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# SOME RESULTS RELATED TO THE CORACH-PORTA-RECHT INEQUALITY

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Abstract. Let $L(H)$ be the algebra of all bounded operators on a complex Hilbert space $H$ and let $S$ be an invertible self-adjoint (or skew-symmetric) operator of $L(H)$. Corach-Porta-Recht proved that

$$(*)\qquad \forall X \in L(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| \geq 2\left\|X\right\|.$$

The problem considered here is that of finding (i) some consequences of the Corach-Porta-Recht Inequality; (ii) a necessary condition (resp. necessary and sufficient condition, when $\sigma(P) = \sigma(Q)$) for the invertible positive operators $P, Q$ to satisfy the operator-norm inequality $\left\|PXP^{-1} + Q^{-1}XQ\right\| \geq 2\left\|X\right\|$, for all $X$ in $L(H)$; (iii) a necessary and sufficient condition for the invertible operator $S$ in $L(H)$ to satisfy $(*)$.

## 1. Introduction

All operators considered here are bounded operators on a complex Hilbert space $H$. The collection of operators in $H$ is denoted by $L(H)$.

For $T \in L(H)$, we denote by $\sigma(T), co(\sigma(T)), r(T), W_0(T), \{T\}'$ and $\{T\}''$ the spectrum, the convex hull of the spectrum, the spectral radius, the numerical range, the commutant and the bicommutant of $T$, respectively.

If $A = (a_{ij})$ and $B = (b_{ij})$ are two complex $n \times n$ matrices, then define the Schur (or Schur-Hadamard) product of $A$ and $B$ to be the matrix $A \circ B = (a_{ij}b_{ij})$.

In [1], Corach, Porta, and Recht have proved that for any invertible self-adjoint or skew-symmetric operator $S$, the operator-norm inequality

$$\left\|SXS^{-1} + S^{-1}XS\right\| \geq 2\left\|X\right\|$$

holds for all operators $X$.

It is also clear that, for any invertible operator $S$ and for any two invertible positive operators $P, Q$, we have

**(a)** $0 < \inf_{\|X\|=1} \left\|PXP^{-1} + Q^{-1}XQ\right\| \leq 2,$

**(b)** $0 \leq \inf_{\|X\|=1} \left\|SXS^{-1} + S^{-1}XS\right\| \leq 2.$

It may be seen by the Corach-Porta-Recht Inequality that the infimum in (a) is 2, if $P = Q$; and the infimum in (b) is also 2, for $S$ an invertible self-adjoint operator,

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or more generally, if $S$ is of the form $S = \lambda M$, where $M$ is an invertible self-adjoint operator and $\lambda$ is a nonzero scalar.

The purpose of this paper is the following:

**(1)** In §2, we give the following consequences of the Corach-Porta-Recht Inequality. For all invertible positive commuting operators $P, Q$ and for all operators $X$, we have

**(i)** $\left\|PXP^{-1} + Q^{-1}XQ\right\| \geq 2\left\|X\right\|$, if $\left\|X\right\| = r(X)$,
**(ii)** $\max\{\left\|PXP^{-1} + Q^{-1}XQ\right\|, \left\|PX^{*}P^{-1} + Q^{-1}X^{*}Q\right\|\} \geq 2\left\|X\right\|$,
**(iii)** $\left\|nX + PXP^{-1} + P^{-1}XP\right\| \geq (n+2)\left\|X\right\|$, for $n = 0, 1, 2$.

**(2)** In §3, we show that the infimum in (a) is 2 only if $\{P\}^{'} = \{Q\}^{'}$; on the other hand, if $\sigma(P) = \sigma(Q)$, then the infimum in (a) is 2 if and only if $P = Q$.

**(3)** In §4, we show that the only operators $S$ for which the infimum in (b) is 2 are those of the form $S = \lambda M$, where $M$ is an invertible self-adjoint operator and $\lambda$ is a nonzero scalar.

