $\forall X \in \mathcal{F}_1(H)$, $\left\|S^* X S^{-1} + S^{-1} X S^*\right\| = 2\|X\|$,
(iv) $\left\|\psi_S\right\|_\lambda = 2$,
(v) $\frac{1}{\|S\|} S$ *is unitary.*

Proof (i) $\Longrightarrow$ (ii). Assume (i) holds and let $S = UP$ be the polar decomposition of S. Let $X \in \mathfrak{B}(H)$. Then, we have

$$\begin{aligned} 2\|X\| &\geq \left\|S^* X S^{-1} + S^{-1} X S^*\right\| \\ &= \left\|PU^* X P^{-1} U^* + P^{-1} U^* X P U^*\right\| \\ &= \left\|P\left(U^* X\right) P^{-1} + P^{-1}\left(U^* X\right) P\right\| \\ &\geq 2\left\|U^* X\right\|, \quad \text{(using (2.1))}, \\ &= 2\|X\|. \end{aligned}$$

This proves (ii).

The two implications (ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (iv) are trivial.

If (iv) holds, using the above proposition, we find $\|S\| \left\|S^{-1}\right\| = 1$, and this proves (v).

The implication (v) $\Longrightarrow$ (i) is trivial. □

Remark 4.9 The inequality given in Proposition 4.7. (iii) may be strict. Indeed, in dimension two, we choose the invertible normal operator $S = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1+i}{2} \end{bmatrix}$. By a simple computation, we find that

$$2 = \left\|\varphi_S\right\|_\lambda < \|S\| \left\|S^{-1}\right\| + \frac{1}{\|S\| \left\|S^{-1}\right\|} = \frac{3\sqrt{2}}{2}.$$

Notation 4.10 *We denote the following*:

$$\mathcal{E}(H) = \left\{T \in \mathcal{N}_0(H) : \left\|\varphi_T\right\|_\lambda = \|T\| \left\|T^{-1}\right\| + \frac{1}{\|T\| \left\|T^{-1}\right\|}\right\}.$$

From above, $\mathcal{E}(H)$ contains every invertible selfadjoint (resp. every unitary) operator in $\mathfrak{B}(H)$, but $\mathcal{E}(H)$ does not contain every invertible normal operator in $\mathfrak{B}(H)$ (see the example in the above remark). In the next proposition, we give a characterization of this class $\mathcal{E}(H)$, where we use the following notations:

- $\sigma_1(S) = \left\{\lambda \in \sigma(S) : |\lambda| = \min_{\mu \in \sigma(S)} |\mu|\right\}$,
- $\sigma_2(S) = \{\lambda \in \sigma(S) : |\lambda| = r(S)\}$,
- D_θ (where $\theta \in [0, \pi)$) is the straight line through the origin with slope $\tan\theta$.

Proposition 4.11 [18] *The two following properties are equivalent*:

(i) $S \in \mathcal{E}(H)$,

(ii) *S is normal, and there exists $\theta \in [0, \pi[$ such that*

$$D_\theta \cap \sigma_1(S) \neq \emptyset, \; D_\theta \cap \sigma_2(S) \neq \emptyset.$$

Proof (i) $\Rightarrow$ (ii). Assume (i) holds. Using Proposition 4.4(ii) and from the compactness of $\sigma(S)$, we may choose $\lambda, \mu \in \sigma(S)$ such that

$$\|S\| \left\|S^{-1}\right\| + \frac{1}{\|S\| \left\|S^{-1}\right\|} = \|\varphi_S\|_\lambda = \left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right|.$$

Hence,

$$\begin{aligned}\|S\| \left\|S^{-1}\right\| + \frac{1}{\|S\| \left\|S^{-1}\right\|} &\leq \left|\frac{\lambda}{\mu}\right| + \left|\frac{\mu}{\lambda}\right| \\ &\leq \|\psi_S\|_\lambda \\ &= \|S\| \left\|S^{-1}\right\| + \frac{1}{\|S\| \left\|S^{-1}\right\|}, \quad \text{(using Proposition 4.4(iii))}.\end{aligned}$$

Thus, $\left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right| = \left|\frac{\lambda}{\mu}\right| + \left|\frac{\mu}{\lambda}\right| = \|S\| \left\|S^{-1}\right\| + \frac{1}{\|S\|\left\|S^{-1}\right\|}$. Put $p = \frac{1}{\|S\|\left\|S^{-1}\right\|}$. Since S is normal, then $\min_{\lambda,\mu\in\sigma(S)} \left|\frac{\lambda}{\mu}\right| = p$, and $\max_{\lambda,\mu\in\sigma(S)} \left|\frac{\lambda}{\mu}\right| = \frac{1}{p}$. The positive function $f(t) = t + \frac{1}{t}, \ \ p \leq t \leq \frac{1}{p}$ is bounded and attain its maximum $p + \frac{1}{p} = \|S\| \left\|S^{-1}\right\| + \frac{1}{\|S\|\left\|S^{-1}\right\|}$ only at $t = p$ and in $t = \frac{1}{p}$. So, we may choose λ in $\sigma_1(S)$ and μ in $\sigma_2(S)$. Since $\left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right| = \left|\frac{\lambda}{\mu}\right| + \left|\frac{\mu}{\lambda}\right|$, then, λ and μ must belong to a straight line passing through the origin. This proves (ii).

(ii) $\Rightarrow$ (i). Assume (ii) holds. Let $\alpha \in D_\theta \cap \sigma_1(S)$ and $\beta \in D_\theta \cap \sigma_2(S)$. Since S is normal, then $\alpha = \frac{e^{i\theta}}{\left\|S^{-1}\right\|}$ and $\beta = e^{i(\theta+k\pi)} \|S\|$, for some $k \in \{0, 1\}$. Thus, $\|\varphi_S\|_\lambda \geq \left|\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right| = \|S\| \left\|S^{-1}\right\| + \frac{1}{\|S\|\left\|S^{-1}\right\|}$. Then, using Proposition 4.7. (iii), (i) holds. □

In the next proposition, we shall give two necessary and sufficient conditions for which $\|\varphi_S\|_\lambda$ gets its minimal value 2.

We need the two following lemmas:

Lemma 4.12 [24] *If* $|\langle Sx, x\rangle| \leq 1$ *and* $\left|\left\langle S^{-1}x, x\right\rangle\right| \leq 1$, *for every unit vector* x *in* H, *then* S *is unitary.*

Lemma 4.13 *The operator* S *is normal if and only if* $S^* S^{-1}$ *is unitary.*

Proof The proof is trivial. □

Proposition 4.14 [19] *The following properties are equivalent*:

(i) $\|\varphi_S\|_\lambda = 2$,

(ii) $\forall X \in \mathcal{F}_1(H)$, $\left\|SXS^{-1} + S^{-1}XS\right\| \leq 2\|X\|$,

(iii) *S is normal and* $\sup_{\lambda,\mu\in\sigma(S)}\left|\frac{\lambda}{\mu}+\frac{\mu}{\lambda}\right|=2.$

Proof (i) $\Leftrightarrow$ (ii). This equivalence follows immediately from Proposition 3.1 and Corollary 4.5(i).

(i) $\Rightarrow$ (iii). Assume (i) holds. From Proposition 4.4, we deduce that $\sup_{\lambda,\mu\in\sigma(S)}\left|\frac{\lambda}{\mu}+\frac{\mu}{\lambda}\right|=2.$

So, it remains to prove that S is normal. By using the same argument as used in [15, Lemma 1], we deduce the following inequality:

$$\forall x, y\in (H)_1,\ \left\|\varphi_S\right\|_{\lambda}\geq 2\left|\langle Sx, y\rangle\left\langle S^{-1}x, y\right\rangle\right|.$$

Hence, the inequality $\left|\langle Sx, y\rangle\left\langle S^{-1}x, y\right\rangle\right|\leq\|x\|\,\|y\|$ holds for every $x,\ y$ in H. So we obtain $\left|\left\langle S^{*}S^{-1}x, x\right\rangle\right|\leq 1$ and $\left|\left\langle\left(S^{*}S^{-1}\right)^{-1}x, x\right\rangle\right|\leq 1$, for every $x,\ y$ in $(H)_1$. Then, using the two above lemmas, we deduce that S is normal. Thus, (iii) holds.

(iii) $\Rightarrow$ (i). This follows immediately from Proposition 4.4. □

Remark 4.15 The class of all operators S for which $\left\|\varphi_S\right\|_{\lambda}$ is minimal contains strictly the class of all unitary operators, and contained strictly in the class of all invertible normal operators. Indeed, it is easy to see that $\left\|\varphi_S\right\|_{\lambda}=2$, if S is unitary, and for an operator $I_1\oplus\left(\frac{1}{2}iI_2\right)$ with respect to some orthogonal direct $H=H_1\oplus H_2$ (where I_i is the identity on H_i, for $i=1,2$) belongs to this class, but it is not unitary; the second inclusion is trivial.

In the next proposition, we shall give some other characterizations of the class of all unitary operators multiplied by nonzero scalars.

Proposition 4.16 [19] *The following properties are equivalent*:

(i) $\forall X\in\mathfrak{B}(H),\ \left\|SXS^{-1}\right\|+\left\|S^{-1}XS\right\|=2\|X\|,$

(ii) $\forall X\in\mathfrak{B}(H),\ \left\|SXS^{-1}\right\|+\left\|S^{-1}XS\right\|\leq 2\|X\|,$

(iii) $\forall X\in\mathfrak{B}(H),\ \left\|SXS^{-1}+S^{-1}XS\right\|\leq 2\|X\|,$

(iv) $\frac{1}{\|S\|}S$ *is unitary.*

Proof The two implications (i) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (iii) are trivial.

(iii) $\Longrightarrow$ (iv). Assume (iii) holds. So, it follows that $\left\|\varphi_S\right\|_{\lambda}=2$. Using Proposition 4.14, then S is normal and $\sup_{\lambda,\mu\in\sigma(S)}\left|\frac{\lambda}{\mu}+\frac{\mu}{\lambda}\right|=2.$

Using the spectral measure of S, there exists a sequence (S_n) of invertible normal operators in $\mathfrak{B}(H)$ with finite spectrum such that:

(a) $S_n\longrightarrow S$ uniformly,
(b) for every $\lambda\in\sigma(S)$, there exists a sequence (λ_n) such that $\lambda_n\in\sigma(S_n)$, for every n, and $\lambda_n\longrightarrow\lambda$.

Let $\lambda,\ \mu \in \sigma(S)$. Then from (b), there exist two sequences $(\lambda_n),\ (\mu_n)$ such that $\lambda_n,\ \mu_n \in \sigma(S_n)$, for every n, and $\lambda_n \longrightarrow \lambda,\ \mu_n \longrightarrow \mu$.

Let $\epsilon > 0$. Then, there exists an integer $N \geq 1$ such that

$$\forall n > N,\ \forall X \in \mathfrak{B}(H),\ \left\| S_n X S_n^{-1} + S_n^{-1} X S_n \right\| \leq (2+\epsilon)\left\| X \right\|. \tag{4.1}$$

Let $n > N$. Since S_n is normal with finite spectrum, there exist p orthogonal projections $E_1, \ldots, E_p$ in $\mathfrak{B}(H)$ such that $E_i E_j = 0$, if $i \neq j$, $\sum_{i=1}^{p} E_i = I$, $S_n = \sum_{i=}^{p} \alpha_i E_i$, where $\sigma(S_n) = \left\{\alpha_1, \ldots, \alpha_p\right\}$, $\alpha_1 = \lambda_n$, $\mu_n = \alpha_2$.

