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ON THE INJECTIVE NORM OF $\sum_{i=1}^{n} A_i \otimes B_i$ AND CHARACTERIZATION OF NORMALOID OPERATORS

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(*communicated by F. Kittaneh*)

Abstract. Let $\mathscr{B}(H)$ denotes the C^*-algebra of all bounded linear operators acting on the complex Hilbert space H. In this note, we shall give some lower estimates for the injective norm of the element $\sum_{i=1}^{n} A_i \otimes B_i$ in the tensor product $\mathscr{B}(H) \otimes \mathscr{B}(H)$, where $A = (A_1, ..., A_n)$ and $B = (B_1, ..., B_n)$ are two n-tuples of elements in $\mathscr{B}(H)$; and we shall characterize the normaloid operators in $\mathscr{B}(H)$ using the injective norm.

1. Introduction

Let $\mathscr{A}$ be a standard operator algebra acting on a (real or complex) normed space (it is a subalgebra of bounded linear operators acting on a normed space that contains all finite rank operators). A linear operator $R : \mathscr{A} \to \mathscr{A}$ defined by $R(X) = \sum_{i=1}^{n} A_i X B_i$, where $A_i,\ B_i \in \mathscr{A}$ $(1 \leqslant i \leqslant n)$ is called an elementary operator on $\mathscr{A}$; and it is denoted by $R = R_{A,B}$, where $A = (A_1, ..., A_n)$ and $B = (B_1, ..., B_n)$.

This concrete class includes many important operators on $\mathscr{A}$, such as the two-sided multiplication $M_{A,B} : X \longrightarrow AXB$, the inner derivation $\delta_A : X \longrightarrow AX - XA$, the generalized derivation $\delta_{A,B} : X \longrightarrow AX - XB$, the symmetrized two-sided multiplication $U_{A,B} : X \longrightarrow AXB + BXA$, for given $A,\ B \in \mathscr{A}$.

The norm problem for elementary operators consists in finding a formula which describes the norm of an elementary operator in terms of its coefficients. It is easy to see that for a two-sided multiplication, we have $\|M_{A,B}\| = \|A\|\,\|B\|$. For a generalized derivation, Stampfli [15] characterized the norm of $\delta_{A,B}$ by the relation $\|\delta_{A,B}\| = \inf\{\|A - \lambda I\| + \|B - \lambda I\| : \lambda \in \mathbb{C}\}$, and he proved that $\|\delta_{A,B}\| = \|A\| + \|B\|$ if and only if $W_N(A) \cap W_N(-B) \neq \varnothing$ (where A and B are nonzero in $\mathscr{A}$, $W_N(.)$ denotes the normalized maximal numerical range and where $\mathscr{A} = \mathscr{B}(H)$ and H is a complex Hilbert space). In the complex Hilbert space case also and for a symmetrized two-sided multiplication, the best lower estimate $\|U_{A,B}\| \geqslant \|A\|\,\|B\|$ was given separately

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in $[3,16]$. But in our general situation of normed space, we obtained in $[11]$ that the lower estimate $\|U_{A,B}\| \geqslant 2(\sqrt{2}-1)\|A\|\|B\|$ holds. Moreover, Blanco-Boumazgour-Ransford $[3]$ showed that this last estimation is the best possible within the class of Banach algebras. The lower estimate for $U_{A,B}$ was studied by several authors, for more details see $[2,3,5-12,14,16]$. In the complex Hilbert space case, Barraa-Boumazgour $[1]$ showed that $\|I+M_{A,B}\| = 1+\|A\|\|B\|$ if and only if $W_N(A)\cap W_N(B^*) \neq \varnothing$. For an arbitrary elementary operator $R_{A,B}$, its norm is obviously bounded from above by $\sum_{i=1}^{n}\|A_i\|\|B_i\|$ (this upper bound is denoted by $D(R_{A,B})$).

