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Integral Equations and Operator Theory

THE NUMERICAL RANGE OF ELEMENTARY OPERATORS

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For n-tuples $A = (A_1, ..., A_n)$ and $B = (B_1, ..., B_n)$ of operators on a Hilbert space H, let $R_{A,B}$ denote the operator on $L(H)$ defined by $R_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i$. In this paper we prove that

$$co\left\{\sum_{i=1}^{n} \alpha_i\beta_i : (\alpha_1, ..., \alpha_n) \in W(A), (\beta_1, ..., \beta_n) \in W(B)\right\}^{-} \subset W_0(R_{A,B})$$

where W is the joint spatial numerical range and W_0 is the numerical range. We will show also that this inclusion becomes an equality when $R_{A,B}$ is taken to be a generalized derivation, and it is strict when $R_{A,B}$ is taken to be an elementary multiplication operator induced by non scalar self-adjoints operators.

Introduction.

All operators considered here are bounded operators on a complex Hilbert space H. The collection of operators in H is denoted by $L(H)$.
We denote by tr the trace map on the Banach space $(\mathcal{C}_1(H), \|.\|_1)$ of operators of class trace on H; and if $M \subset C$, we denote by M^- and coM respectively the closure and the convex hull of M.
If $\mathcal{A}$ is a complex unital Banach algebra and $A \in \mathcal{A}$, we denote by $W_0(A)$, the numerical range of A given by:

$$W_0(A) = \{f(A) : f \in \mathcal{P}(\mathcal{A})\}$$

where $\mathcal{P}(\mathcal{A}) = \left\{f \in \mathcal{A}' : f(I) = \|f\| = 1\right\}$ is the set of all states on $\mathcal{A}$. It is known that $W_0(A)$ is convex and compact, this result follows at once from the corresponding properties of the set of states. A is called Hermitian if $W_0(A)$ is real.
If $\mathcal{A} = L(H)$, then $W_0(A)$ is the closure of the usual numerical range $W(A)$ of A, where $W(A) = \{\langle Ax, x\rangle : x \in H, \|x\| = 1\}$, this result follows immediately from [2] and [6].
For more details, see [3] and [8].
For n-tuples $A = (A_1, ..., A_n)$ and $B = (B_1, ..., B_n)$ of operators on H, we define:

(i) the joint spatial numerical range of A (see [5]) by:

$$W(A_1, ..., A_n) = \{\langle A_1x, x\rangle, ..., \langle A_nx, x\rangle : x \in H, \|x\| = 1\}$$

(ii) the elementary operator $R_{A,B} : L(H) \longrightarrow L(H)$ by:

$$\forall X \in L(H) : R_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i$$

For $A, B \in L(H)$, we also define the particular elementary operator:
(iii) the left multiplication operator $L_A : L(H) \longrightarrow L(H)$ by:

$$\forall X \in L(H) : L_A(X) = AX$$

(iv) the right multiplication operator $R_B : L(H) \longrightarrow L(H)$ by:

$$\forall X \in L(H) : R_B(X) = XB$$

(v) the generalized derivation $\delta_{A,B} = L_A - R_B$ induced by A, B.
(vi) the inner derivation $\delta_A = L_A - R_A$ induced by A.
(vii) the elementary multiplication operator $\mathcal{M}(A, B) = L_A R_B$ induced by A, B.
For $x, y \in L(H)$, we define the operator $(x \otimes y)$ on H by:

$$\forall z \in H : (x \otimes y)(z) = \langle z, y \rangle x$$

Curto [4] proved that, if A and B are n-tuples of commuting operators on H then:

$$\sigma(R_{A,B}) = \left\{ \sum_{i=1}^{n} \alpha_i \beta_i : (\alpha_1, ..., \alpha_n) \in \sigma_T(A), (\beta_1, ..., \beta_n) \in \sigma_T(B) \right\}$$

where σ_T is the joint spectrum (the spectrum in the sense of J. L. Taylor, see [9]).
In this note, we give a similar result with the numerical range and the joint spatial numerical range without the assumption of the commutativity, more precisely we will show that:

$$co \left\{ \sum_{i=1}^{n} \alpha_i \beta_i : (\alpha_1, ..., \alpha_n) \in W(A), (\beta_1, ..., \beta_n) \in W(B) \right\}^{-} \subset W_0(R_{A,B})$$

and we obtain that this inclusion becomes an equality when $R_{A,B}$ is taken to be a derivation, and it is strict when $R_{A,B}$ is taken to be an elementary multiplication operator induced by non scalar self-adjoints operators.