## 2. Some consequences of the Corach-Porta-Recht Inequality

**Lemma 2.1** ([1])**.** *For an invertible self-adjoint or skew-symmetric operator $S$, we have $\forall X \in L(H) : \left\|SXS^{-1} + S^{-1}XS\right\| \geq 2\left\|X\right\|$.*

**Theorem 2.2.** *For any pair $(P, Q)$ of commuting invertible positive operators and for any $X \in L(H)$ such that $\left\|X\right\| = r(A)$, we have*

$$\left\|PXP^{-1} + Q^{-1}XQ\right\| \geq 2\left\|X\right\|.$$

*Proof.* Let $X \in L(H)$ such that $\left\|X\right\| = r(A)$ and put $Y = P^{\frac{1}{2}}Q^{-\frac{1}{2}}XQ^{\frac{1}{2}}P^{-\frac{1}{2}}$. Then since $P^{\frac{1}{2}}Q^{\frac{1}{2}} = Q^{\frac{1}{2}}P^{\frac{1}{2}}$ is self-adjoint, we have by Lemma 2.1

$$\begin{aligned} \left\|PXP^{-1} + Q^{-1}XQ\right\| &= \left\|(P^{\frac{1}{2}}Q^{\frac{1}{2}})Y(P^{\frac{1}{2}}Q^{\frac{1}{2}})^{-1} + (P^{\frac{1}{2}}Q^{\frac{1}{2}})^{-1}Y(P^{\frac{1}{2}}Q^{\frac{1}{2}})\right\| \\ &\geq 2\left\|Y\right\| \\ &\geq 2r(X) \\ &\geq 2\left\|X\right\|. \end{aligned}$$

□

**Theorem 2.3.** *For any pair $(P, Q)$ satisfying the condition of Theorem 2.2 and for any operator $X$, we have*

$$\max\{\left\|PXP^{-1} + Q^{-1}XQ\right\|, \left\|PX^{*}P^{-1} + Q^{-1}X^{*}Q\right\|\} \geq 2\left\|X\right\|.$$

*Proof.* For $X \in L(H)$, let

$$A = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}, \; B = \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 & X \\ X^{*} & 0 \end{bmatrix}.$$

The pair $(A, B)$ satisfies the condition of Theorem 2.2 and $\left\|Y\right\| = r(Y)$ (since $Y$ is self-adjoint). Then we have

$$\begin{aligned} \left\|AXA^{-1} + B^{-1}XB\right\| &= \left\| \begin{matrix} 0 & PXP^{-1} + Q^{-1}XQ \\ PX^{*}P^{-1} + Q^{-1}X^{*}Q & 0 \end{matrix} \right\| \\ &\geq 2\left\|Y\right\| = 2\left\|X\right\| l, \end{aligned}$$

i.e.

$$\max\left\{\left\|PXP^{-1} + Q^{-1}XQ\right\|, \left\|PX^{*}P^{-1} + Q^{-1}X^{*}Q\right\|\right\} \geq 2\left\|X\right\|.$$

□

**Theorem 2.4.** *For any invertible positive operator $P$, and for $n = 0, 1, 2$, we have*

$$\forall X \in L(H) : \left\| nX + PXP^{-1} + P^{-1}XP \right\| \geq (n+2) \left\| X \right\| . \tag{1}$$

*Proof.* If $n = 0$, (1) follows from Lemma 2.1.

For all $X$, we have

$$\begin{aligned} \left\| 2X + PXP^{-1} + P^{-1}XP \right\| &= \Big\| P^{\frac{1}{2}} \left( P^{\frac{1}{2}} X P^{-\frac{1}{2}} + P^{-\frac{1}{2}} X P^{\frac{1}{2}} \right) P^{-\frac{1}{2}} \\ &\quad + P^{-\frac{1}{2}} \left( P^{\frac{1}{2}} X P^{-\frac{1}{2}} + P^{-\frac{1}{2}} X P^{\frac{1}{2}} \right) P^{\frac{1}{2}} \Big\| \\ &\geq 2 \left\| P^{\frac{1}{2}} X P^{-\frac{1}{2}} + P^{-\frac{1}{2}} X P^{\frac{1}{2}} \right\| \\ &\geq 4 \left\| X \right\| , \end{aligned}$$

that is, (1) is true for $n = 2$.