Recently $[10]$, we have been interested (in the general situation of standard operator algebra of normed space) in the supremum $d(R_{A,B})$ of the norm of $R_{A,B}(X)$ over all unit rank one operators instead of $\|R_{A,B}\|$ (It is clear that $d(R_{A,B}) \leqslant \|R_{A,B}\| \leqslant D(R_{A,B})$). We have characterized $d(R_{A,B})$ when it gets its maximal value. In particular, we have obtained the following characterization:

(i) $d(U_{A,B}) = D(U_{A,B})$ $(= 2\|A\|\|B\|)$ if and only if $A \parallel B$ (that means $\|A+\lambda B\| = \|A\|+\|B\|$ for some unit scalar λ),

(ii) $d(L_A+R_A) = D(L_A+R_A)$ $(= 2\|A\|)$ if and only if A is normaloid,

(iii) $d(\delta_{A,B}) = D(\delta_{A,B})$ $(= \|A\|+\|B\|)$ if and only if $\|I+\lambda A\| = 1+\|A\|$ and $\|I-\lambda B\| = 1+\|B\|$ for some unit scalar λ.

On the other hand, we have established the two following lower estimates (where $N(.)$ denotes either $d(.)$ or $\|.\|$ and $V(.)$ is the algebraic numerical range):

$$\begin{aligned} N(\delta_{A,B}) &\geqslant \max\left\{\sup_{\lambda\in V(B)}\|A-\lambda I\|,\ \sup_{\lambda\in V(A)}\|B-\lambda I\|\right\} \\ N(U_{A,B}) &\geqslant 2(\sqrt{2}-1)\|A\|\|B\| \end{aligned}$$

In this note we shall show that the first lower estimate is sharp in the complex Hilbert space case. We shall give also another lower estimates for other elementary operators and some characterization of normaloid operators.

Through this paper $\mathscr{A}$ denotes a standard operator algebra of a normed space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), and $\mathscr{B}(H)$ the C^*-algebra of all bounded linear operators acting on the complex Hilbert space H. All elementary operators $R_{A,B}$ given in this note are defined on $\mathscr{A}$ (if one of the operators A_i, B_i is equal to the identity I, $\mathscr{A}$ is assumed to be unital).

In this note we adopt the following notations and definitions:

(i) If F is a normed space, we denote by F' the topological dual space of F, and by $(F)_1$ the unit sphere of F.

(ii) Let Ω be a unital normed algebra over the field $\mathbb{K}$ and let $A=(A_1,...,A_n) \in \Omega^n$. The n-algebraic numerical range (or joint numerical range) of A is $V(A) = \{(f(A_1),...,f(A_n)) : f \in \mathscr{P}(\Omega)\}$, where $\mathscr{P}(\Omega) = \{f \in \Omega' : f(I) = \|f\| = 1\}$ is the set of all states on Ω. The n-numerical radius of A is $w(A) = \sup\left\{\sqrt{\sum_{i=1}^{n}|\lambda_i|^2} : (\lambda_1,...,\lambda_n) \in V(A)\right\}$. The n-norm of A is $\|A\| = \sqrt{\sum_{i=1}^{n}\|A_i\|^2}$. It is known that $V(A)$ is non empty, closed and convex of $\mathbb{K}^n$. A is called n-normaloid if $\|A\| = w(A)$. For $n=1$, we say only the algebraic numerical range, the numerical radius, the norm of A, and A is normaloid; in this case we denote by $V_N(A) = V(\frac{A}{\|A\|})$ the normalized algebraic range of A if $A \neq 0$. For more details, see [4].

(iii) If R is an elementary operator on $\mathscr{A}$, we denote by $N(R)$ either $d(R)$ or $\|R\|$.

(iv) If F is an inner product space and $x,\ y \in F,$ then the relation $x \parallel y$ (that means $x,\ y$ are linearly dependent) holds if and only if $\|x+\lambda y\| = \|x\| + \|y\|$ for some unit scalar λ. The two conditions make sense on a normed space and the first condition implies the second but the converse is false in general. So, we may adopt as definition of the norm-parallelism relation in normed space as follows $x \parallel y$ (we say that x is norm-parallel to y) if and only if $\|x+\lambda y\| = \|x\| + \|y\|$ for some unit scalar λ (this relation is reflexive, symmetric but in general not transitive).

