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On some operator norm inequalities

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Abstract

Let $\mathscr{B}(H)$ be the C^*-algebra of all bounded linear operators on a complex Hilbert space H, S be an invertible and selfadjoint operator in $\mathscr{B}(H)$ and let $(I, \|.\|_I)$ denote a norm ideal of $\mathscr{B}(H)$. In this note, we shall show the following inequality:

$$\forall X \in I : \|SXS^{-1} - S^{-1}XS\|_I \leqslant (\|S\|\|S^{-1}\| - 1)\|SXS^{-1} + S^{-1}XS\|_I.$$

Keywords: Hilbert space; Norm ideal; Operator norm inequality; Selfadjoint operator

1. Introduction

Let $\mathscr{B}(H)$ be the C^*-algebra of all bounded linear operators on a complex Hilbert space H and let $\|.\|$ denote the usual norm on $\mathscr{B}(H)$. In their work on the geometry of the space of selfadjoint invertible elements of a C^*-algebra, Corach–Porta–Recht proved in [1] that if S is invertible and selfadjoint in $\mathscr{B}(H)$, then

$$\forall X \in \mathscr{B}(H) : \|SXS^{-1} + S^{-1}XS\| \geqslant 2\|X\|. \tag{1}$$

In [2], Kittaneh obtained the inequality (1) in any norm ideal of $\mathscr{B}(H)$.

Note that a norm ideal of $\mathscr{B}(H)$ is a two-sided ideal I of $\mathscr{B}(H)$ associated with a norm $\|.\|_I$ such that

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(i) I is a Banach space under the norm $\|.\|_I$,
(ii) $\|X\| = \|X\|_I$ for every rank one operator X in $\mathscr{B}(H)$,
(iii) $\|AXB\|_I \leqslant \|A\|\|X\|_I\|B\|$ for every $A, B \in \mathscr{B}(H)$, $X \in I$.

So the general version of Corach–Porta–Recht inequality which was given by Kittaneh is the following inequality:

$$\forall X \in I : \|SXS^{-1} + S^{-1}XS\|_I \geqslant 2\|X\|_I. \tag{2}$$

Examples of norm ideals are the Schatten p-ideals $\mathfrak{C}_p$ of $\mathscr{B}(H)$ associated with the p-norms $\|.\|_p$ $(1 \leqslant p \leqslant \infty)$ and $\mathscr{B}(H)$ associated with the usual norm.

Recently in [5], we have obtained that the set of all invertible operators S in $\mathscr{B}(H)$ satisfying the inequality (1) is given by

$$\{\lambda M : \lambda \in \mathbb{C}^*,\ M \text{ is an invertible and selfadjoint operator in } \mathscr{B}(H)\}.$$

Corach–Porta–Recht have mentioned in [1, Remark 2] that the (usual) norm of $SXS^{-1} + S^{-1}XS$ is in general unrelated to the (usual) norm of $SXS^{-1} - S^{-1}XS$, for $S, X \in \mathscr{B}(H)$ such that S is invertible and selfadjoint.

From the work of McIntoch [3], Kittaneh deduced immediately in [2] that for $1 < p < \infty$, there exists a constant $\gamma_p > 0$ such that

$$\forall X \in \mathfrak{C}_p : \|SXS^{-1} - S^{-1}XS\|_p \leqslant \gamma_p\|SXS^{-1} + S^{-1}XS\|_p, \tag{3}$$

where S is invertible and selfadjoint in $\mathscr{B}(H)$.

Here, we shall prove that for any norm ideal $(I, \|.\|_I)$ of $\mathscr{B}(H)$, we have

$$\forall X \in I : \|SXS^{-1} - S^{-1}XS\|_I \leqslant (\|S\|\|S^{-1}\| - 1)\|SXS^{-1} + S^{-1}XS\|_I, \tag{4}$$

where S is invertible and selfadjoint in $\mathscr{B}(H)$.