It follows from the case $n = 2$, that for all $X$, we have

$$\begin{aligned} \left\| X + PXP^{-1} + P^{-1}XP \right\| &\geq \left\| 2X + PXP^{-1} + P^{-1}XP \right\| - \left\| X \right\| \\ &\geq 3 \left\| X \right\| . \end{aligned}$$

□

*Remark* 2.1. In the cases $n = 1$ and $n = 2$, the relation (1) is false in general if we replace the condition “positive” by the condition “self-adjoint”; this may be seen by the following example:

$$P = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

Then

$$\begin{cases} \left\| X + PXP^{-1} + P^{-1}XP \right\| = 1 < 3 = 3 \left\| X \right\| , \\ \left\| 2X + PXP^{-1} + P^{-1}XP \right\| = 0 < 4 = 4 \left\| X \right\| . \end{cases}$$

## 3. Operator-norm inequality and positive operators

**Definition 3.1.** An operator $A$ in $L(H)$ is called convexoid if $W_0(A) = co\sigma(A)$.

**Lemma 3.1** ([2]). *Let $A \in L(H)$. If $\|A - \alpha\| = r(A - \alpha)$, for all complex $\alpha$, then $A$ is convexoid.*

**Lemma 3.2.** *Let $P$ and $Q$ be in $L(H)$ such that $P > 0$ and $Q > 0$. If we have*

$$\forall X \in L(H) : \left\| X \right\| + \left\| PXP^{-1} \right\| \geq 2 \left\| QXQ^{-1} \right\| , \tag{2}$$

*then $\{P\}^{'} \subset \{Q\}^{'}$.*

*Proof.* (i) Let $X$ be self-adjoint such that $PX = XP$, and let $\alpha$ be a complex number. Then, by (2), $\|X - \alpha\| \geq \left\| Q(X - \alpha)Q^{-1} \right\|$, and since $X - \alpha$ is normal, we also have $\left\| Q(X - \alpha)Q^{-1} \right\| \geq \|X - \alpha\|$, so that $\left\| Q(X - \alpha)Q^{-1} \right\| = \|X - \alpha\|$. Then, by Lemma 3.1, we have $W_0(QXQ^{-1}) = co\sigma(X)$, and since $X$ is self-adjoint, we obtain $QXQ^{-1} = Q^{-1}XQ$ , and also $QX = XQ$.

(ii) Now let $X = X_1 + iX_2$, where $X_1 = ReX$ and $X_2 = ImX$, such that $PX = XP$. Then, we have $PX_1 = X_1P$ and $PX_2 = X_2P$; from (i) it follows that $QX_1 = X_1Q$ and $QX_2 = X_2Q$, and also $QX = XQ$; we conclude that $\{P\}^{'} \subset \{Q\}^{'}$ . □

**Theorem 3.3.** *Let $P$ and $Q$ be in $L(H)$ such that $P>0$ and $Q>0$. If we have*

$$\forall X \in L(H): \ \left\|PXP^{-1}\right\| + \left\|Q^{-1}XQ\right\| \geq 2\left\|X\right\|, \tag{3}$$

*then* $\{P\}^{'} = \{Q\}^{'}$.

*Proof.* From (3), we have

$$\forall X \in L(H): \ \left\|X\right\| + \left\|PQXQ^{-1}P^{-1}\right\| \geq 2\left\|QXQ^{-1}\right\|. \tag{4}$$

Let $UM$ be the polar decomposition of $PQ$ ($U$ is unitary and $M=(QP^2Q)^{\frac{1}{2}}$). Then, from (4), we obtain

$$\forall X \in L(H): \ \left\|X\right\| + \left\|MXM^{-1}\right\| \geq 2\left\|QXQ^{-1}\right\|$$

and, by Lemma 3.2, we have $MQ=QM$; then $PQ=QP$.

Now let $X$ be self-adjoint such that $PX=XP$ and let $\alpha$ be a complex number. Therefore, $QXQ^{-1} \in \{P\}^{'}$ and, from (3), we obtain

$$\forall X \in L(H): \ \left\|Q(X-\alpha)Q^{-1}\right\| \leq \left\|X\right\|.$$

It follows that $QX=XQ$ , so that $\{P\}^{'} \subset \{Q\}^{'}$ .

The symmetric roles of $P,Q$ in (3) also give $\{Q\}^{'} \subset \{P\}^{'}$, and finally we have $\{P\}^{'} = \{Q\}^{'}$. □

**Corollary 3.4.** *Let $P$ and $Q$ be in $\mathcal{L}(H)$ such that $P>0$ and $Q>0$. If we have*

$$\forall X \in \mathcal{L}(H): \ \left\|PXP^{-1} + Q^{-1}XQ\right\| \geq 2\left\|X\right\|,$$

*then* $\{P\}^{'} = \{Q\}^{'}$.