(v) If F is an inner product space and $x,\ y\ \in\ F,$ the relation $x \perp y$ holds if and only if $\inf_{\lambda \in \mathbb{K}} \|y+\lambda x\| = \|y\|$ (or equivalently to $\|y+\lambda x\| \geqslant \|y\|$, for every λ in $\mathbb{K}$). This last condition makes sense in any normed space and therefore may be taken as a definition of the relation of orthogonality in this general situation. In this general case, it is clear by using the Hahn-Banach Theorem, that the relation $x \perp y$ holds if and only if there exists a unit element f in F' such that $f(x) = 0$ and $f(y) = \|y\|$ (this relation is not symmetric in a general situation of a normed space).

(vi) For a n-tuple $A = (A_1, ..., A_n)$ of elements in $\mathscr{B}(H)$, we denote by A^*, the n-tuple $A^* = (A_1^*, ..., A_n^*)$.

(vii) Let $A \in \mathscr{B}(H)$. The numerical range of A is $W(A) = \{\langle Ax, x\rangle\ : x \in (H)_1\}$. It is known that $V(A) = W(A)^-$ (the closure of $W(A)$) and $2w(A) \geqslant \|A\|$. We denote by $|A|$ the positive square root of A^*A.

($viii$) For $A \in \mathscr{A}$, we denote by Δ_A the operator defined on $\mathscr{A}$ by $\Delta_A(X) = AX + XA$.

For the sake of completeness, we refer the reader to the two following theorems given in [10] which represents the basic sources of all results obtained in our recent paper and in this note:

THEOREM 1. *Let $A = (A_1, ..., A_n)$ and $B = (B_1, ..., B_n)$ be two n-tuples of elements in $\mathscr{A}$. The following equalities hold:*

$$\begin{aligned} d(R_{A,B}) &= \sup_{f,\ g \in (\mathscr{A}')_1} \left| \sum_{i=1}^{n} f(A_i) g(B_i) \right| \\ &= \sup_{f,\ g \in (\mathscr{A}')_1} \left\| \sum_{i=1}^{n} f(B_i) A_i \right\| \\ &= \sup_{f,\ g \in (\mathscr{A}')_1} \left\| \sum_{i=1}^{n} f(A_i) B_i \right\| \end{aligned}$$

REMARK 1. In the case where $\mathscr{A} = \mathscr{B}(H)$, the injective norm of $R = \sum_{i=1}^{n} A_i \otimes B_i \in \mathscr{A} \otimes \mathscr{A}$ is defined by $\|R\|_\lambda = \sup_{f,\ g \in (\mathscr{A}')_1} |(f \otimes g)(R)|$, and where $(f \otimes g)(R) = \sum_{i=1}^{n} (A_i) g(B_i)$. So it follows in this situation that $d(R_{A,B}) = \|R\|_\lambda$. This equality was given also by Magajna-Turnsek [7] using some properties of the predual of $\mathscr{B}(H)$. But our proof was given in a general situation by a direct proof.

THEOREM 2. *Let $A=(A_1,...,A_n)$ and $B=(B_1,...,B_n)$ be two n-tuples of non zero elements in $\mathscr{A}$. The following properties are equivalent:*

(i) $d(R_{A,B})=D(R_{A,B})$,

(ii) *there exist two unit elements f, g in $\mathscr{A}'$ and n unit scalars $\lambda_1,...,\lambda_n$ such that $f(A_i)=\lambda_i\|A_i\|$ and $g(B_i)=\overline{\lambda_i}\|B_i\|$ for $i=1,...,n$,*

(iii) $\left\|\sum_{i=1}^{n}\lambda_i A_i\right\|=\sum_{i=1}^{n}\|A_i\|$ *and* $\left\|\sum_{i=1}^{n}\overline{\lambda_i}B_i\right\|=\sum_{i=1}^{n}\|B_i\|$ *for some unit scalars $\lambda_1,...,\lambda_n$.*

The following corollaries follows immediately from the above Theorem.