Theorem 1 . *Let $A = (A_1, ..., A_n)$ and $B = (B_1, ..., B_n)$ be n-tuples of operators on H. Then we have:*

$$co \left\{ \sum_{i=1}^{n} \alpha_i \beta_i : (\alpha_1, ..., \alpha_n) \in W(A), (\beta_1, ..., \beta_n) \in W(B) \right\}^{-} \subset W_0(R_{A,B})$$

Proof. The proof of this Theorem is based on the construction of a special state on $L(L(H))$, we perform as follows: for $x, y \in H$ such that:$\|x\| = \|y\| = 1$, we define the linear functional f on $L(L(H))$ by:

$$\forall F \in L(L(H)) : \quad f(F) = tr\left[(y \otimes x) F (x \otimes y)\right]$$

Since $\|x \otimes y\|_1 = \|x \otimes y\| = \|x\| . \|y\| = 1$, then:

$$\begin{aligned} |f(F)| &\leq \|(y \otimes x) F(x \otimes y)\|_1 \\ &\leq \|y \otimes x\|_1 \times \|F(x \otimes y)\| \\ &\leq \|F(x \otimes y)\| \\ &\leq \|F\| \end{aligned}$$

since $f(I) = 1$, then we have : $\|f\| = f(I) = 1$; therefore f is a state on $L(L(H))$,and since we have for $i = 1, ..., n$:

$$\begin{aligned} f(L_{A_i} R_{B_i}) &= tr\left[(y \otimes x) L_{A_i} R_{B_i} (x \otimes y)\right] \\ &= tr\left[(y \otimes x)(A_i x \otimes B_i^* y)\right] \\ &= tr\left[\langle A_i x, x\rangle (y \otimes B_i^* y)\right] \\ &= \langle A_i x, x\rangle \langle B_i y, y\rangle \end{aligned}$$

so we obtain :

$$f(R_{A,B}) = \sum_{i=1}^{n} \langle A_i x, x\rangle \langle B_i y, y\rangle$$

then:

$$\left\{\sum_{i=1}^{n} \alpha_i \beta_i : (\alpha_1, ..., \alpha_n) \in W(A), (\beta_1, ..., \beta_n) \in W(B)\right\} \subset W_0(R_{A,B})$$

and since $W_0(R_{A,B})$ is compact and convex, we deduce:

$$co\left\{\sum_{i=1}^{n} \alpha_i \beta_i : (\alpha_1, ..., \alpha_n) \in W(A), (\beta_1, ..., \beta_n) \in W(B)\right\}^{-} \subset W_0(R_{A,B})$$

Corollary 2 . *Let $A, B \in L(H)$. Then we have $W_0(\delta_{A,B}) = W_0(A) - W_0(B)$.*

Proof. By Theorem 1, we obtain $W(A) - W(B) \subset W_0(\delta_{A,B})$, and since $W_0(\delta_{A,B})$ is closed, then we have $(W(A) - W(B))^{-} = W_0(A) - W_0(B) \subset W_0(\delta_{A,B})$, and on the other hand we obtain by [1, Theorem 5.2] that $W_0(\delta_{A,B}) \subset W_0(L_A) - W_0(R_B) = W_0(A) - W_0(B)$, so we obtain the equality.

Remark 1 . *This corollary shows that the inclusion of the Theorem 1 is in fact an equality when the elementary operator is taken to be a generalized derivation. Also this corollary is a generalization of [1, Theorem 5.2], where Anderson and Foias proved that $W_0(L_A) = W_0(R_A) = W_0(A)$, for any A in $L(H)$. Note that the proof of the $W_0(\delta_{A,B}) \subset W_0(L_A) - W_0(R_B)$ can be obtained only by using the immediate inclusions $W_0(L_A) \subset W_0(A)$, $W_0(R_B) \subset W_0(B)$. The converse inclusion $W_0(A) - W_0(B) \subset W_0(\delta_{A,B})$ is obtained only by using Theorem 1 and the elementary properties of the numerical range. Therefore, our proof makes no appeal to [1, Theorem 5.2] (the Anderson and Foias theorem makes appeal to Hahn-Banach theorem).*

Theorem 3 . *Let A, B are two non zero operators in $L(H)$. Then $\mathcal{M}(A, B)$ is a Hermitian operator if and only if there exist $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that one of the operators $e^{-i\theta}A$, $e^{i\theta}B$ is self-adjoint and the other is real scalar.*

The proof of this theorem is based on the following Lemmas.