Then, using (4.1) and putting $A = \begin{bmatrix} 2 & \gamma_n \\ \gamma_n & 2 \end{bmatrix}$, where $\gamma_n = \frac{\lambda_n}{\mu_n} + \frac{\mu_n}{\lambda_n}$, we obtain

$$\forall X \in \mathfrak{B}(\mathbb{C}^2),\ \left\| A \circ X \right\| \leq (2+\epsilon)\left\| X \right\|.$$

Put $X = \begin{bmatrix} t\mathrm{Im}\gamma_n & i \\ i & t\mathrm{Im}\gamma_n \end{bmatrix}$ (where $t > 0$) in this last inequality, we obtain

$$(2t\mathrm{Im}\gamma_n)^2 + |\gamma_n|^2 + 4t\,(\mathrm{Im}\gamma_n)^2 \leq (2t\mathrm{Im}\gamma_n)^2 + 4 + \left(4\epsilon + \epsilon^2\right)\left((t\mathrm{Im}\gamma_n)^2 + 1\right).$$

Put $\gamma = \lim \gamma_n = \frac{\lambda}{\mu} + \frac{\mu}{\lambda}$, and letting $n \longrightarrow \infty$ in this last inequality, it follows that

$$|\gamma|^2 + 4t\,(\mathrm{Im}\gamma)^2 \leq 4 + \left(4\epsilon + \epsilon^2\right)\left((t\mathrm{Im}\gamma)^2 + 1\right).$$

Now, letting $\epsilon \longrightarrow 0$, we deduce that $4t\,(\mathrm{Im}\gamma)^2 \leq 4 - |\gamma|^2$, for every $t > 0$. Hence, $\mathrm{Im}\gamma = 0$, and $|\gamma| \leq 2$. Then, by a simple computation, we find that $|\lambda| = |\mu|$. Then $\sigma(S)$ is included in the circle centered at the origin and of radius $\|S\|$. Since S is normal, this proves (iv).

(iv) $\Longrightarrow$ (i). This implication is trivial. □

Conclusion 4.17 (1) *The class of all invertible operators* $S \in \mathfrak{B}(H)$ *for which* $\left\|\varphi_S\right\|_\lambda$ *is minimal is characterized by each of the two following properties*:

$$\forall X \in \mathcal{F}_1(H),\ \left\| TXT^{-1} + T^{-1}XT \right\| \leq 2\left\| X \right\|,\ (T \in \mathfrak{I}(H)),$$
$$T \in \mathcal{N}_0(H) \textit{ and } \sup_{\lambda,\mu\in\sigma(T)} \left| \frac{\lambda}{\mu} + \frac{\mu}{\lambda} \right| = 2, \qquad (T \in \mathfrak{I}(H)).$$

(2) *The class of all unitary operators* $\mathfrak{U}(H)$ *is characterized by each of the following properties*:

$$\begin{aligned}
&\forall X \in \mathfrak{B}(H), && \left\| TXT^{-1} + T^{-1}XT \right\| \leq 2\|X\|, && T \in (\mathfrak{I}(H))_1, \\
&\forall X \in \mathfrak{B}(H), && \left\| TXT^{-1} \right\| + \left\| T^{-1}XT \right\| \leq 2\|X\|, && T \in (\mathfrak{I}(H))_1, \\
&\forall X \in \mathfrak{B}(H), && \left\| TXT^{-1} \right\| + \left\| T^{-1}XT \right\| = 2\|X\|, && T \in (\mathfrak{I}(H))_1, \\
&\forall X \in \mathfrak{B}(H), && \left\| T^{*}XT^{-1} \right\| + \left\| T^{-1}XT^{*} \right\| = 2\|X\|, && T \in (\mathfrak{I}(H))_1, \\
&\forall X \in \mathfrak{B}(H), && \left\| T^{*}XT^{-1} + T^{-1}XT^{*} \right\| = 2\|X\|, && T \in (\mathfrak{I}(H))_1, \\
&\forall X \in \mathcal{F}_1(H), && \left\| T^{*}XT^{-1} + T^{-1}XT^{*} \right\| = 2\|X\|, && T \in (\mathfrak{I}(H))_1, \\
& && \|\psi_T\|_\lambda = 2, && T \in (\mathfrak{I}(H))_1 .
\end{aligned}$$

5 *N*-arithmetic–geometric mean inequality, normal operators, and characterizations

In this section, we shall present some characterizations of the class $\mathcal{N}(H)$ of all normal operators in $\mathfrak{B}(H)$ in terms of operator inequalities, and also its two subclasses $\mathcal{N}_0(H)$, and $\mathcal{N}_{cr}(H)$. These operator inequalities are related to the *N*-arithmetic–geometric mean inequality which will be introduced in the next subsection.

We start with the following remark which contains two trivial characterizations of the class $\mathcal{N}(H)$.

Remark 5.1 Let $S \in \mathfrak{B}(H)$. It is easy to see that the three following properties are equivalent:

(i) S is normal,
(ii) $\forall X \in \mathfrak{B}(H),\ \|S^*X\| = \|SX\|$,
(iii) $\forall X \in \mathfrak{B}(H),\ \|XS^*\| = \|XS\|$.

5.1 Operator inequality related to the *N*-arithmetic–geometric mean inequality

In this subsection, we consider the *N*-arithmetic–geometric mean inequality given by (2.3)

This inequality follows immediately from the known arithmetic–geometric mean inequality (2.2). In the next proposition, we present a family of operator inequalities generated by normal operators that are equivalent to the inequality (2.3), and we shall prove (2.3) independently in (2.2).

Proposition 5.2 [4] *The following operator inequalities hold and are mutually equivalent*:

(i) $\forall X \in \mathfrak{B}(H),\ \|A^*AX\| + \|XBB^*\| \geq 2\|AXB\|$, *for every* $A,\ B \in \mathfrak{B}(H)$,
(ii) $\forall X \in \mathfrak{B}(H),\ \left\|SXR^+\right\| + \left\|S^+XR\right\| \geq 2\left\|SS^+XR^+R\right\|$, *for every* $S, R \in \mathcal{N}_{cr}(H)$,

(iii) $\forall X \in \mathfrak{B}(H),\ \left\|S^2X\right\| + \left\|XR^2\right\| \geq 2\left\|SXR\right\|$, *for every* $S, R \in \mathcal{N}_{cr}(H)$,
(iv) $\forall X \in \mathfrak{B}(H),\ \left\|SXR^{-1}\right\| + \left\|S^{-1}XR\right\| \geq 2\left\|X\right\|$, *for every* $S, R \in \mathcal{N}_0(H)$,
(v) $\forall X \in \mathfrak{B}(H),\ \left\|S^2X\right\| + \left\|XR^2\right\| \geq 2\left\|SXR\right\|$, *for every* $S, R \in \mathcal{N}(H)$,
(vi) $\forall X \in \mathfrak{B}(H),\ \left\|A^*AX\right\| + \left\|XAA^*\right\| \geq 2\left\|AXA\right\|$, *for every* $A \in \mathfrak{B}(H)$,
(vii) $\forall X \in \mathfrak{B}(H),\ \left\|SXS^+\right\| + \left\|S^+XS\right\| \geq 2\left\|SS^+XS^+S\right\|$, *for every* $S \in \mathcal{N}_{cr}(H)$,
(viii) $\forall X \in \mathfrak{B}(H),\ \left\|S^2X\right\| + \left\|XS^2\right\| \geq 2\left\|SXS\right\|$, *for every* $S \in \mathcal{N}_{cr}(H)$,
(ix) $\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geq 2\left\|X\right\|$, *for every* $S \in \mathcal{N}_0(H)$,
(x) $\forall X \in \mathfrak{B}(H),\ \left\|S^2X\right\| + \left\|XS^2\right\| \geq 2\left\|SXS\right\|$, *for every* $S \in \mathcal{N}(H)$.

Proof (i) $\Rightarrow$ (ii). Assume (i) holds. Let $S, R \in \mathcal{N}_{cr}(H)$, $X \in \mathfrak{B}(H)$. Since $S^* = S^*SS^+$ and $R^* = R^+RR^*$, then from (i) and Remark 5.1, it follows that

$$\begin{aligned}\left\|SXR^+\right\| + \left\|S^+XR\right\| &= \left\|S^*S\left(S^+XR^+\right)\right\| + \left\|\left(S^+XR^+\right)RR^*\right\| \\ &\geq 2\left\|SS^+XR^+R\right\|.\end{aligned}$$

Hence, (ii) holds.

(ii) $\Rightarrow$ (iii). Assume (ii) holds. Let $S, R \in \mathcal{N}_{cr}(H)$, $X \in \mathfrak{B}(H)$. Then from (ii) and since $SS^+S = S,\ RR^+R = R$, and $S^+S,\ RR^+$ are orthogonal projections, it follows that

$$\begin{aligned}\left\|S^2X\right\| + \left\|XR^2\right\| &\geq \left\|S\left(SXR\right)R^+\right\| + \left\|S^+\left(SXR\right)R\right\| \\ &\geq 2\left\|SS^+\left(SXR\right)R^+R\right\| \\ &= 2\left\|SXR\right\|.\end{aligned}$$

Thus, (iii) holds.

(iii) $\Rightarrow$ (iv). This implication is trivial.

(iv) $\Rightarrow$ (i). Assume (iv) holds. Then the following inequality holds:

$$\forall S, R \in \mathcal{N}_0(H), \forall X \in \mathfrak{B}(H),\ \left\|S^2X\right\| + \left\|XR^2\right\| \geq 2\left\|SXR\right\|.$$

Let $A, B, X \in \mathfrak{B}(H)$. Put $P = |A|,\ Q = |B^*|$. It is clear that the two operators $P + \epsilon I$ and $Q + \epsilon I$ are normal and invertible, for every $\epsilon > 0$. So, using the last inequality, we obtain

$$\forall \epsilon > 0,\ \left\|(P + \epsilon I)^2 X\right\| + \left\|X(Q + \epsilon I)^2\right\| \geq 2\left\|(P + \epsilon I)X(Q + \epsilon I)\right\|.$$

By letting $\epsilon \to 0$, we deduce (i).

(i) $\Rightarrow$ (v). This follows immediately by using Remark 5.1.

(v) $\Rightarrow$ (iii). This implication is trivial.

Therefore, the operator inequalities (i)–(v) are equivalent.

From a pair of operators to a single operator, we deduce that the operator inequalities (vi)–(x) are also equivalent.

(i) $\Rightarrow$ (vi). This implication is trivial.

(vi) $\Rightarrow$ (i). Assume (vi) holds (here we use the Berberian technique).

Let $A, B, X \in \mathfrak{B}(H)$. Consider now the two bounded linear operators C, Y defined on the Hilbert space $H \oplus H$ given by $C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, $Y = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$. By a simple computation, we obtain $C^*CY = \begin{bmatrix} 0 & A^*AX \\ 0 & 0 \end{bmatrix}$, $YCC^* = \begin{bmatrix} 0 & XBB^* \\ 0 & 0 \end{bmatrix}$, and $CYC = \begin{bmatrix} 0 & AXB \\ 0 & 0 \end{bmatrix}$. Applying (vi) for the Hilbert space $H \oplus H$, we obtain $\|A^*AX\| + \|XBB^*\| = \|C^*CY\| + \|YCC^*\| \geq 2\,\|CYC\| = 2\,\|AXB\|$. This proves (i).

Therefore, the inequalities (i)–(v) and (vi)–(x) are mutually equivalent. It remains to prove that one of them holds. It is clear that (i) is an immediate consequence of the known arithmetic–geometric mean inequality (2.2). But here, we shall give a direct proof of (i) independently of (2.2) by using the numerical arithmetic–geometric mean inequality. Let $A, B, X \in \mathfrak{B}(H)$. The following inequalities hold:

$$
\begin{aligned}
\frac{1}{2}\left(\left\|A^*AX\right\| + \left\|XBB^*\right\|\right) &\geq \sqrt{\|A^*AX\|\,\|XBB^*\|} \\
&\geq \sqrt{\|BB^*X^*A^*AX\|} \\
&\geq \sqrt{r(BB^*X^*A^*AX)} \\
&= \sqrt{r\left(B^*X^*A^*AXB\right)} \\
&= \|AXB\|\,.
\end{aligned}
$$

□

Corollary 5.3 *The following operator inequalities hold*:

(i) *For every* $S, R \in \mathfrak{I}(H)$, *the following holds*:

$$\forall X \in \mathfrak{B}(H),\ \left\|S^*XR^{-1}\right\| + \left\|S^{-1}XR^*\right\| \geq 2\,\|X\|\,.$$

(ii) *For every* $S, R \in \mathcal{R}(H)$, *the following holds*:

$$\forall X \in \mathfrak{B}(H),\ \left\|S^*XR^+\right\| + \left\|S^+XR^*\right\| \geq 2\left\|SS^+XR^+R\right\|.$$

(iii) *For every* $S \in \mathfrak{I}(H)$, *the following holds*:

$$\forall X \in \mathfrak{B}(H),\ \left\|S^*XS^{-1}\right\| + \left\|S^{-1}XS^*\right\| \geq 2\,\|X\|\,.$$

(iv) *For every* $S \in \mathcal{R}(H)$, *the following holds*:

$$\forall X \in \mathfrak{B}(H),\ \left\|S^*XS^+\right\| + \left\|S^+XS^*\right\| \geq 2\left\|SS^+XS^+S\right\|.$$

Proof It suffices to prove (ii) and the three others follow immediately from (ii).

