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Integral Equations and Operator Theory

DERIVATION AND JORDAN OPERATORS

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For $A \in \mathcal{L}(H)$ (the algebra of all operators on the complex Hilbert space H), let δ_A denote the operator on $\mathcal{L}(H)$ defined by : $\delta_A(X) = AX - XA$.

We show here that for all Jordan operators A : $R(\delta_A) \cap \{A^*\}' = \{0\}$, where $R(\delta_A)$ is the range of δ_A and $\{A^*\}'$ is the commutant of the adjoint of A.

Introduction.

Let $\mathcal{L}(H)$ be the algebra of bounded linear operators on the infinite-dimensional complex Hilbert space H. For $A \in \mathcal{L}(H)$, we define the linear operator δ_A on $\mathcal{L}(H)$ by :

$$\forall X \in \mathcal{L}(H) \quad \delta_A(X) = AX - XA;$$

we denote $R(\delta_A)$, $R(\delta_A)^-$ and $\{A\}'$ respectively the range, the norm closure of the range and the kernel of δ_A.

We denote $\mathcal{N} = \{A \in \mathcal{L}(H) : R(\delta_A)^- \cap \{A^*\}' = \{0\}\}$.

If H is finite-dimensional, $\mathcal{N} = \mathcal{L}(H)$. If H is infinite-dimensional, this equality does not hold. So a reasonable purpose is to determine what elements are in $\mathcal{N}$.

When H is a separable Hilbert space, $\mathcal{N}$ contains the operators A for which $p(A)$ is normal for some quadratic polynomial $p(z)$ [2],the subnormal operators with cyclic vectors [2] and the isometries [3]. In this paper, we show that $\mathcal{N}$ contains also all the operators unitarily equivalent to Jordan operators.

Notation : see [4].

For a complex Hilbert space H and an integer n strictly greater than 1, an operator A on $H^{(n)} = \underbrace{H \oplus H \oplus \ldots \oplus H}_{n\ times}$, with matrix $[A_{i,j}]_{1 \leq i,j \leq n}$, ($i$ is the row index), is said to be a Jordan block of order n if we have, for all $i \in \{1, \ldots, n-1\}$, $A_{i,i+1} = I_H$ (where I_H is the identity operator on H), and $A_{i,j} = 0$ in the other cases.

We denote by 0_H, the null-operator defined on H, the Jordan block of order 1.

Let m be a strictly positive integer, $H_1, \ldots, H_m$ m complex Hilbert spaces and $\alpha_1, \ldots \alpha_m$ m strictly positive integers.

Set $\widetilde{H}_k = H_k^{(\alpha_k)}$ and J_k the Jordan block of order α_k operating on $\widetilde{H}_k$, for $k = 1, .., m$.

Every operator of the form $J_1 \oplus ... \oplus J_m$ operating on $\widetilde{H}_1 \oplus ... \oplus \widetilde{H}_m$ is called a Jordan operator of order $sup\{\alpha_k : k = 1, .., m\}$.

For two complex Hilbert spaces H and K, and $A \in \mathcal{L}(H)$, $B \in \mathcal{L}(K)$, we denote by $\delta_{A,B}$ the linear operator defined on $\mathcal{L}(K, H)$ (the space of bounded linear operators defined from K into H) by :

$$\forall\, X \in \mathcal{L}(K, H), \quad \delta_{A,B}(X) = AX - XB.$$

Recall that for every operator $A \in \mathcal{L}(H)$, similar to a Jordan operator, $R(\delta_A)$ is closed [1].

Lemma 1 . *Let $A, B \in \mathcal{L}(H)$ with B unitarily equivalent to A and $A \in \mathcal{N}$. Then we have $B \in \mathcal{N}$.*

Proof . Let $A, B \in \mathcal{L}(H)$ such that $A \in \mathcal{N}$ and such that there exists a unitary operator $U \in \mathcal{L}(H)$ verifying $B = U^*AU$.
If $C^* \in R(\delta_B)^- \cap \{B^*\}'$, there exists $(X_n)_{n\in\mathbb{N}^*} \subset \mathcal{L}(H)$ such that :

$$C^* = lim_n(BX_n - X_nB) \quad \text{and} \quad BC = CB.$$

So we have :

$$C^* = lim_n(U^*AUX_n - X_nU^*AU) \quad \text{and} \quad U^*AUC = CU^*AU.$$

Let us operate on the left with U and on the right with U^* the two members of these last equalities, we obtain :

$$UC^*U^* = lim_n(A(UX_nU^*) - (UX_nU^*)A) \quad \text{and} \quad A(UCU^*) = (UCU^*)A,$$

hence $UC^*U^* \in R(\delta_A)^- \cap \{A^*\}'$; and taking into account that $A \in \mathcal{N}$, we deduce that $C = 0$; so $B \in \mathcal{N}$.