Lemma 4 (7) . *Let $F \in L(L(H))$. Then F is Hermitian if and only if there exist two self-adjoints operators A, B in $L(H)$ such that $F = L_A + \delta_B$.*

Lemma 5 . *Let S, T be a non zero self-adjoints operators in $L(H)$. Then the operator $F = \mathcal{M}(S,T)$ is Hermitian if and only if one of the operators S , T is a scalar.*

Proof. Assume that F Hermitian. By Lemma 4, we obtain $F - L_{F(I)}$ is a derivation induced by a self-adjoint operator, and since $F(I) = ST$, then:

$$\forall X \in L(H): \; SX^*T - STX^* = X^*TS - TX^*S,$$

so that:

$$\forall X \in L(H): \; S(TX - XT) - (TX - XT)S = 0,$$

Hence $\delta_S.\delta_T = \delta_I$, and finally, by using [10], one of the operators S , T is scalar. On the other hand if one of the operators S , T is scalar, then by [1, Theorem 5.2] F is Hermitian.

Lemma 6 . *Let Γ, Ω be two subsets of C such that $\Gamma \neq \{0\}, \Omega \neq \{0\}$ and $\Gamma \times \Omega$ is real (where $\Gamma \times \Omega = \{\gamma\omega : \gamma \in \Gamma, \omega \in \Omega\}$). Then there exist $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\Gamma \subset D_\theta$, $\Omega \subset D_{-\theta}$ (where $D_\alpha = \{ke^{i\alpha} : k \in R\}$, for α in $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right]$).*

Proof. We can choose γ in $\Gamma - \{0\}$, and then we can write $\gamma = ke^{i\theta}$, for some non zero real k and some θ in $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then $kre^{i(\alpha+\theta)}$ is real, for all element $re^{i\alpha}$ in Ω; then $\alpha = -\theta \; (mod\pi)$, so we obtain $\Omega \subset D_{-\theta}$; likewise we obtain also $\Gamma \subset D_\theta$.

Proof of Theorem 3. Assume that $\mathcal{M}(A,B)$ is Hermitian. By Theorem 1 and since $W_0(\mathcal{M}(A,B))$ is real, then $W_0(A) \times W_0(B)$ is real, so that by Lemma 6, $W_0(A) \subset D_\theta$, $W_0(B) \subset D_{(-\theta)}$ for some θ in $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right]$. It follows that $W_0(e^{-i\theta}A)$ and $W_0(e^{i\theta}B)$ are real, this give that $e^{-i\theta}A$ and $e^{i\theta}B$ are self-adjoints and since $L_AR_B = L_{e^{-i\theta}A}R_{e^{i\theta}B}$, by Lemma 5, we obtain that one of the operators $e^{-i\theta}A$, $e^{i\theta}B$ is self-adjoint and the other is real scalar. The converse implication is trivial.

Corollary 7 . *For $A, B \in L(H)$, we have $co(W_0(A) \times W_0(B)) \subset W_0(\mathcal{M}(A,B))$, and this inclusion is strict if A and B are non scalar self-adjoints operators.*

Proof. The proof is immediate by Theorem 1 and Theorem 3.

Remark 2 . *(1)The Corollary 7 shows that the inclusion of Theorem 1 may be stritct when the elementary operator is taken to be an elementary multiplication operator induced by non scalar self-adjoints operators.*
(2) Note that Anderson and Foias [1, Theorem 5.8] proved that if $P^2 = P = P^* \in L(H)$, then L_PR_P is Hermitian if and only if P is 0 or I. Therefore we may say that Theorem 3 is a generalization of [1, Theorem 5.8].

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