Let $S, R \in \mathcal{R}(H)$, and $X \in \mathfrak{B}(H)$. Since $SS^+S = S$ and $RR^+R = R$, then we have

$$\begin{aligned}\left\|S^*XR^+\right\| + \left\|S^+XR^*\right\| &= \left\|S^*S\left(S^+XR^+\right)\right\| + \left\|\left(S^+XR^+\right)RR^*\right\| \\ &\geq 2\left\|SS^+XR^+R\right\|, \quad \text{(from (2.3))}.\end{aligned}$$

□

Note that the eight operator inequalities (ii)–(v) and (vii)–(x) given in Proposition 5.2 are generated by a pair of normal operators and a single normal operator, respectively.

It will be of interest to describe the class of all operators $S \in \mathfrak{I}(H)$ (resp. $S \in \mathfrak{R}(H)$, $S \in \mathfrak{B}(H)$) satisfying the operator inequality (ix) (resp. (vii), (x)).

We shall prove that the largest class of

$(\cdot)$ all operators $S \in \mathfrak{I}(H)$ satisfying (ix) is the class $\mathcal{N}_0(H)$ of all normal operators $S \in \mathfrak{I}(H)$,

$(\cdot\cdot)$ all operators $S \in \mathfrak{R}(H)$ satisfying the operator inequality (vii) is the class $\mathcal{N}_{cr}(H)$ of all normal operators $S \in \mathfrak{R}(H)$,

$(\cdot\cdot\cdot)$ all operators $S \in \mathfrak{B}(H)$ satisfying the operator inequality (x) is the class $\mathcal{N}(H)$ of all normal operators $S \in \mathfrak{B}(H)$.

In the next subsection, we shall present all these characterizations and others.

5.2 Normal operators and characterizations

We need the following lemmas.

Lemma 5.4 [25] *Let $A \in \mathfrak{B}(H)$. If $\|A - \lambda I\| = r(A - \lambda I)$, for all complex λ, A is convexoid.*

Lemma 5.5 [14] *Let P, Q be two invertible positive operators in $\mathfrak{B}(H)$ satisfying the following operator inequality*:

$$\forall X \in \mathfrak{B}(H), \ \|X\| + \left\|PXP^{-1}\right\| \geq 2\left\|QXQ^{-1}\right\|.$$

Then, we have $\{P\}' \subset \{Q\}'$.

Proof (i) Let X be a selfadjoint operator in $\mathfrak{B}(H)$ such that $PX = XP$, and let α be an arbitrary complex number. Replace X by $X - \alpha I$ in the inequality given by the lemma, and since $X - \alpha I$ is normal, we obtain

$$\|X - \alpha I\| \geq \left\|Q(X - \alpha I)Q^{-1}\right\| \geq r\left(Q(X - \alpha I)Q^{-1}\right) = \|X - \alpha I\|.$$

Hence, $\left\| QXQ^{-1} - \alpha I \right\| = r\left(QXQ^{-1} - \alpha I\right)$, for all complex number α. Using the above lemma, we obtain that

$$\begin{aligned} V\left(QXQ^{-1}\right) &= co\left(\sigma\left(QXQ^{-1}\right)\right) \\ &= co\sigma\left(X\right) \\ &\subset \mathbb{R}. \end{aligned}$$

This give us that QXQ^{-1} is selfadjoint. Hence, $QX = XQ$.

(ii) Now, let X be an arbitrary operator in $\mathfrak{B}(H)$. Put $X = X_1 + iX_2$, where $X_1 = \mathrm{Re}(X)$, and $X_2 = \mathrm{Im}X$. Assume that $PX = XP$. Then, $PX_1 = X_1P$ and $PX_2 = X_2P$. From (i), we deduce that, $QX_1 = X_1Q$ and $QX_2 = X_2Q$. Thus, $QX = XQ$. Therefore, $\{P\}' \subset \{Q\}'$.

□

Lemma 5.6 [14] *Let $P,\ Q$ be two invertible positive operators in $\mathfrak{B}(H)$ satisfying the following operator inequality*:

$$\forall X \in \mathfrak{B}(H),\ \left\| PXP^{-1} \right\| + \left\| Q^{-1}XQ \right\| \geq 2\left\| X \right\|.$$

Then, we have $\{P\}' = \{Q\}'$.

Proof From the inequality given in the lemma, we have

$$\forall X \in \mathfrak{B}(H),\ \|X\| + \left\| PQXQ^{-1}P^{-1} \right\| \geq 2\left\| QXQ^{-1} \right\|. \tag{5.1}$$

Put $M = |PQ|$. So, from this last inequality, we obtain

$$\forall X \in \mathfrak{B}(H),\ \|X\| + \left\| MXM^{-1} \right\| \geq 2\left\| QXQ^{-1} \right\|.$$

Hence, from the above lemma, we deduce that $MQ = QM$. Then, $PQ = QP$.

Now, let X be a selfadjoint operator in $\mathfrak{B}(H)$ such that $PX = XP$, and let α be an arbitrary complex number. Replace in (5.1), X by $X - \alpha I$, so we have $\|X - \alpha I\| \geq \left\| Q(X - \alpha I)Q^{-1} \right\|$, for every complex number α. Hence, $QX = XQ$, and thus $\{P\}' \subset \{Q\}'$, this follows by using the same argument as used in the proof of the above lemma.

Using again the inequality given in the lemma, we obtain also that $\{Q\}' = \left\{Q^{-1}\right\}' \subset \left\{P^{-1}\right\}' = \{P\}'$. Therefore, $\{P\}' = \{Q\}'$. □

Lemma 5.7 *Let $\epsilon > 0$, and let $\alpha_1, \ldots, \alpha_n,\ \beta_1, \ldots, \beta_n$ (where $n \geq 1$) be real numbers such that $0 < \alpha_1 \leq \cdots \leq \alpha_n \leq 1$, $\left\{\alpha_1, \ldots, \alpha_n\right\} \subset \left\{\beta_1, \ldots, \beta_n\right\}$, and $\frac{\alpha_i}{\alpha_j} + \frac{\beta_j}{\beta_i} \geq 2 - \epsilon$, for every $i,\ j$. Then, we have $\left|\alpha_i - \beta_i\right| \leq \epsilon$, for $i = 1, \ldots, n$.*

Proof From the hypothesis, we deduce easily that $\beta_i - \beta_j \leq \epsilon$, if $i < j$.

Let $i \in \{1, \ldots, n\}$ such that $\alpha_i \neq \beta_i$ (in the case $\alpha_i = \beta_i$, of course, we have $\left|\alpha_i - \beta_i\right| = 0 \leq \epsilon$).

There are three cases: $i = 1,\ i = n$, and $1 < i < n$.

Case 1. $i = 1$. There exists $j \geq 2$ such that $\beta_j = \alpha_1$. So, we have $|\alpha_1 - \beta_1| = \beta_1 - \beta_j \leq \epsilon$, since $j > 1$.

Case 2. $i = n$. There exists $j < n$ such that $\beta_j = \alpha_n$. Hence, $|\alpha_n - \beta_n| = \beta_j - \beta_n \leq \epsilon$, since $j < n$.

Case 3. $1 < i < n$.

If $\alpha_i < \beta_i$, then there exists $j > i$, such that $\beta_j \leq \alpha_i$. Hence, $\left|\alpha_i - \beta_i\right| \leq \beta_i - \beta_j \leq \epsilon$, since $i < j$.

If $\alpha_i > \beta_i$, then there exists $j < i$, such that $\beta_j \geq \alpha_i$. Hence, $\left|\alpha_i - \beta_i\right| \leq \beta_j - \beta_i \leq \epsilon$, since $i > j$. □

Remark 5.8 In the original paper [14], the above lemma is Lemma 3.5, but it is given in a particular case with equality instead of inclusion, and where the sequence $\left\{\alpha_1, \ldots, \alpha_n\right\}$ is increasing instead of non-decreasing. In this particular case, the lemma is needed only for invertible case.

Lemma 5.9 [14] *Let $P,\ Q$ be two invertible positive operators in $\mathfrak{B}(H)$ such that $\sigma(Q) \subset \sigma(P)$ or $\sigma(P) \subset \sigma(Q)$. Then, the two following properties are equivalent:*

(i) $\forall X \in \mathfrak{B}(H),\ \left\|PXP^{-1}\right\| + \left\|Q^{-1}XQ\right\| \geq 2\left\|X\right\|,$

(ii) $P = Q$.

Proof We may assume without loss of the generality that $\|P\| = \|Q\| = 1$.

(i) ⇒ (ii). Assume (i) holds. Decompose P and Q using their spectral measure:

$$P = \int \lambda dE_\lambda, \quad Q = \int \lambda dF_\lambda,$$

and consider

$$P_n = \int h_n(\lambda)\, dE_\lambda = h_n(P), \quad Q_n = \int h_n(\lambda)\, dF\lambda = h_n(Q).,$$

where $h_n(\lambda)$ is the function defined by

$$h_n(\lambda) = \frac{k}{n}, \text{ if } \frac{k}{n} \leq \lambda < \frac{k+1}{n}, \quad \text{for } k = 1, 2, 3, \ldots$$

Case 1. $\sigma(Q) \subset \sigma(P)$. Using the spectral theorem with the function h_n, we have

$$\sigma\left(Q_n\right) = \sigma\left(h_n(Q)\right) = h_n(\sigma(Q)) \subset h_n(\sigma(P)) = \sigma\left(h_n(P)\right) = \sigma\left(P_n\right).$$

Then $P_n,\ Q_n$ are invertible positive operators in $\mathfrak{B}(H)$ with finite spectrum such that $\sigma\left(Q_n\right) \subset \sigma\left(P_n\right),\ P_n \to P,\ Q_n \to Q$ uniformly, and $P_n \in \{P\}'',\ Q_n \in \{Q\}''$ (where $\{P\}'' = \{Q\}''$, from the Lemma 5.6).

Hence, $P_n Q_n = Q_n P_n$, for every $n \geq 1$. Then, $Q_n = \sum_{i=}^{p} \alpha_i E_i$, $P_n = \sum_{i=}^{p} \beta_i E_i$, where $\sigma\left(Q_n\right) = \left\{\alpha_1, \ldots, \alpha_p\right\}$ such that $0 < \alpha_1 \leq \cdots \leq \alpha_p \leq 1$, $\sigma\left(P_n\right) = \left\{\beta_1, \ldots, \beta_p\right\}$, and $E_1, \ldots, E_p$ are orthogonal projections in $\mathfrak{B}(H)$ such that $E_i E_j = 0$, if $i \neq j$, $\sum_{i=1}^{p} E_i = I$. Thus, $\left\{\alpha_1, \ldots, \alpha_p\right\} \subset \left\{\beta_1, \ldots, \beta_p\right\}$.