Lemma 2 . *Let H, K be two complex Hilbert spaces and n, m two strictly positive integers. For all $A \in \mathcal{L}(H^{(n)})$, Jordan block of order n, and for all $B \in \mathcal{L}(K^{(m)})$, Jordan block of order m , we have :*

$$R(\delta_{A,B}) \cap ker(\delta_{A^*,B^*}) = \{0\}.$$

Proof . We consider the two cases : $m \leq n$ and $n < m$.

Case 1 : $m \leq n$.
We have : $A = [A_{i,j}]_{1\leq i,j\leq n}$, $B = [B_{\alpha,\beta}]_{1\leq\alpha,\beta\leq m}$
with $A_{i,i+1} = I_H$, for $i = 1, ..., n-1$; $B_{\alpha,\alpha+1} = I_K$, for $\alpha = 1, ..., m-1$;
and $A_{i,j} = 0_H$, $B_{\alpha,\beta} = 0_K$ in the other cases.
Let $C^* \in R(\delta_{A,B}) \cap ker(\delta_{A^*,B^*})$; there exists $X \in \mathcal{L}(K^{(m)}, H^{(n)})$ such that :

$$C^* = AX - XB \quad \text{and} \quad CA = BC.$$

We denote by $[X_{i,\alpha}\ ]$ and $[C_{\alpha,j}\ ]$ the matrices of X and C respectively.
Then we have three possible cases :
i) $n = m = 1$.This case is trivial because $A = 0_H$ and $B = 0_K$.
ii) $n \geq 2$ and $m = 1$. We have $B = 0_K$, $C^* = AX$ and $CA = 0$. Then $CC^* = 0$, so $C = 0$ and $R(\delta_{A,B}) \cap ker(\delta_{A^*,B^*}) = \{0\}$.
iii) $n \geq 2$ and $m \geq 2$. For all $i \in \{1, ..., n\}$ and for all $\alpha \in \{1, ..., m\}$ we can write :

$$(I) \qquad \left\{ \begin{array}{rcl} C^*_{\alpha,i} & = & \sum_{j=1}^{n} A_{i,j} X_{j,\alpha} \ - \ \sum_{\beta=1}^{m} X_{i,\beta} B_{\beta,\alpha} \\ \sum_{j=1}^{n} C_{\alpha,j} A_{j,i} & = & \sum_{\beta=1}^{m} B_{\alpha,\beta} C_{\beta,i} \end{array} \right. .$$

Using the first line of (I), we write :

$$(II) \quad \left\{ \begin{array}{lcccc} \text{For } 1 \leq i \leq n-1 \ \text{ and } \ 2 \leq \alpha \leq m : C^*_{\alpha,i} & = & X_{i+1,\alpha} & - & X_{i,\alpha-1} \\ \text{For } i = n \ \text{ and } \ 2 \leq \alpha \leq m : \qquad C^*_{\alpha,n} & = & -X_{n,\alpha-1} & & \\ \text{For } 1 \leq i \leq n-1 \ \text{ et } \ \alpha = 1 : \qquad C^*_{1,i} & = & X_{i+1,1} & & \\ \text{For } i = n \ \text{ and } \ \alpha = 1 : \qquad\qquad C^*_{1,n} & = & 0 & & \end{array} \right.$$

At last using the second line of (I), we have :

$$(III) \qquad \left\{ \begin{array}{lcc} \text{For } 2 \leq i \leq n \ \text{ and } \ 1 \leq \alpha \leq m-1 : C_{\alpha,i-1} & = & C_{\alpha+1,i} \\ \text{For } 2 \leq i \leq n \ \text{ and } \ \alpha = m : \qquad C_{m,i-1} & = & 0 \\ \text{For } i = 1 \ \text{ et } \ 1 \leq \alpha \leq m-1 : \qquad C_{\alpha+1,1} & = & 0 \end{array} \right. .$$

The first line of (III) means that for all $i \in \{1, ..., n\}$ and for all $\alpha \in \{1, .., m\}$, all the terms of the diagonal of the matrix $[C_{\beta,j}\ ]$ containing $C_{\alpha,i}$ are equal. We now turn our attention on the position of the first term of each diagonal.