*Proof.* Since we have $\left\|PXP^{-1}\right\| + \left\|Q^{-1}XQ\right\| \geq \left\|PXP^{-1}+Q^{-1}XQ\right\|$, for all operators $X$, the result follows immediately by Theorem 3.3. □

**Lemma 3.5.** *Let $\varepsilon>0$ and let $\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n$ (for $n\in\mathbb{N}^*$) such that $0<\alpha_1<\cdots<\alpha_n\leq 1$, $\{\alpha_1,\dots,\alpha_n\}=\{\beta_1,\dots,\beta_n\}$ and $\frac{\alpha_i}{\alpha_j}+\frac{\beta_j}{\beta_i}\geq 2-\varepsilon$, for all $i,j$. Then we have $|\alpha_i-\beta_i|<\varepsilon$, for all $i$.*

*Proof.* From the hypothesis, we obtain $\beta_i-\beta_j<\varepsilon$, if $i<j$.

Let $i\in\{1,...,n\}$ such that $\alpha_i\neq\beta_i$ (in the case $\alpha_i=\beta_i$, of course we have $|\alpha_i-\beta_i|=0<\varepsilon$).

There are three cases $i=1$, $i=n$ and $1<i<n$.

*Case* 1. $i=1$. There exists $j\geq 2$, such that $\beta_j=\alpha_1$, so we have $|\beta_1-\alpha_1|=\beta_1-\beta_j<\varepsilon$, since $j>1$.

*Case* 2. $i=n$. There exists $j<n$, such that $\beta_j=\alpha_n$, so we have $|\beta_n-\alpha_n|=\beta_j-\beta_n<\varepsilon$, since $n>j$.

*Case* 3. $1<i<n$. If $\beta_i>\alpha_i$, then there exists $j>i$, such that $\beta_j\leq\alpha_i$, and we have $|\beta_i-\alpha_i|=\beta_i-\beta_j<\varepsilon$, since $j>i$. If $\beta_i<\alpha_i$, then there exists $j<i$, such that $|\beta_i-\alpha_i|=\beta_j-\beta_i<\varepsilon$, since $i>j$. □

**Theorem 3.6.** *Let $P$ and $Q$ be in $L(H)$ such that $P>0$, $Q>0$ and $\sigma(P)=\sigma(Q)$. Then the following properties are equivalent:*

(i) $\forall X\in L(H), \left\|PXP^{-1}\right\| + \left\|Q^{-1}XQ\right\| \geq 2\left\|X\right\|$.

(ii) $P=Q$.

*Proof.* We may assume, without loss of the generality, that $\|P\| = \|Q\| = 1$.

(i) implies (ii). Decompose $P$ and $Q$ using the spectral measure

$$P = \int \lambda dE_\lambda, \;\; Q = \int \lambda dF_\lambda$$

and consider

$$P_n = \int h_n(\lambda)\, dE_\lambda = h_n(P), \;\; Q_n = \int h_n(\lambda)\, dF_\lambda = h_n(Q)$$

where $h_n(\lambda)$ is a function of the form

$$h_n(\lambda) = \frac{k}{n}, \text{ for } \frac{k}{n} \leq \lambda < \frac{k+1}{n}, \text{ and } k = 0, 1, 2, \dots.$$

Then by the spectral theorem and by the form of $h_n(\lambda)$, we have $\sigma(P_n) = \sigma(Q_n) = h_n(\sigma(P))$ is finite, $P_n \longrightarrow P$, $Q_n \longrightarrow Q$ (uniformly) and $P_n \in \{P\}''$, $Q_n \in \{Q\}''$ (where $\{P\}'' = \{Q\}''$, by Theorem 3.3).

Put $\sigma(P_n) = \{\alpha_1, \dots, \alpha_p\}$ such that $0 < \alpha_1 < \dots < \alpha_p \leq 1$. Then there exist $p$ orthogonal projections $E_1, ..., E_p$ such that $E_iE_j = E_jE_i = 0$ if $i \neq j$, $E_1 \oplus ... \oplus E_p = I$ and $P_n = \sum\limits_{i=1}^{p} \alpha_i E_i$.

Since $\sigma(P_n) = \sigma(Q_n)$, $P_nQ_n = Q_nP_n$ and $Q_n$ is normal, there exist $p$ scalar $\beta_1, ..., \beta_p$ such that $Q_n = \sum\limits_{i=1}^{p} \beta_i E_i$ and $\{\alpha_1, ..., \alpha_p\} = \{\beta_1, ..., \beta_p\}$.