Let $\epsilon > 0$. Then, there exists an integer $N \geq 1$ such that

$$\forall n > N,\ \forall X \in \mathfrak{B}(H),\ \left\| P_n X P_n^{-1} \right\| + \left\| Q_n^{-1} X Q_n \right\| \geq (2-\epsilon)\|X\|.$$

Let $n > N$, and replace X by $E_i X E_j$ (where $X \in \mathfrak{B}(H)$) in this last inequality, then we deduce that

$$\frac{\alpha_i}{\alpha_j} + \frac{\beta_j}{\beta_i} \geq 2-\epsilon, \quad \text{for } i, j = 1, \ldots, p.$$

From these last inequalities, and since $0 < \alpha_1 \leq \cdots \leq \alpha_p \leq 1$, $\left\{\alpha_1, \ldots, \alpha_p\right\} \subset \left\{\beta_1, \ldots, \beta_p\right\}$, and using the above lemma, we obtain $\left|\alpha_i - \beta_i\right| \leq \epsilon$, for $i = 1, \ldots, p$.

Since $P_n = \sum_{i=}^{p} \alpha_i E_i$, and $Q_n = \sum_{i=}^{p} \beta_i E_i$, then

$$\begin{aligned} \left\| P_n - Q_n \right\| &= \max_{1 \leq i \leq p_n} \left|\alpha_i - \beta_i\right| \\ &\leq \epsilon. \end{aligned}$$

Therefore, $P = Q$.

Case 2. $\sigma(Q) \subset \sigma(P)$. Using the same argument as used before, we find also $P = Q$.

The implication (ii) $\Rightarrow$ (i) follows immediately from (2.3). □

Remark 5.10 The above lemma in the original paper [14] is the Theorem 3.6, but with equality between spectrum of P and Q instead of the inclusion. The equality condition is enough for the invertible case. But for the non-invertible case, the lemma presented here with inclusion is needed.

In the next proposition, we shall present the first characterization of the class of all invertible normal operators in $\mathfrak{B}(H)$.

Proposition 5.11 [18] *Let S be an invertible operator in $\mathfrak{B}(H)$. Then, the two following properties are equivalent*:

(i) *S is normal*,
(ii) $\forall X \in \mathfrak{B}(H),\ \left\|S X S^{-1}\right\| + \left\|S^{-1} X S\right\| \geq 2\|X\|$.

Proof (i) $\Rightarrow$ (ii). Assume (i) holds. Let $X \in \mathfrak{B}(H)$, then we have

$$\begin{aligned} \left\|S X S^{-1}\right\| + \left\|S^{-1} X S\right\| &= \left\|S^* X S^{-1}\right\| + \left\|S^{-1} X S^*\right\|, \text{ (using Remark 5.1)} \\ &\geq 2\|X\| \text{ (using Corollary 5.3(iii))}. \end{aligned}$$

(ii) $\Rightarrow$ (i). Assume (ii) holds. Let $S = UP$, $S^* = U^*Q$ be the polar decompositions of S and S^*. From (ii), it follows that

$$\forall X \in \mathfrak{B}(H), \ \left\| PXP^{-1} \right\| + \left\| Q^{-1}XQ \right\| \geq 2\left\| X \right\|.$$

Since $\sigma(P^2) = \sigma(S^*S) = \sigma(SS^*) = \sigma(Q^2)$, so from the Spectral Theorem, $\sigma(P) = \sigma(Q)$. Using the last Lemma, we obtain $P = Q$. Therefore, S is normal. □

Corollary 5.12 [18] *Let S be an invertible operator in $\mathfrak{B}(H)$. Then, the following properties are equivalent*:

(i) *S is normal*,
(ii) $\forall X \in \mathfrak{B}(H),\ \left\| SXS^{-1} \right\| + \left\| S^{-1}XS \right\| = \left\| S^*XS^{-1} \right\| + \left\| S^{-1}XS^* \right\|$,
(iii) $\forall X \in \mathfrak{B}(H),\ \left\| SXS^{-1} \right\| + \left\| S^{-1}XS \right\| \geq \left\| S^*XS^{-1} \right\| + \left\| S^{-1}XS^* \right\|$.

Proof (i) $\Rightarrow$ (ii). This follows immediately from Remark 5.1.
(ii) $\Rightarrow$ (iii). This implication is trivial.
(iii) $\Rightarrow$ (i). Assume (iii) holds. Using the Corollary 5.3(iii), we have

$$\forall X \in \mathfrak{B}(H), \ \left\| SXS^{-1} \right\| + \left\| S^{-1}XS \right\| \geq 2\left\| X \right\|.$$

From the last theorem, we obtain that S is normal. □

Proposition 5.13 [20] *Let $S \in \mathfrak{I}(H)$. The following properties are equivalent*:

(i) $S \in \mathcal{N}_0(H)$,
(ii) $\forall X \in \mathfrak{B}(H), \left\| SXS^{-1} \right\| + \left\| S^{-1}XS \right\| \leq \left\| S^*XS^{-1} \right\| + \left\| S^{-1}XS^* \right\|$,
(iii) $\forall X \in \mathcal{F}_1(H), \left\| SXS^{-1} \right\| + \left\| S^{-1}XS \right\| \leq \left\| S^*XS^{-1} \right\| + \left\| S^{-1}XS^* \right\|$.

Proof (i) $\Rightarrow$ (ii). This implication follows immediately from Remark 5.1.
(ii) $\Rightarrow$ (iii). This implication is trivial.
(iii) $\Rightarrow$ (i). From (iii), it follows that the following inequality holds:

$$\forall x,\ y \in (H)_1,\ \|Sx\| \left\| (S^*)^{-1}y \right\| + \left\| S^{-1}x \right\| \left\| S^*y \right\| \leq \left\| S^*x \right\| \left\| (S^*)^{-1}y \right\| + \left\| S^{-1}x \right\| \|Sy\|.$$

Hence,

$$\forall x,\ y \in (H)_1,\ \left(\|Sx\| - \left\| S^*x \right\|\right) \left\| (S^*)^{-1}y \right\| \leq \left(\|Sy\| - \left\| S^*y \right\|\right) \left\| S^{-1}x \right\|. \quad (5.2)$$

Thus,

$$\left(\forall x \in (H)_1,\ \|Sx\| \geq \left\| S^*x \right\|\right) \vee \left(\forall x \in (H)_1,\ \|Sx\| \leq \left\| S^*x \right\|\right).$$

Assume that the inequality $\|Sx\| \geq \|S^*x\|$ holds for every $x \in (H)_1$.

Since the relation $\frac{1}{\|T^{-1}\|} \leq \|Tx\| \leq \|T\|$ holds for every $T \in \mathfrak{I}(H)$ and for every $x \in (H)_1$, then from (5.2), it follows that

$$\forall x,\ y \in (H)_1,\ \|Sx\| - \left\|S^* x\right\| \leq k \left(\|Sy\| - \left\|S^* y\right\|\right),$$

where $k = \|S\| \left\|S^{-1}\right\|$. So we have

$$\forall x,\ y \in (H)_1,\ \|Sx\| + k \left\|S^* y\right\| \leq \left\|S^* x\right\| + k \|.Sy\|.$$

Hence,

$$\forall x \in (H)_1,\ \sup_{\|y\|=1} \left(\|Sx\| + k \left\|S^* y\right\|\right) \leq \sup_{\|y\|=1} \left(\left\|S^* x\right\| + k \|Sy\|\right).$$

Thus,

$$\forall x \in (H)_1,\ \|Sx\| + k \|S\| \leq \left\|S^* x\right\| + k \|S\|.$$

So, it follows that the inequality $\|Sx\| \leq \|S^* x\|$ holds for every vector x in $(H)_1$. Hence, the equality $\|Sx\| = \|S^* x\|$ holds for every vector x in $(H)_1$. Therefore, $S \in \mathcal{N}_0(H)$.

With the second assumption and by the same argument, we find also that $S \in \mathcal{N}_0(H)$. □

In the next proposition, we shall give a complete characterization of the class of all normal operators in $\mathfrak{B}(H)$ in terms of operator inequality. To prove this, we need the following results of Halmos (see [9]) that says: the set

$$\mathfrak{D}(H) = \{S \in \mathfrak{B}(H) : S \text{ is left invertible or right invertible}\}$$

is dense in $\mathfrak{B}(H)$, and from the fact that for $S \in \mathfrak{B}(H)$, we have:

(i) S is left invertible if and only if S is injective with closed range,
(ii) S is right invertible if and only if S is surjective.

Proposition 5.14 [23] *Let $S \in \mathfrak{B}(H)$. Then, the following properties are equivalent*:

(i) *S is normal*,
(ii) $\forall X \in \mathfrak{B}(H),\ \left\|S^2 X\right\| + \left\|X S^2\right\| \geq 2 \|SXS\|$.

Proof We may assume that $S \neq 0$.

(i) ⇒ (ii). Assume (i) holds. Let $X \in \mathfrak{B}(H)$. Then we have

$$\begin{aligned}\left\|S^2 X\right\| + \left\|X S^2\right\| &= \left\|S^* S X\right\| + \left\|X S S^*\right\|, \text{ (from Remark 5.1)},\\ &\geq 2 \|SXS\|, \text{ (from (2.3))}.\end{aligned}$$

This proves (ii).

(ii) $\Rightarrow$ (i). Assume (ii) holds. Put $P = |S|$. We prove this implication with four cases.

Case 1. Assume that S is injective with closed range.

Hence, $S^+S = I$, $\ker P = \ker S = \{0\}$, and $R(P) = R(S^*S)$ is closed (since $R(S^*)$ is also closed). Thus, $\ker P = \{0\}$ and $R(P) = (\ker P)^\perp = H$. So, P is invertible. From (ii), the two following inequalities hold:

$$\forall X \in \mathfrak{B}(H),\ \left\|S^2S^+XS^+\right\| + \left\|S^+XS\right\| \geq 2\left\|SS^+X\right\|. \tag{5.3}$$

$$\forall X \in \mathfrak{B}(H),\ \|XS\| + \left\|S^2XS^+\right\| \geq 2\left\|SX\right\|. \tag{5.4}$$

The proof is given in four steps.

Step 1. Prove that $\left(S^2\right)^+ S = S^+$.

It is known that S^+ is the unique solution of the following four equations: $SXS = S,\ XSX = X,\ (XS)^* = XS,\ (SX)^* = SX$. It is easy to see that $\left(S^2\right)^+ S$ satisfies the first three equations.

Now, we prove that $\left(S^2\right)^+ S$ also satisfies the last equation. Since the operator $S\left(S^2\right)^+ S$ is a projection, it suffices to prove that its norm is less than or equal to one. By taking $X = \left(S^2\right)^+ S$ in (5.4), we obtain

$$2 \geq \left\|\left(S^2\right)^+ S^2\right\| + \left\|S^2\left(S^2\right)^+ SS^+\right\| \geq 2\left\|S\left(S^2\right)^+ S\right\|.$$

Hence, $\left\|S\left(S^2\right)^+ S\right\| \leq 1$. Therefore, $\left(S^2\right)^+ S = S^+$.

Step 2. Prove that $\left(S^2\right)^+ = (S^+)^2$.

Since $S^2\left(S^2\right)^+ = SS^+S^2\left(S^2\right)^+$, then $S^2\left(S^2\right)^+ = S^2\left(S^2\right)^+ SS^+$. So from step 2, we obtain $S^2\left(S^2\right)^+ = S^2(S^+)^2$. Since S^2 is injective, we have $\left(S^2\right)^+ = (S^+)^2$.

Step 3. Prove that $\ker S^* = \{0\}$.

All the 2×2 matrices used in this step are given with respect to the orthogonal direct sum $H = R(S) \oplus \ker S^*$. Then, $S = \begin{bmatrix} S_1 & S_2 \\ 0 & 0 \end{bmatrix}$. We put $Q = |S^*|$, $P_1 = \left|S_1\right|$, $P_2 = \left|S_2\right|$, $Q_1 = \left(S_1S_1^* + S_2S_2^*\right)^{\frac{1}{2}}$. So we have $S^*S = P^2 = \begin{bmatrix} P_1^2 & S_1^*S_2 \\ S_2^*S_1 & P_2^2 \end{bmatrix}$, $SS^* = Q^2 = \begin{bmatrix} Q_1^2 & 0 \\ 0 & 0 \end{bmatrix}$. It is clear that $Q = \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix}$, Q_1 is invertible, and $Q^+ = \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$.