Note that Kittaneh has obtained the inequality (3) in a particular norm ideal where the implicit constant γ_p depends only on the p-norm and not on the operator S. But the inequality (4) is obtained in any norm ideal where the explicit constant $c_S = \|S\|\|S^{-1}\| - 1$ depends only on the operator S and not on the given unitarily invariant norm.

In this note, we consider $(I, \|.\|_I)$ to be a norm ideal of $\mathscr{B}(H)$ and $S \in \mathscr{B}(H)$ denotes an invertible and selfadjoint operator.

For $A, B \in \mathscr{B}(H)$, define the two operators $U_{I,A,B}$ and $V_{I,A,B}$ on I by

$$\begin{cases} U_{I,A,B}(X) = AXB + BXA, \\ V_{I,A,B}(X) = AXB - BXA. \end{cases}$$

We denote by $\Phi_{I,S}$ and $\Psi_{I,S}$ the operators $U_{I,S,S^{-1}}$ and $V_{I,S,S^{-1}}$ respectively. It is clear that $U_{I,A,B}, V_{I,A,B}, \Phi_{I,S}, \Psi_{I,S} \in \mathscr{B}(I)$.

In the case, where I is the Schatten p-ideal ($1 \leqslant p \leqslant \infty$), we denote $\Phi_{I,S}$ and $\Psi_{I,S}$ by $\Phi_{p,S}$ and $\Psi_{p,S}$ respectively.

More recently in [4], we are interested to know whether there exists a uniform lower bound for the operator $U_{I,A,B}$. We have obtained that $\|U_{I,A,B}\| \geqslant 2(\sqrt{2} - 1)\|A\|\|B\|$, for any $A, B \in \mathscr{B}(H)$, and $\|\Phi_{I,S}\| \geqslant \|S\|\|S^{-1}\| + (1/\|S\|\|S^{-1}\|)$.

In this note, we shall give an alternative proof for the last estimation and we shall also give a similar estimation for $\Psi_{I,S}$. Precisely, we find that $\|\Psi_{I,S}\| \geqslant \|S\|\|S^{-1}\| - (1/\|S\|\|S^{-1}\|)$. For the upper estimate, we show that $\|\Psi_{I,S}\| \leqslant 2(\|S\|\|S^{-1}\| - 1)$.

2. Some operator inequalities

Theorem 2.1. *We have the following inequalities*:

$$\begin{cases} \|\Phi_{I,S}\| \geqslant \|S\|\|S^{-1}\| + (1/\|S\|\|S^{-1}\|), & (5)\\ \|\Psi_{I,S}\| \geqslant \|S\|\|S^{-1}\| - (1/\|S\|\|S^{-1}\|), & (6)\\ \|\Psi_{I,S}\| \leqslant 2(\|S\|\|S^{-1}\| - 1). & (7) \end{cases}$$

Proof. Let $S = UP$ be the polar decomposition of S, where $P = |S|$ and $U = U^* = U^{-1}$.

So for every $X \in I$, we have $\|SXS^{-1} \pm S^{-1}XS\|_I = \|PXP^{-1} \pm P^{-1}XP\|_I$. Thus $\|\Phi_{I,S}\| = \|\Phi_{I,P}\|$ and $\|\Psi_{I,S}\| = \|\Psi_{I,P}\|$. Let F denote the operator on I defined by $F(X) = PXP^{-1}$ and let $\mathscr{A}$ denote the maximal commutative Banach algebra containing F and F^{-1}. We denote by $M_{\mathscr{A}}$ the set of all multiplicative functionals on $\mathscr{A}$. Since $F, F^{-1} \in \mathscr{A}$ and $\min\{\varphi(F) : \varphi \in M_{\mathscr{A}}\} = \frac{1}{\|P\|\|P^{-1}\|}$, $\max\{\varphi(F) : \varphi \in M_{\mathscr{A}}\} = \|P\|\|P^{-1}\|$, so the numerical radius of $F \pm F^{-1}$ is given by

$$r(F \pm F^{-1}) = \max\left\{|\varphi(F) \pm \frac{1}{\varphi(F)}| : \varphi \in M_{\mathscr{A}}\right\} = \|P\|\|P^{-1}\| \pm \frac{1}{\|P\|\|P^{-1}\|}.$$