If the first term is on the first column but not on the first row, it is a $C_{\alpha,1}, 2 \leq \alpha \leq m$, and using the third line of (III), we find that it is zero. So, if we call null-diagonal every diagonal whose all the terms are null, all the diagonals under the diagonal containing $C_{1,1}$ are null-diagonals .
With the notations $C_{1,j} = C_j$ for $j \in \{1, ..., n-1\}$, the matrix $[C_{\beta,j}\ ]$ is

$$[C_{\beta}, j\ ] = \begin{pmatrix} C_1 & C_2 & \cdots & C_{m-1} & C_m & C_{m+1} & \cdots & C_{n-1} & 0 \\ 0 & C_1 & C_2 & \cdots & C_{m-1} & C_m & \cdots & \cdots & C_{n-1} \\ \vdots & 0 & \ddots & \ddots & \cdots & \ddots & \ddots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \cdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & C_1 & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

If the first term is on the first row, it is a $C_{1,i}, i \in \{1, .., n-1\}$. So we have :

$$(IV) \qquad [C_{1,i} = C_{2,i+1} = ... = C_{\alpha-1,j-1} = C_{\alpha,j}],$$

where $C_{\alpha,j}$ denotes the last term of the diagonal containing $C_{1,i}$.
a) If $j \in \{m, ..., n-1\}$ we have $\alpha = m$, the last term of the diagonal is on the last row but not on the last column; so by using the first line of (III) we obtain $C_{m,j} = 0$; then the diagonal containing $C_{1,i}$ is null.
b) If $j = n$, we have $\alpha \in \{2, ..., m\}$, the last term of the diagonal is on the last column; we use the adjoint in (IV) and use (II), so we obtain :

$$C^*_{1,i} = X_{i+1,1} = X_{i+2,2} - X_{i+1,1} = ... = X_{n,\alpha-1} - X_{n-1,\alpha-2} = -X_{n,\alpha-1}.$$

Remark that the sum of all these equal terms is zero, so each term is zero.
From a) and b),we deduce that all the diagonals of $[C_{\beta,j}]$ are null-diagonals.
This ends the proof. We have proved that $C = 0$ and $R(\delta_{A,B}) \cap ker(\delta_{A^*,B^*}) = \{0\}$.

Case 2 : $n \leq m$.
Let $C^* \in R(\delta_{A,B}) \cap ker(\delta_{A^*,B^*})$.
We have $C \in R(\delta_{B^*,A^*}) \cap ker(\delta_{B,A})$, and using the result of the case 1, we obtain $C = 0$ and we are done. We have obtained $R(\delta_{A,B}) \cap ker(\delta_{A^*,B^*}) = \{0\}$.

Theorem 3 . *Let A be a Jordan operator (of any order). Then we have :*

$$R(\delta_A) \cap \{A^*\}' = \{0\}.$$

Proof . With the notations precised at the beginning we write

$$A = J_1 \oplus \cdots \oplus J_m.$$

Let $B^* \in R(\delta_A) \cap \{A^*\}'$ and $[B_{\alpha,\beta}]$ the matrix associated to B.
There exists $X = [X_{\alpha,\beta}]$ such that :

$$B^* = AX - XA \quad \text{and} \quad AB = BA.$$

So we have for all $\alpha \in \{1, ..., m\}$ and for all $\beta \in \{1, ..., m\}$,

$$\begin{cases} B^*_{\beta,\alpha} = J_\alpha X_{\alpha,\beta} - X_{\alpha,\beta} J_\beta \\ J_\beta B_{\beta,\alpha} = B_{\beta,\alpha} J_\alpha \end{cases}$$

Then : $B^*_{\beta,\alpha} \in R(\delta_{J_\alpha,J_\beta}) \cap ker(\delta_{J^*_\alpha,J^*_\beta})$,
and since the J_γ $(\gamma = 1, ..., m)$ are Jordan blocks, using lemma 2, we have $B_{\beta,\alpha} = 0$, for all $\alpha, \beta \in \{1, ..., m\}$; then $B = 0$, and $R(\delta_A) \cap \{A^*\}' = \{0\}$.

Corollary 4 . *The class $\mathcal{N}$ contains all the operators unitarily equivalent to Jordan operators.*

Proof . $R(\delta_A)$ is closed for each operator A similar to a Jordan operator, so this corollary follows immediatly from theorem 3 and lemma 1.

This result induces the next question :

Question : *Are all the operators similar to Jordan operators in the class $\mathcal{N}$? (or equivalently, using the equivalence in* [1] , *are the nilpotent operators A such that $R(\delta_A)$ is closed in the class $\mathcal{N}$?)*

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