Since S is injective, then $\ker S^* = \{0\}$ if and only if $S_2 = 0$. Assume that $S_2 \neq 0$.

Since $\left(S^2\right)^+ = (S^+)^2$, then the two operators S^*S and SS^+ commute (see [3, 11]). Thus, $P^2 = \begin{bmatrix} P_1^2 & 0 \\ 0 & P_2^2 \end{bmatrix}$, so that $P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$.

Since $\ker S^* \neq \{0\}$, then $\sigma(Q^2) = \sigma(Q_1^2) \cup \{0\}$. From the fact that $\sigma(P^2) = \sigma(Q^2) - \{0\}$, we have $\sigma(P^2) = \sigma(Q_1^2)$. Then, $\sigma(P_1^2) \cup \sigma(P_2^2) = \sigma(Q_1^2)$. Hence, $\sigma(P_1^2) \subset \sigma(Q_1^2)$. Thus, $\sigma(P_1) \subset \sigma(Q_1)$.

Using the polar decomposition of S and S^* in the inequality (5.3) , we obtain the following inequality:

$$\forall X \in \mathfrak{B}(H), \ \left\| S^2 S^+ X P^{-1} \right\| + \left\| Q^+ X Q \right\| \geq 2 \left\| S S^+ X \right\| .$$

By taking $X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}$ (resp. $X = \begin{bmatrix} 0 & X_2 \\ 0 & 0 \end{bmatrix}$), where $X_1 \in \mathfrak{B}(R(S))$ (resp. $X_2 \in \mathfrak{B}(\ker S^*, R(S))$) in the last inequality and since $S^2 S^+ = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix}$, we deduce the two following inequalities:

$$\forall X_1 \in \mathfrak{B}(R(S)), \ \left\| P_1 X_1 P_1^{-1} \right\| + \left\| Q_1^{-1} X_1 Q_1 \right\| \geq 2 \left\| X_1 \right\| . \tag{5.5}$$

$$\forall X_2 \in \mathfrak{B}(\ker S^*, R(S)), \ \left\| P_1 X_2 P_2^{-1} \right\| \geq 2 \left\| X_2 \right\| . \tag{5.6}$$

By taking $X_2 = x \otimes y$ (where $x \in (R(S))_1$, $y \in \ker S^*$) in (5.6), we obtain

$$\forall x \in (R(S))_1, \forall y \in \ker S^*, \ \left\| P_1 x \right\| \left\| P_2^{-1} y \right\| \geq 2 \left\| y \right\| .$$

So, we have

$$\forall x \in (R(S))_1, \forall y \in \left(\ker S^*\right)_1, \ \left\| P_1 x \right\| \geq 2 \left\| P_2 y \right\| .$$

Thus $\| P_2 y \| \leq \frac{k}{2}$, for every $y \in (\ker S^*)_1$ (where $k = \inf_{\|x\|=1} \| P_1 x \| > 0$). Then $\left\langle P_2^2 y, y \right\rangle \leq \frac{k^2}{4}$, for every $y \in (\ker S^*)_1$. So we obtain $\sigma(P_2^2) \subset (0, \frac{k^2}{4}]$ and $\sigma(P_1^2) \subset [k^2, \infty)$.

Since $\sigma(P_1) \subset \sigma(Q_1)$, and P_1, Q_1 satisfy the inequality (5.5), then using Lemma 5.9, we obtain $P_1 = Q_1$. Hence $\sigma(Q_1^2) = \sigma(P_1^2) = \sigma(P_1^2) \cup \sigma(P_2^2)$. Then $\sigma(P_2^2) \subset \sigma(P_1^2)$, that is impossible, since $(0, \frac{k^2}{4}] \cap [k^2, \infty) = \varnothing$. Therefore, $\ker S^* = \{0\}$.

Step 4. Prove that S is normal.

Since $\ker S^* = \{0\}$, then $R(S) = H$, so that S is invertible and satisfies the inequality (ii). Hence, S satisfies the following inequality:

$$\forall X \in \mathfrak{B}(H), \ \left\| S X S^{-1} \right\| + \left\| S^{-1} X S \right\| \geq 2 \left\| X \right\| .$$

Therefore S is normal, by using Proposition 5.11.

Case 2. Assume S surjective.

Then, S^* is injective with a closed range satisfying also the inequality (ii). So that from case 1, S^* is normal. Hence, S is normal.

Case 3. General situation.

We may assume without loss of generality that $\|S\| = 1$. Then $\left\| S^2 \right\| = \|S\|^2 = 1$. There exists a sequence $(S_n)_{n \geq 1}$ of elements in $\mathfrak{D}(H)$ such that $S_n \to S$ uniformly.

Define the real function F on the complete metric space $(\mathfrak{B}(H))_1$ by

$$\forall X \in (\mathfrak{B}(H))_1\,,\ F(X) = \left\| S^2 X \right\| + \left\| X S^2 \right\| - 2\left\| SXS \right\|,$$

and for $n \geq 1$, define the real function F_n on $(\mathfrak{B}(H))_1$ by

$$\forall X \in (\mathfrak{B}(H))_1\,,\ F_n(X) = \left\| S_n^2 X \right\| + \left\| X S_n^2 \right\| - 2\left\| S_n X S_n \right\|.$$

Put $D = \{X \in (\mathfrak{B}(H))_1 : F(X) > 0\}$. Then there are two cases, $D = \varnothing$, $D \neq \varnothing$.
(1) $D = \varnothing$. So, it follows that

$$\forall X \in \mathfrak{B}(H),\ \left\| S^2 X \right\| + \left\| X S^2 \right\| = 2\left\| SXS \right\|. \tag{5.7}$$

From this equality, we have

$$\forall x, y \in H,\ \left\| S^2 x \right\| \|y\| + \|x\| \left\| S^{*2} y \right\| = 2\left\| Sx \right\| \left\| S^* y \right\|.$$

Since $S^2 \neq 0$, and from this last inequality, we deduce easily that S and S^* are injective, and then S is with dense range.

Prove now that S is with closed range. Let (x_n) be a sequence of vectors in H such that (Sx_n) converges to a vector $y \in H$. We may choose a vector $u \in (H)_1$ such that $S^{*2}u \neq 0$. From the above inequality, we obtain

$$\forall n, m \geq 1,\ \left\| S^2 x_n - S^2 x_m \right\| + \left\| x_n - x_m \right\| \left\| S^{*2} u \right\| = 2\left\| Sx_n - Sx_m \right\| \left\| S^* u \right\|.$$

Hence, (x_n) is a Cauchy sequence, and then it converges to some vector $x \in H$. So that $Sx_n \to y = Sx$. This proves $R(S)$ is closed. Then, S is invertible.

So, from (5.7), it follows that

$$\forall X \in \mathfrak{B}(H),\ \left\| SXS^{-1} \right\| + \left\| S^{-1} X S \right\| = 2\left\| X \right\|.$$

From Proposition 5.11, (i) holds.

(2) $D \neq \varnothing$. From the fact that F is a positive continuous map on $(\mathfrak{B}(H))_1$, it follows that

$$\overline{D} = \overline{F^{-1}\left((0, \infty)\right)} = F^{-1}\left([0, \infty)\right) = \{X \in (\mathfrak{B}(H))_1 : F(X) \geq 0\} = (\mathfrak{B}(H))_1\,.$$

Let $X \in D$, and $\epsilon > 0$. Since $S_n \to S$ uniformly, then there exists an integer $N \geq 1$ (depends only in ϵ) such that

$$\forall n \geq N,\ \forall Y \in (\mathfrak{B}(H))_1\,,\ |F(Y) - F_n(Y)| \leq \epsilon.$$

If there exists $n \geq N$ such that $F_n(X) < 0$, then using this last inequality, we have $0 \leq F(X) < \epsilon$, for every $\epsilon > 0$; thus $F(X) = 0$, leading to a contradiction with $X \in D$.

From this fact, it follows that

$$\forall X \in D,\ \forall n \geq N, \quad F_n(X) \geq 0.$$

Since each F_n is a continuous map on $(\mathfrak{B}(H))_1$ and D is dense in $(\mathfrak{B}(H))_1$, then

$$\forall X \in (\mathfrak{B}(H))_1\,,\ \forall n \geq N,\ F_n(X) \geq 0.$$

So, it follows that

$$\forall X \in \mathfrak{B}(H),\ \forall n \geq N,\ \left\| S_n^2 X \right\| + \left\| X S_n^2 \right\| \geq 2 \left\| S_n X S_n \right\| .$$

Since for each $n \geq 1$, $S_n \in \mathfrak{D}(H)$, then from the two above cases, we obtain that S_n is a normal operator, for every $n \geq N$. Since $S_n \to S$ uniformly and the class of all normal operators in $\mathfrak{B}(H)$ is closed, then S is a normal. □

Remark 5.15 In the above proof, case 1 is the lemma presented in the corrigendum [22]. Note that in the proof of this lemma in the corrigendum, we have used Theorem 3.6 of [14] with equality between the spectrum (that is not suffice) and we mentioned that with the inclusion between spectrum the theorem remains true (without argument). In this survey, we have present this argument in Lemma 5.9.

Corollary 5.16 *Let $S \in \mathfrak{B}(H)$. Then, the following properties are equivalent.*

(i) *S is normal,*
(ii) $\forall X \in \mathfrak{B}(H),\ \left\| S^2 X \right\| \left\| X S^2 \right\| \geq \left\| S X S \right\|^2 .$

Proof (i) $\Rightarrow$ (ii). Assume (i) holds, and let $X \in \mathfrak{B}(H)$. Then we have

$$\begin{aligned} \left\| S^2 X \right\| \left\| X S^2 \right\| &= \left\| S^* S X \right\| \left\| X S S^* \right\| \text{ (using Remark 5.1)} \\ &\geq \left\| S X S \right\|^2 \text{ (see the proof of (2.3) in Proposition 5.2).} \end{aligned}$$

(ii) $\Rightarrow$ (i). Assume (ii) holds, and let $X \in \mathfrak{B}(H)$. Then we have:

$$\begin{aligned} \frac{\left\| S^2 X \right\| + \left\| X S^2 \right\|}{2} &\geq \sqrt{\left\| S^2 X \right\| \left\| X S^2 \right\|} \text{ (from the numerical AGMI)} \\ &\geq \left\| S X S \right\| \text{ (using (ii)).} \end{aligned}$$

From the last proposition, (i) holds. □

Corollary 5.17 [21] *Let $S \in \mathcal{R}(H)$. Then the following properties are equivalent*:

$$\begin{array}{ll} \text{(i)} & S \in \mathcal{N}(H), \\ \text{(ii)} & \forall X \in \mathfrak{B}(H),\ \left\|SXS^{+}\right\| + \left\|S^{+}XS\right\| = \left\|S^{*}XS^{+}\right\| + \left\|S^{+}XS^{*}\right\|, \\ \text{(iii)} & \forall X \in \mathfrak{B}(H),\ \left\|SXS^{+}\right\| + \left\|S^{+}XS\right\| \geq \left\|S^{*}XS^{+}\right\| + \left\|S^{+}XS^{*}\right\|, \\ \text{(iv)} & \forall X \in \mathfrak{B}(H),\ \left\|SXS^{+}\right\| + \left\|S^{+}XS\right\| \geq 2\left\|SS^{+}XS^{+}S\right\|. \end{array}$$

Proof We may assume that $S \neq 0$.

(i) $\Rightarrow$ (ii). This follows immediately from Remark 5.1.

The implication (ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (iv). This follows immediately from Corollary 5.3(iv).

(iv) $\Rightarrow$ (i). Assume (iv) holds. Then the following inequality holds:

$$\forall X \in \mathfrak{B}(H), \quad \left\|S^{2}XSS^{+}\right\| + \left\|S^{+}SXS^{2}\right\| \geq 2\left\|SS^{+}SXSS^{+}S\right\|.$$

From this inequality and since $\left\|SS^{+}\right\| = \left\|S^{+}S\right\| = 1$, and $SS^{+}S = S$, it follows that

$$\forall X \in \mathfrak{B}(H), \quad \left\|S^{2}X\right\| + \left\|XS^{2}\right\| \geq 2\left\|SXS\right\|.$$

Using Proposition 5.14, S is normal. □

Remark 5.18 In the original paper [21], the above corollary is presented before the characterization of $\mathcal{N}(H)$ in its general situation, and for this reason its proof is very strong. But, in this survey, we have adopted a new strategy, where this corollary follows immediately from the general situation.