Let $\varepsilon > 0$. Then there exists an integer $N$ such that

$$(*) \qquad \forall n > N,\ \forall X \in L(H),\ \left\|PXP^{-1}\right\| + \left\|Q^{-1}XQ\right\| \geq (2-\varepsilon)\|X\|.$$

Let $n > N$ and $X_{ij} = E_iXE_j$, for $X \in L(H)$. Then, by using $(*)$, we have

$$\frac{\alpha_i}{\alpha_j} + \frac{\beta_j}{\beta_i} \geq 2 - \varepsilon.$$

By Lemma 3.5, this implies $|\alpha_i - \beta_i| < \varepsilon$, for all $i$; therefore

$$\|P_n - Q_n\| = \max_{1 \leq i \leq p} |\alpha_i - \beta_i| < \varepsilon,$$

so we obtain $P = Q$.

(ii) implies (i) is immediate from Lemma 2.1. □

**Corollary 3.7.** *Let the pair $(P, Q)$ of operators satisfy the condition of Theorem 3.6. Then the following properties are equivalent:*

(i) $\forall X \in L(H), \left\|PXP^{-1} + Q^{-1}XQ\right\| \geq 2\|X\|$.
(ii) $P = Q$.

## 4. Characterization of the Corach-Porta-Recht Inequality

*Notation.* For $\theta \in [0, \pi[$, we denote by $D_\theta$ the straight line through the origin in the complex plane.

**Lemma 4.1.** *Let $\lambda, \mu \in \mathbb{C}^*$ such that $\frac{\lambda}{\mu} + \frac{\mu}{\lambda} \in \mathbb{R}$ and $\left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right| \geq 2$. Then there exists a scalar $\theta \in [0, \pi[$ such that $\lambda, \mu \in D_\theta$.*

*Proof.* Let $\lambda = r_1 e^{i\theta_1}$ and $\mu = r_2 e^{i\theta_2}$ be the polar decompositions respectively of $\lambda$ and $\mu$. Then we have

$$\frac{\lambda}{\mu} + \frac{\mu}{\lambda} = \left(\frac{r_1}{r_2} + \frac{r_2}{r_1}\right)\cos(\theta_1 - \theta_2) + i\left(\frac{r_1}{r_2} - \frac{r_2}{r_1}\right)\sin(\theta_1 - \theta_2),$$

so we obtain $r_1 = r_2$ or $\theta_1 - \theta_2 \equiv 0$ (mod.$\pi$).

The case $r_1 = r_2$ also gives $\theta_1 - \theta_2 \equiv 0$ (mod.$\pi$), therefore $\lambda, \mu \in D_\theta$, for some $\theta \in [0, \pi[$. □

**Lemma 4.2.** *All invertible operators $S$ satisfying the condition*

$$\forall X \in L(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| \geq 2\left\|X\right\| \tag{5}$$

*are normal.*

Let $S$ be an invertible operator satisfying (5) and let $UP$, $VQ$ be the polar decompositions respectively of $S$ and $S^*$. Then, by (5), we obtain

$$\forall X \in L(H),\ \left\|PXP^{-1}\right\| + \left\|Q^{-1}XQ\right\| \geq 2\left\|X\right\|.$$

Since $P^2 = S^*S$ and $Q^2 = SS^*$, then $\sigma(P^2) = \sigma(Q^2)$, and by the spectral theorem, we obtain $\sigma(P) = \sigma(Q)$; so we have, by Theorem 3.6, $P = Q$, and also $S^*S = SS^*$. Therefore $S$ is normal.

**Lemma 4.3.** *Let $S$ be an invertible normal operator. Then the following properties are equivalent:*

- **(i)** $\forall X \in L(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| \geq 2\left\|X\right\|$.
- **(ii)** $\sigma(S) \subset D_\theta$, *for some* $\theta \in [0, \pi[$.
- **(iii)** $S = \lambda M$, *for some nonzero scalar $\lambda$ and for some invertible self-adjoint operator $M$.*

*Proof.* (i) implies (ii).