Therefore

$$\begin{cases} \|\Phi_{I,P}\| = \|F + F^{-1}\| \geqslant r(F + F^{-1}) = \|P\|\|P^{-1}\| + \frac{1}{\|P\|\|P^{-1}\|},\\ \|\Psi_{I,P}\| = \|F - F^{-1}\| \geqslant r(F - F^{-1}) = \|P\|\|P^{-1}\| - \frac{1}{\|P\|\|P^{-1}\|}. \end{cases}$$

In view of $\|\Phi_{I,S}\| = \|\Phi_{I,P}\|, \|\Psi_{I,S}\| = \|\Psi_{I,P}\|, \|S\| = \|P\|$ and $\|S^{-1}\| = \|P^{-1}\|$, we have

$$\begin{cases} \|\Phi_{I,S}\| \geqslant \|S\|\|S^{-1}\| + \frac{1}{\|S\|\|S^{-1}\|},\\ \|\Psi_{I,S}\| \geqslant \|S\|\|S^{-1}\| - \frac{1}{\|S\|\|S^{-1}\|}. \end{cases}$$

On the other hand, we obtain the inequality (7) in two steps:

Step 1. Suppose $\|P\| = 1$.

Since $\Psi_{I,P} = V_{I,P,P^{-1}-\lambda P}$, for all complex λ, it follows that

$$\|\Psi_{I,P}\| \leqslant 2\|P^{-1} - \|P^{-1}\|P\| = 2(\|P^{-1}\| - 1).$$

Step 2. From the step 1, we obtain

$$\|\Psi_{I,P}\| = \|\Psi_{I,\frac{P}{\|P\|}}\| \leqslant 2(\|P\|\|P^{-1}\| - 1).$$

Thus, we find that

$$\|\Psi_{I,S}\| \leqslant 2(\|S\|\|S^{-1}\| - 1). \qquad \square$$

Theorem 2.2. *We have the following equalities*:

$$\begin{cases} \|\Phi_{2,S}\| = \|S\|\|S^{-1}\| + \frac{1}{\|S\|\|S^{-1}\|}, & (8) \\ \|\Psi_{2,S}\| = \|S\|\|S^{-1}\| - \frac{1}{\|S\|\|S^{-1}\|}. & (9) \end{cases}$$

Proof. Employing the notation used in the proof of the preceding theorem (where $I = \mathfrak{C}_2$) and since $F \pm F^{-1}$ is a bounded selfadjoint operator on the Hilbert space $\mathfrak{C}_2$ and $\sigma(F) = \sigma(F^{-1}) = \sigma(P)\,\sigma(P^{-1})$, we get

$$\|F \pm F^{-1}\| = r(F \pm F^{-1}) = \|P\|\|P^{-1}\| \pm \frac{1}{\|P\|\|P^{-1}\|}.$$

Then the result follows immediately. $\square$

Theorem 2.3. *We have the following inequality*:

$$\forall X \in I : \|SXS^{-1} - S^{-1}XS\|_I \leqslant (\|S\|\|S^{-1}\| - 1)\|SXS^{-1} + S^{-1}XS\|_I.$$

Proof. The inequality follows immediately from the inequalities (2) and (7). $\square$

Corollary 2.1. *If* $\|S\|\|S^{-1}\| \leqslant 2$, *then we have the following inequality*:

$$\forall X \in I : \|SXS^{-1} - S^{-1}XS\|_I \leqslant \|SXS^{-1} + S^{-1}XS\|_I.$$

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