Conclusion 5.19 (1) *The class of all invertible normal operators in $\mathfrak{B}(H)$ is characterized by each of the following properties*:

$$\begin{array}{ll} \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geq 2\left\|X\right\|, & (S \in \mathfrak{I}(H)). \\ \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| = \left\|S^{*}XS^{-1}\right\| + \left\|S^{-1}XS^{*}\right\|, & (S \in \mathfrak{I}(H)). \\ \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geq \left\|S^{*}XS^{-1}\right\| + \left\|S^{-1}XS^{*}\right\|, & (S \in \mathfrak{I}(H)). \\ \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \leq \left\|S^{*}XS^{-1}\right\| + \left\|S^{-1}XS^{*}\right\|, & (S \in \mathfrak{I}(H)). \\ \forall X \in \mathcal{F}_{1}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \leq \left\|S^{*}XS^{-1}\right\| + \left\|S^{-1}XS^{*}\right\|, & (S \in \mathfrak{I}(H)). \end{array}$$

(2) *The class of all normal operators with closed ranges in $\mathfrak{B}(H)$ is characterized by each of the following properties*:

$$\begin{array}{ll} \forall X \in \mathfrak{B}(H),\ \left\|SXS^{+}\right\| + \left\|S^{+}XS\right\| \geq 2\left\|SS^{+}XS^{+}S\right\|, & (S \in \mathfrak{R}(H)) \\ \forall X \in \mathfrak{B}(H),\ \left\|SXS^{+}\right\| + \left\|S^{+}XS\right\| = \left\|S^{*}XS^{+}\right\| + \left\|S^{+}XS^{*}\right\|, & (S \in \mathfrak{R}(H)) \\ \forall X \in \mathfrak{B}(H),\ \left\|SXS^{+}\right\| + \left\|S^{+}XS\right\| \geq \left\|S^{*}XS^{+}\right\| + \left\|S^{+}XS^{*}\right\|, & (S \in \mathfrak{R}(H)) \end{array}$$

(3) The class of all normal operators in $\mathfrak{B}(H)$ is characterized by:

$$\forall X \in \mathfrak{B}(H),\ \left\|S^{2}X\right\| + \left\|XS^{2}\right\| \geq 2\left\|SXS\right\|, \quad (S \in \mathfrak{B}(H)).$$

5.3 Characterizations related to Magajna–Petkovsek–Turnsek

For $S, T \in \mathfrak{B}(H)$:

(i) we say that S and T are unitarily equivalent, if there exists a unitary operator $U \in \mathfrak{B}(H)$ such that $S = U^* T U$,
(ii) it is easy to see that if $S \in \mathfrak{I}(H)$, then $|S|$ and $|S^*|$ are unitarily equivalent,
(iii) if S and T are unitarily equivalent, then S and T have the same spectrum, but the converse is false (in general),
(iv) S is paranormal if $\|x\| \left\|S^2 x\right\| \geq \|Sx\|^2$, for every $x \in H$,
(v) we say that S belongs to class $\mathbf{A}$, if $\left|S^2\right| \geq |S|^2$,
(vi) if $S \geq 0$, $T \geq 0$, and $S \geq T$, then $S^\alpha \geq T^\alpha$, for every $\alpha \in [0, 1]$ (Löwner–Heinz inequality [10]),
(vii) if S belongs to class $\mathbf{A}$, then it is paranormal (see [8]).

Using the Theorem 2.1 of Magajna–Petkovsek–Turnsek [12], we shall present in this subsection other characterizations of the class $\mathcal{N}_0(H)$, and then, we deduce some general characterizations of the class $\mathcal{N}(H)$.

Lemma 5.20 *Let P and Q be two invertible positive unitarily equivalent operators in $\mathfrak{B}(H)$. The two following properties are equivalent*:

(i) $\forall X \in \mathcal{F}_1(H),\ \left\|PXP^{-1}\right\| + \left\|Q^{-1}XQ\right\| \geq 2\|X\|$,
(ii) $P = Q$.

Proof The implication (i) $\Longrightarrow$ (ii) represents a particular case of [12, Theorem 2.1].
The implication (ii) $\Longrightarrow$ (i) follows immediately from Proposition 5.11. □

Proposition 5.21 *Let $S \in \mathfrak{I}(H)$. The following properties are equivalent*:

(i) $S \in \mathcal{N}_0(H)$,
(ii) $\forall X \in \mathcal{F}_1(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| = \left\|S^*XS^{-1}\right\| + \left\|S^{-1}XS^*\right\|$,
(iii) $\forall X \in \mathcal{F}_1(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geq \left\|S^*XS^{-1}\right\| + \left\|S^{-1}XS^*\right\|$,
(iv) $\forall X \in \mathcal{F}_1(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geq 2\|X\|$.

Proof The implication (i) $\Longrightarrow$ (ii) follows immediately from Remark 5.1, the implication (ii) $\Longrightarrow$ (iii) is trivial, and the implication (iii) $\Longrightarrow$ (iv) follows from Corollary 5.3(iii).

(iv) $\Longrightarrow$ (i). Assume (iv) holds.

Put $P = |S|$ and $Q = |S^*|$. Using (iv) and the polar decomposition of S and S^*, we deduce the following inequality:

$$\forall X \in \mathcal{F}_1(H),\ \left\|PXP^{-1}\right\| + \left\|Q^{-1}XQ\right\| \geq 2\|X\|.$$

Since P and Q are unitarily equivalent, then using the above lemma, we find that $P = Q$. This proves (i). □

In [2], Ando proved that for $S \in \mathfrak{B}(H)$, S is normal if and only if S and S^* are paranormal, and $\ker S = \ker S^*$. In the next proposition, we present some new general characterizations of the class $\mathcal{N}(H)$, and we shall show that the Ando result remains true without the kernel assumption.

Proposition 5.22 *Let $S \in \mathfrak{B}(H)$. The following properties are equivalent.*

(i) *S is normal,*
(ii) $\left|S^2\right| = |S|^2,\ \left|S^{*2}\right| = |S^*|^2$,
(iii) $\left|S^2\right|^2 \geq |S|^4,\ \left|S^{*2}\right|^2 \geq |S^*|^4$,
(iv) *S and S^* belong to class* $\mathbf{A}$,
(v) *S and S^* are paranormal.*

Proof The implications (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii) are trivial.

(iii) $\Rightarrow$ (iv). This follows from Löwner–Heinz inequality with $\alpha = \frac{1}{2}$.

(iv) $\Rightarrow$ (v). This follows from [10].

(v) $\Rightarrow$ (i). Assume (v) holds. Then, we have

$$\begin{cases} \forall x \in H,\ \|x\| \left\|S^2 s\right\| \geq \|Sx\|^2, & (5.8)\\ \forall x \in H,\ \|x\| \left\|(S^*)^2 x\right\| \geq \|S^* x\|^2. & (5.9)\end{cases}$$

So, it follows that

$$\forall X \in \mathcal{F}_1(H),\ \left\|S^2 X\right\| + \left\|X S^2\right\| \geq 2\,\|SXS\|. \tag{5.10}$$

The proof is given in three cases.

Case 1. Assume that S surjective.

From (5.8), it follows that

$$\forall x \in H,\ \left\|S^+ x\right\| \|Sx\| \geq \|x\|^2.$$

Then S is injective. Hence, S is invertible. So using (5.10), we obtain

$$\forall X \in \mathcal{F}_1(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geq 2\,\|X\|.$$

So from the above proposition, S is normal.

Case 2. Assume that S is injective with closed range.

Then S^* is surjective. So, from (5.9) and using the same argument as used in the case 1, we find that S is invertible, so that S is normal.

Case 3. General situation.

From (5.10), and by using the same argument as used in the case 3 of the proof of Proposition 5.14 (where $\mathcal{F}_1(H)$ takes the place of $\mathfrak{B}(H)$), and using [12, Theorem 2.1], we obtain that S is normal. □

Remark 5.23 (1) The equivalences between (i), (ii), and (iii) in the above corollary were given in [23], and follow from Proposition 5.22. The equivalences between (i), (iv), and (v) are new.

(2) The Lemma 5.20 must be used only for the invertible case, but Lemma 5.9 is needed for the general situation. The hypothesis of Lemma 5.9 is more general than the hypothesis of Lemma 5.20, but the condition (i) of Lemma 5.9 is a particular case of the condition (i) of Lemma 5.20.

Conclusion 5.24 (1) *The class of all invertible normal operators in* $\mathfrak{B}(H)$ *is characterized by each of the following properties*:

$$\begin{array}{lll} \forall X \in \mathcal{F}_1(H), \ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geq & 2\left\|X\right\|, & (S \in \mathfrak{I}(H)), \\ \forall X \in \mathcal{F}_1(H), \ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| = & \left\|S^*XS^{-1}\right\| + \left\|S^{-1}XS^*\right\|, & (S \in \mathfrak{I}(H)), \\ \forall X \in \mathcal{F}_1(H), \ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geq & \left\|S^*XS^{-1}\right\| + \left\|S^{-1}XS^*\right\|, & (S \in \mathfrak{I}(H)). \end{array}$$

(2) *The class of all normal operators in* $\mathfrak{B}(H)$ *is characterized by each of the following properties*:

$$\begin{array}{ll} \left|S^2\right| = |S|^2, \ \left|S^{*2}\right| = |S^*|^2, & (S \in \mathfrak{B}(H), \\ \left|S^2\right|^2 \geq |S|^4, \ \left|S^{*2}\right|^2 \geq |S^*|^4, & (S \in \mathfrak{B}(H), \\ S, \ S^* \textit{belong to class } \mathbf{A}, & (S \in \mathfrak{B}(H), \\ S, \ S^* \textit{are paranormal}, & (S \in \mathfrak{B}(H). \end{array}$$

6 Arithmetic–geometric mean inequality, selfadjoint operators, and characterization

6.1 Operator inequality related to the S-arithmetic–geometric mean inequality

In [10], Heinz proved that for every two positive operators P and Q in $\mathfrak{B}(H)$, and for every $\alpha \in [0, 1]$, the following operator inequality holds:

$$\forall X \in \mathfrak{B}(H), \quad \|PX + XQ\| \geq \left\|P^\alpha X Q^{1-\alpha} + P^{1-\alpha} X Q^\alpha\right\|. \tag{6.1}$$

As a particular case of this, for $\alpha = \frac{1}{2}$, the well-known arithmetic–geometric mean inequality is given by (2.2).

Note that the proof of (6.1) given by Heinz is somewhat complicated. For this reason, McIntosh [13], with an elegant proof, proved that the operator inequality (2.2) holds, and deduced from it from the Heinz inequality by the iteration method.

Independently of the work of Heinz and McIntosh, Corach et al. proved in [6], that for every invertible selfadjoint operator S in $\mathfrak{B}(H)$, the inequality (2.1) holds.

In [7], it was proved that the three above operator inequalities are equivalent, and proving (2.1) with an easy proof, this gives us an easier proof of Heinz inequality.

In the following proposition, we shall give a family of operator inequalities that are equivalent to the Heinz inequality and present the proof of (2.1) given in [7].

Proposition 6.1 [4, 7] *The following operator inequalities hold and are mutually equivalent*:

(i) $\forall X \in \mathfrak{B}(H), \ \|A^*AX + XBB^*\| \geq 2\|AXB\|$, *for every* $A, B \in \mathfrak{B}(H)$.
(ii) $\forall X \in \mathfrak{B}(H), \ \left\|SXR^+ + S^+XR\right\| \geq 2\left\|SS^+XR^+R\right\|$, *for every* $S, R \in \mathbb{S}_{cr}(H)$.
(iii) $\forall X \in \mathfrak{B}(H), \ \left\|S^2X + XR^2\right\| \geq 2\|SXR\|$, *for every* $S, R \in \mathbb{S}_{cr}(H)$.
(iv) $\forall X \in \mathfrak{B}(H), \ \left\|SXR^{-1} + S^{-1}XR\right\| \geq 2\|X\|$, *for every* $S, R \in \mathbb{S}_0(H)$.
(v) $\forall X \in \mathfrak{B}(H), \ \left\|S^2X + XR^2\right\| \geq 2\|SXR\|$, *for every* $S, R \in \mathbb{S}(H)$.