From (i) and Lemma 4.2, $S$ is normal. Then, by the spectral measure of $S$, there exists a sequence $(S_n)$ of invertible normal operators with finite spectrum such that

(a) $S_n \longrightarrow S$ uniformly,

(b) for all $\lambda$ in $\sigma(S)$, there exists a sequence $(\lambda_n)$ such that $\lambda_n \in \sigma(S_n)$, for all $n$ and $\lambda_n \longrightarrow \lambda$.

Let $\lambda, \mu \in \sigma(S)$ and let $\varepsilon > 0$. Then by (i), (a) and (b), there exists an integer $N$ such that

$$\forall n > N, \forall X \in L(H)\quad \left\|S_nXS_n^{-1} + S_n^{-1}XS_n\right\| \geq (2 - \varepsilon)\left\|X\right\| \tag{6}$$

and there exist two sequences $(\lambda_n)$ and $(\mu_n)$ such that

$$\forall n,\ \lambda_n, \mu_n \in \sigma(S_n);\ \ \lambda_n \longrightarrow \lambda,\ \ \mu_n \longrightarrow \mu.$$

Let $n > N$ and since $S_n$ is normal with finite spectrum, there exist $p$ orthogonal projections $E_1, ..., E_p$ such that $E_kE_j = E_jE_k = 0$, if $k \neq j$, $E_1 \oplus ... \oplus E_p = I$ and $S_n = \sum_{k=1}^{p} \alpha_k E_k$, where $\sigma(S_n) = \{\alpha_1, ..., \alpha_p\}$, $\alpha_1 = \lambda_n, \alpha_2 = \mu_n$.

Then by (6) and if we put $A = \begin{bmatrix} 2 & \gamma_n \\ \gamma_n & 2 \end{bmatrix}$, where $\gamma_n = \frac{\lambda_n}{\mu_n} + \frac{\mu_n}{\lambda_n}$, we obtain

$$\forall X \in L(\mathbb{C}^2),\ \left\|A \circ X\right\| \geq (2 - \varepsilon)\left\|X\right\|, \tag{7}$$

and if we put $\delta_n = \frac{1}{\gamma_n}$ and $B = \begin{bmatrix} \frac{1}{2} & \delta_n \\ \delta_n & \frac{1}{2} \end{bmatrix}$, then from (7), we also have

$$\forall X \in L(\mathbb{C}^2),\ \|B \circ X\| \leq \frac{\|X\|}{(2-\varepsilon)}. \tag{8}$$

From (7), we deduce $\left|\frac{\lambda_n}{\mu_n} + \frac{\mu_n}{\lambda_n}\right| \geq (2-\varepsilon)$, so we obtain $\left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right| \geq 2$.

On the other hand, if in (8) we put $X = \left[\begin{smallmatrix} 1 & ia \\ ia & 1 \end{smallmatrix}\right]$, where $a > 0$, we obtain $\frac{1}{4} + a^2 |\gamma_n|^2 + a|\beta_n| \leq \frac{1+a^2}{(2-\varepsilon)^2}$, where $\beta_n = Im\,\gamma_n$; so that $\frac{1}{4} + a^2|\alpha|^2 + a|\beta| \leq \frac{1+a^2}{(2-\varepsilon)^2}$. Therefore $a|\alpha|^2 + |\beta| \leq \frac{a}{4}$; then $\beta = 0$ and $\frac{\lambda}{\mu} + \frac{\mu}{\lambda} \in \mathbb{R}$. This implies condition (ii) by Lemma 4.1.

(ii) implies (iii).

If we put $M = e^{-i\theta}S$, then $M$ is an invertible normal operator with real spectrum, so we have $S = e^{i\theta}M$, where $M$ is an invertible self-adjoint operator.

(iii) implies (i) is immediate by Lemma 2.1. □

**Theorem 4.4.** *The set of all invertible operators $S$, for which*

$$\forall X \in L(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| \geq 2\|X\|$$

*is the set* $\{\lambda M \,:\, \lambda \in \mathbb{C}^*,\ M \text{ an invertible self-adjoint operator}\}$.

*Proof.* This follows immediately by Lemma 4.2 and Lemma 4.3. □

*Remark* 4.1. The extremal class of invertible operators $S$ satisfying the condition

$$\inf_{\|X\|=1} \left\|SXS^{-1} + S^{-1}XS\right\| = 2$$

has been characterized. So it remains the characterization of the second extremal class of all invertible operators $S$ satisfies the condition

$$\inf_{\|X\|=1} \left\|SXS^{-1} + S^{-1}XS\right\| = 0.$$

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