(vi) $\forall X \in \mathfrak{B}(H)$, $\|A^*AX + XAA^*\| \geq 2\|AXA\|$, *for every* $A \in \mathfrak{B}(H)$.
(vii) $\forall X \in \mathfrak{B}(H)$, $\|SXS^+ + S^+XS\| \geq 2\|SS^+XS^+S\|$, *for every* $S \in \mathbb{S}_{cr}(H)$.
(viii) $\forall X \in \mathfrak{B}(H)$, $\|S^2X + XS^2\| \geq 2\|SXS\|$, *for every* $S \in \mathbb{S}_{cr}(H)$.
(ix) $\forall X \in \mathfrak{B}(H)$, $\|SXS^{-1} + S^{-1}XS\| \geq 2\|X\|$, *for every* $S \in \mathbb{S}_0(H)$.
(x) $\forall X \in \mathfrak{B}(H)$, $\|S^2X + XS^2\| \geq 2\|SXS\|$, *for every* $S \in \mathbb{S}(H)$.

Proof (i) $\Rightarrow$ (ii). Assume (i) holds. Let $S, R \in \mathbb{S}_{cr}(H)$, $X \in \mathfrak{B}(H)$. Since $S = S^*SS^+$ and $R = R^+RR^*$, then from (i) it follows that

$$\begin{aligned}\|SXR^+ + S^+XR\| &= \|S^*S\left(S^+XR^+\right) + \left(S^+XR^+\right)RR^*\| \\ &\geq 2\|SS^+XR^+R\|.\end{aligned}$$

Hence, (ii) holds.
(ii) $\Rightarrow$ (iv). Trivial.
(iv) $\Rightarrow$ (v). Assume (iv) holds. Let $S, R \in \mathbb{S}(H)$, and put $P = |S|$, $Q = |R|$.
Let $\epsilon > 0$. From (iv), the following inequality holds:

$$\forall X \in \mathfrak{B}(H), \ \left\|(P+\epsilon I)^2 X + X(Q+\epsilon I)^2\right\| \geq 2\|(P+\epsilon I)X(Q+\epsilon I)\|.$$

Letting $\epsilon \to \infty$, we obtain

$$\forall X \in \mathfrak{B}(H), \ \left\|S^2X + XR^2\right\| \geq 2\|SXR\|.$$

(v) $\Rightarrow$ (i). This follows immediately by using the polar decomposition of an operator.
(v) $\Rightarrow$ (iii). Trivial.
(iii) $\Rightarrow$ (iv). Trivial.
Hence, the equivalences (i)–(v) hold.
From a pair of operators to a single operator, the equivalences (vi)–(x) hold.
(i) $\Rightarrow$ (vi). Trivial.
(vi) $\Rightarrow$ (i). This follows using Berberian technique as used in Proposition 5.2.
Hence, the ten properties are mutually equivalent.
Prove now that the operator inequality (iv) holds.
Step 1. Let $S, X \in \mathfrak{B}(H)$ such that S and X are selfadjoint, and S invertible.
Then, there exists $\lambda \in \sigma(X)$ such that $|\lambda| = \|X\|$. Since $\sigma(X) = \sigma(SXS^{-1}) \subset V(SXS^{-1})$, there exists a state f on $\mathfrak{B}(H)$ such that $\lambda = f\left(SXS^{-1}\right) = f\left(S^{-1}XS\right)$. This gives us $2\|X\| = \left|f\left(SXS^{-1} + S^{-1}XS\right)\right| \leq \left\|SXS^{-1} + S^{-1}XS\right\|$.
Step 2. Let $S, X \in \mathfrak{B}(H)$ such that S is selfadjoint invertible.
Let the two following operators on the Hilbert space $H \oplus H$ be given by $M = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}$, so that M and Y are selfadjoint operators in $\mathfrak{B}(H \oplus H)$ and where

M is invertible. Applying step 1 for this pair of operators, we obtain

$$\begin{aligned}\left\|SXS^{-1}+S^{-1}XS\right\| &= \left\|MXM^{-1}+M^{-1}XM\right\| \\ &\geq 2\,\|Y\| \\ &= 2\,\|X\|\,.\end{aligned}$$

□

Remark 6.2 In [1], we find generalized versions of the arithmetic–geometric mean inequality.

Corollary 6.3 *The following operator inequalities hold*:

(i) *For every* $S, R \in \mathfrak{I}(H)$, *the following holds*:

$$\forall X \in \mathfrak{B}(H),\ \left\|S^*XR^{-1}+S^{-1}XR^*\right\| \geq 2\,\|X\|\,.$$

(ii) *For every* $S, R \in \mathcal{R}(H)$, *the following holds*:

$$\forall X \in \mathfrak{B}(H),\ \left\|S^*XR^{+}+S^{+}XR^*\right\| \geq 2\left\|SS^{+}XR^{+}R\right\|.$$

(iii) *For every* $S \in \mathfrak{I}(H)$, *the following holds*:

$$\forall X \in \mathfrak{B}(H),\ \left\|S^*XS^{-1}+S^{-1}XS^*\right\| \geq 2\,\|X\|\,.$$

(iv) *For every* $S \in \mathcal{R}(H)$, *the following holds*:

$$\forall X \in \mathfrak{B}(H),\ \left\|S^*XS^{+}+S^{+}XS^*\right\| \geq 2\left\|SS^{+}XS^{+}S\right\|.$$

Proof It suffices to prove (ii) and the three others follow immediately from (ii).

Let $S, R \in \mathcal{R}(H)$, and $X \in \mathfrak{B}(H)$. Since $SS^+S = S$ and $RR^+R = R$, then we have

$$\begin{aligned}\left\|S^*XR^{+}+S^{+}XR^*\right\| &= \left\|S^*S\left(S^{+}XR^{+}\right)+\left(S^{+}XR^{+}\right)RR^*\right\| \\ &\geq 2\left\|SS^{+}XR^{+}R\right\|, \quad \text{(from (2.2))}.\end{aligned}$$

□

Note that the eight operator inequalities (ii)–(v) and (vii)–(x) given in Proposition 6.1 are generated by a pair of selfadjoint operators and a single selfadjoint operator, respectively.

It will be of interest to describe the class of all operators $S \in \mathfrak{I}(H)$ (resp. $S \in \mathfrak{R}(H)$, $S \in \mathfrak{B}(H)$) satisfying the operator inequality (ix) (resp. (vii), (x)).

We shall prove that the largest class of

$(\cdot)$ all operators $S \in \mathfrak{I}(H)$ satisfying (ix) is the class $(\mathbb{C})_1\, \mathcal{S}_0(H)$ (the class of all rotation of all selfadjoint operators $S \in \mathfrak{I}(H)$),

$(\cdot\cdot)$ all operators $S \in \mathfrak{R}(H)$ satisfying the operator inequality (vii) is the class $(\mathbb{C})_1\, \mathcal{S}_{cr}(H)$ (the class of all rotation of all selfadjoint operators $S \in \mathfrak{R}(H)$),

$(\cdot\cdot\cdot)$ all operators $S \in \mathfrak{B}(H)$ satisfying the operator inequality (x) is the class $(\mathbb{C})_1\, \mathcal{S}(H)$ (the class of all rotation of all selfadjoint operators $S \in \mathfrak{B}(H)$).

In the next subsection, we shall present all these characterizations and others.

6.2 Selfadjoint operators and characterizations

In this section, we shall present some characterizations of the class of all invertible selfadjoint operators multiplied by nonzero scalars, the class of all selfadjoint operators with closed ranges multiplied by scalars, and the class of all selfadjoint operators multiplied by scalars.

We need the following lemma.

Lemma 6.4 [14] *Let* $\lambda,\ \mu \in \mathbb{C}^*$ *such that* $\frac{\lambda}{\mu} + \frac{\mu}{\lambda} \in \mathbb{R}$, *and* $\left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right| \geq 2$. *Then there exists* $\theta \in [0, \pi)$ *such that* $\lambda,\ \mu \in D_\theta$.

Proof Let $\lambda = re^{i\alpha},\ \mu = le^{i\beta}$ be the polar decomposition of $\lambda,\ \mu$. Then we have

$$\frac{\lambda}{\mu} + \frac{\mu}{\lambda} = \left(\frac{r}{l} + \frac{l}{r}\right)\cos(\alpha - \beta) + i\left(\frac{r}{l} - \frac{l}{r}\right)\sin(\alpha - \beta).$$

Thus, $r = l$ or $\alpha - \beta \equiv 0\ (\text{mod}.\pi)$. The case $r = l$ also gives $\alpha - \beta \equiv 0\ (\text{mod}.\pi)$. Hence, the prof is completed. ◻

Proposition 6.5 [14] *Let* $S \in \mathfrak{I}(H)$. *Then the two following properties are equivalent*:

(i) $S \in (\mathbb{C})_1\, \mathbb{S}_0(H)$,

(ii) $\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| \geq 2\,\|X\|$.

Proof The implication (i) ⇒ (ii) follows immediately from Proposition 6.1.

(ii) ⇒ (i). Assume (ii) holds. So, we have

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geq 2\,\|X\|.$$

Using Proposition 5.11, then S is normal. Using the spectral measure of S, there exists a sequence (S_n) of invertible normal operators with finite spectrum such that:

(a) $S_n \to S$ uniformly,

(b) for all $\lambda \in \sigma(S)$, there exists a sequence (λ_n) such that $\lambda_n \in \sigma(S_n)$, for all n and $\lambda_n \to \lambda$.

Let $\lambda,\ \mu \in \sigma(S)$ and let $\epsilon > 0$. Using (ii), (a), and (b), there exists an integer $N \geq 1$ such that

$$\forall n > N,\ \forall X \in \mathfrak{B}(H),\ \left\|S_nXS_n^{-1} + S_n^{-1}XS_n\right\| \geq (2 - \epsilon)\,\|X\|, \tag{6.2}$$

and there exist two sequences (λ_n), (μ_n) such that λ_n, $\mu_n \in \sigma(S_n)$, for all n, and $\lambda_n \to \lambda$, $\mu_n \to \mu$.

Let $n > N$ and since S_n is normal, with finite spectrum, there exist p orthogonal projections $E_1, \ldots, E_p$ in $\mathfrak{B}(H)$ such that $E_i E_j = 0$, if $i \neq j$, $\sum_{i=1}^{p} E_i = I$, $S_n = \sum_{i=}^{p} \alpha_i E_i$, where $\sigma(S_n) = \{\alpha_1, \ldots, \alpha_p\}$, $\alpha_1 = \lambda_n$, $\alpha_2 = \mu_n$.

Then by (6.2), and if we put $A = \begin{bmatrix} 2 & \gamma_n \\ \gamma_n & 2 \end{bmatrix}$, where $\gamma_n = \frac{\lambda_n}{\mu_n} + \frac{\mu_n}{\lambda_n}$, we obtain

$$\forall X \in \mathfrak{B}(\mathbb{C}^2),\ \|A \circ X\| \geq (2 - \epsilon)\, \|X\|. \tag{6.3}$$

If we put $\delta_n = \frac{1}{\gamma_n}$, and $B = \begin{bmatrix} \frac{1}{2} & \delta_n \\ \delta_n & \frac{1}{2} \end{bmatrix}$, then from the last inequality, we also have

$$\forall X \in \mathfrak{B}(\mathbb{C}^2),\ \|B \circ X\| \leq \frac{\|X\|}{(2 - \epsilon)}. \tag{6.4}$$

From (6.3), we deduce $\left|\frac{\lambda_n}{\mu_n} + \frac{\mu_n}{\lambda_n}\right| \geq 2 - \epsilon$. Hence, $\left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right| \geq 2$. Put, $\beta_n = \mathrm{Im}\gamma_n$, $\gamma = \lim \gamma_n$, $\beta = \lim \beta_n$.

On the other hand, if in (6.4), we put $X = \begin{bmatrix} 1 & ia \\ ia & 1 \end{bmatrix}$, for $a > 0$, we obtain

$$\frac{1}{4} + a^2 \left|\gamma_n\right|^2 + a \left|\beta_n\right| \leq \frac{1 + a^2}{(2 - \epsilon)^2}.$$

Hence,

$$\frac{1}{4} + a^2 \left|\gamma\right|^2 + a \left|\beta\right| \leq \frac{1 + a^2}{(2 - \epsilon)^2}.$$

Thus, $a\left|\gamma\right|^2 + |\beta| \leq \frac{a}{4}$, for every $a > 0$. This gives us $\mathrm{Im}\left(\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right) = \beta = 0$. So, from the above lemma, λ and μ belongs to a straight line through the origin. Then there exists $\theta \in [0, \pi)$ such that $\sigma(S) \subset D_\theta$. Therefore, $M = e^{-i\theta} S$ is selfadjoint, and $S = e^{i\theta} M$. □

Remark 6.6 In [5], we find the class of all invertible operators S in $\mathfrak{B}(H)$ satisfying the following inequality (with $k \in (-2, 2]$):

$$\forall X \in \mathfrak{B}(H),\ \|SXS^{-1} + S^{-1}XS + kX\| \geq (k + 2)\|X\|.$$

Note that this problem is still open, and for $k = 0$, we find the previous characterizations (Proposition 6.5).

Corollary 6.7 [18] *Let* $S \in \mathfrak{I}(H)$. *Then the following properties are equivalent*:

(i) $S \in (\mathbb{C})_1\, \mathbb{S}_0(H)$,

(ii) $\forall X \in \mathfrak{B}(H),\ \left\| SXS^{-1} + S^{-1}XS \right\| = \left\| S^*XS^{-1} + S^{-1}XS^* \right\|,$
(iii) $\forall X \in \mathfrak{B}(H),\ \left\| SXS^{-1} + S^{-1}XS \right\| \geq \left\| S^*XS^{-1} + S^{-1}XS^* \right\|.$

Proof The two implications (i) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (iii) are trivial.
The implication (iii) $\Longrightarrow$ (i) follows from Corollary 6.3(iii) and the last proposition. □

Corollary 6.8 *Let $S \in \mathfrak{I}(H)$. Then the two following properties are equivalent*:

(i) $S \in \mathbb{C}^*\mathcal{U}_r(H),$
(ii) $\forall X \in \mathfrak{B}(H),\ \left\| SXS^{-1} + S^{-1}XS \right\| = 2\left\| X \right\|.$

Proof This corollary follows immediately from Propositions 4.16 and 6.5. □

In the next proposition, and from the last proposition concerning the invertible case, we conclude for the characterization of the class $(\mathbb{C})_1\,\mathbb{S}(H)$.

Proposition 6.9 [23] *Let $S \in \mathfrak{B}(H)$. The two following properties are equivalent*:

(i) $S \in (\mathbb{C})_1\,\mathbb{S}(H),$
(ii) $\forall X \in \mathfrak{B}(H),\ \left\| S^2X + XS^2 \right\| \geq 2\left\| SXS \right\|.$

Proof We may assume without loss of generality that $\|S\| = 1$.
(i) $\Rightarrow$ (ii). This implication follows immediately from (2.2).
(ii) $\Rightarrow$ (i). Assume (ii) holds. Then, we have

$$\forall X \in \mathfrak{B}(H),\ \left\| S^2X \right\| + \left\| XS^2 \right\| \geq 2\left\| SXS \right\|.$$

Hence, from Proposition 5.14, S is normal. So, we prove (i) in two cases.
Case 1. $S \in \mathfrak{D}(H)$.
Then, S is invertible, so from (ii), we obtain

$$\forall X \in \mathfrak{B}(H),\ \left\| SXS^{-1} + S^{-1}XS \right\| \geq 2\left\| X \right\|.$$

Using the last proposition, we deduce (i).
Case 2. General situation.
Applying triangular inequality in (ii), we deduce that $\left\| S^2 \right\| = \|S\|^2 = 1$.
Define the real function F on the complete metric space $(\mathfrak{B}(H))_1$ by

$$\forall X \in (\mathfrak{B}(H))_1,\ F(X) = \left\| S^2X + XS^2 \right\| - 2\left\| SXS \right\|,$$

and for $n \geq 1$, define the real function F_n on $(\mathfrak{B}(H))_1$ by

$$\forall X \in \mathfrak{B}(H),\ F_n(X) = \left\| S_n^2X + XS_n^2 \right\| - 2\left\| S_nXS_n \right\|.$$

Put $D = \{X \in (\mathfrak{B}(H))_1 : F(X) > 0\}$. Then there are two cases, $D = \varnothing$, $D \neq \varnothing$.

(1) $D = \varnothing$. So, it follows that

$$\forall X \in \mathfrak{B}(H),\ \left\| S^2 X + X S^2 \right\| = 2\left\| SXS \right\|. \tag{6.5}$$

From this equality, we have

$$\forall x, y \in H,\ \left\| S^2 x \otimes y + x \otimes S^{*2} y \right\| = 2\left\| Sx \right\| \left\| S^* y \right\|.$$

Using this last equality and since $S^2 \neq 0$, we deduce that $\ker S^* = \{0\}$. Hence, S is with dense range. Using again this last equality, we obtain the following inequality:

$$\forall x, y \in (H)_1,\ \left\| S^2 x \right\| + 2\left\| Sx \right\| \left\| S^* y \right\| \geq \left\| S^{*2} y \right\|.$$

By taking the supremum over $y \in (H)_1$, we obtain that $\left\| Sx \right\| \geq \frac{1}{3}\left\| x \right\|$, for every $x \in H$. Thus, S is bounded below with dense range. Hence, S is invertible. So, from (6.5), it follows that

$$\forall X \in \mathfrak{B}(H),\ \left\| SXS^{-1} + S^{-1} X S \right\| = 2\left\| X \right\|.$$

Then from the last proposition, (i) holds.

(2) $D \neq \varnothing$. From the fact that F is a positive continuous map on $(\mathfrak{B}(H))_1$, it follows that

$$\overline{D} = \overline{F^{-1}\left((0, \infty)\right)} = F^{-1}\left([0, \infty)\right) = \{X \in (\mathfrak{B}(H))_1 : F(X) \geq 0\} = (\mathfrak{B}(H))_1.$$

Let $X \in D$, and $\epsilon > 0$. Since $S_n \to S$ uniformly, then there exists an integer $N \geq 1$ (depends only in ϵ) such that

$$\forall n \geq N,\ \forall Y \in (\mathfrak{B}(H))_1,\ |F(Y) - F_n(Y)| \leq \epsilon.$$

Using the same argument as used in Proposition 5.14, it follows that

$$\forall X \in D,\ \forall n \geq N, \quad F_n(X) \geq 0.$$

Since each F_n is a continuous map on $(\mathfrak{B}(H))_1$ and D is dense in $(\mathfrak{B}(H))_1$, then

$$\forall X \in (\mathfrak{B}(H))_1,\ \forall n \geq N,\ F_n(X) \geq 0.$$

So, it follows that

$$\forall X \in \mathfrak{B}(H),\ \forall n \geq N,\ \left\| S_n^2 X + X S_n^2 \right\| \geq 2\left\| S_n X S_n \right\|.$$

Since for each $n \geq 1$, $S_n \in \mathfrak{D}(H)$, using the case 1, we obtain that $S_n \in (\mathbb{C})_1\, \mathbb{S}(H)$, for every $n \geq N$. Since $S_n \to S$ uniformly, and $(\mathbb{C})_1\, \mathbb{S}(H)$ is closed in $\mathfrak{B}(H)$, then $S \in (\mathbb{C})_1\, \mathbb{S}(H)$. □

Corollary 6.10 [21] *Let $S \in \mathcal{R}(H)$. Then the following properties are equivalent*:

(i) $S \in (\mathbb{C})_1 \mathbb{S}_{cr}(H)$,

$$\begin{array}{llcl}
\text{(ii)} & \forall X \in \mathfrak{B}(H), & \left\| SXS^+ + S^+XS \right\| = & \left\| S^*XS^+ + S^+XS^* \right\|, \\
\text{(iii)} & \forall X \in \mathfrak{B}(H), & \left\| SXS^+ + S^+XS \right\| \geq & \left\| S^*XS^+ + S^+XS^* \right\|, \\
\text{(iv)} & \forall X \in \mathfrak{B}(H), & \left\| SXS^+ + S^+XS \right\| \geq & 2\left\| SS^+XS^+S \right\| .,
\end{array}$$

Proof The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are trivial.

The implication (iii) $\Rightarrow$ (iv) follows immediately from Corollary 6.3(iv).

(iv) $\Rightarrow$ (i). Assume (iv) holds. Applying the triangular inequality in (iv), we obtain from Corollary 5.17 that S is normal (with a closed range), so that S is an EP operator satisfying (iv). Then, $S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R(S) \\ \ker S^* \end{bmatrix}$, where S_1 is invertible on $R(S)$. Hence, we obtain the following inequality:

$$\forall X \in \mathfrak{B}(R(S)), \ \left\| S_1XS_1^{-1} + S_1^{-1}XS_1 \right\| \geq 2\left\| X \right\|.$$

Hence, S_1 is a selfadjoint operator in $\mathfrak{B}(R(S))$ multiplied by a nonzero scalar. Thus, $S \in (\mathbb{C})_1 \mathbb{S}_{cr}(H)$. □

Conclusion 6.11 (1) *The class of all invertible selfadjoint operators in $\mathfrak{B}(H)$ multiplied by nonzero scalar is characterized by each of the following properties*:

$$\begin{array}{llcll}
\forall X \in \mathfrak{B}(H), & \left\| SXS^{-1} + S^{-1}XS \right\| & \geq & 2\left\| X \right\|, & (S \in \mathfrak{I}(H)), \\
\forall X \in \mathfrak{B}(H), & \left\| SXS^{-1} + S^{-1}XS \right\| & = & \left\| S^*XS^{-1} + S^{-1}XS^* \right\|, & (S \in \mathfrak{I}(H)), \\
\forall X \in \mathfrak{B}(H), & \left\| SXS^{-1} + S^{-1}XS \right\| & \geq & \left\| S^*XS^{-1} + S^{-1}XS^* \right\|, & (S \in \mathfrak{I}(H)).
\end{array}$$

(2) *The class of all selfadjoint operators with closed ranges in $\mathfrak{B}(H)$ multiplied by scalar is characterized by each of the following properties*:

$$\begin{array}{llcll}
\forall X \in \mathfrak{B}(H), & \left\| SXS^+ + S^+XS \right\| & \geq & 2\left\| SS^+XS^+S \right\|, & (S \in \mathfrak{R}(H)), \\
\forall X \in \mathfrak{B}(H), & \left\| SXS^+ + S^+XS \right\| & = & \left\| S^*XS^+ + S^+XS^* \right\|, & (S \in \mathfrak{R}(H)), \\
\forall X \in \mathfrak{B}(H), & \left\| SXS^+ + S^+XS \right\| & \geq & \left\| S^*XS^+ + S^+XS^* \right\|, & (S \in \mathfrak{R}(H)).
\end{array}$$

(3) *The class of all selfadjoint operators in $\mathfrak{B}(H)$ multiplied by nonzero scalar is characterized by*

$$\forall X \in \mathfrak{B}(H), \ \left\| S^2X + XS^2 \right\| \geq 2\left\| SXS \right\|, \quad (S \in \mathfrak{B}(H)).$$

(4) *The class of all unitary reflection operators in $\mathfrak{B}(H)$ multiplied by nonzero scalars is characterized by*

$$\forall X \in \mathfrak{B}(H), \ \left\| SXS^{-1} + S^{-1}XS \right\| = 2\left\| X \right\|, \ (S \in \mathfrak{I}(H)).$$

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