COROLLARY 1. *Let $A=(A_1,...,A_n)$ and $B=(B_1,...,B_n)$ be two n-tuples of non zero elements in $\mathscr{A}$ such that $A_k=I$ (resp. $B_k=I$) for some index k and $d(R_{A,B})=D(R_{A,B})$. Then all operators A_i (resp. B_i) are normaloid in $\mathscr{A}$.*

COROLLARY 2. *Let A and B be two non-zero elements in $\mathscr{B}(H)$. The two following properties are equivalent:*

(i) $d(I+M_{A,A}+M_{A^*,A^*})=1+2\|A\|^2$,

(ii) $V_N(A)\cap V_N(A^*)\cap(\mathbb{C})_1\neq\varnothing$.

COROLLARY 3. *Let A, B be two non-zero elements in $\mathscr{A}$. The following properties are equivalent*

(i) $d(I+M_{A,B})=1+\|A\|\|B\|$,

(ii) $V_N(A)\cap\overline{V_N(B)}\cap(\mathbb{K})_1\neq\varnothing$,

(iii) $\|I+\lambda A\|=1+\|A\|$ *and* $\left\|I+\overline{\lambda}B\right\|=1+\|B\|$, *for some unit scalar λ.*

2. Lower estimates for elementary operators

THEOREM 3. *Let A, $B\in\mathscr{A}$. The following lower estimate holds*

$$N(\delta_{A,B})\geqslant\max\left\{\sup_{\lambda\in V(B)}\|A-\lambda I\|,\ \sup_{\lambda\in V(A)}\|B-\lambda I\|\right\}.$$

Proof. The proof is given in [10]. □

REMARK 2. Let $A\in\mathscr{A}$. So it follows that $\|A-\mu I\|\leqslant N(\delta_A)\leqslant 2\|A-\lambda I\|$, for every scalar λ and for every μ in $V(A)$.

Therefore we can deduce that

$$\sup_{\mu\in V(A)}\|A-\mu I\|\leqslant N(\delta_A)\leqslant 2\inf_{\lambda\in\mathbb{K}}\|A-\lambda I\|.$$

THEOREM 4. *Assume that $\mathscr{A} \subset \mathscr{B}(H)$ and let $A,\ B \in \mathscr{A}$. If A and B are singular and positive in $\mathscr{B}(H)$, then the following equalities hold*

$$N(\delta_{A,B}) = \max\left\{ \sup_{\lambda \in V(B)} \|A-\lambda I\|,\ \sup_{\lambda \in V(A)} \|B-\lambda I\| \right\} = \max\left\{\|A\|,\ \|B\|\right\}.$$

Proof. It is clear that for $X = A$ and $X = B$, $V(X) = V(X,\ \mathscr{B}(H))$ coincides with the convex hull of $\sigma(X)$, and so $V(X) = [0, \|X\|]$. Since $V(\delta_{A,B,\mathscr{B}(H)}) = V(A,\ \mathscr{B}(H)) - V(B,\ \mathscr{B}(H)) \subset \mathbb{R}$ (see [13]), then $\delta_{A,B,\mathscr{B}(H)}$ is Hermitian. So it follows that $\left\|\delta_{A,B,\mathscr{B}(H)}\right\| = r\left(\delta_{A,B,\mathscr{B}(H)}\right)$. From the known result $\sigma\left(\delta_{A,B,\mathscr{B}(H)}\right) = \sigma(A) - \sigma(B)$ (spectrums in $\mathscr{B}(H)$), it is easy to see that $r\left(\delta_{A,B,\mathscr{B}(H)}\right) = \max\left\{\|A\|, \|B\|\right\}$. On the other hand, from the above Theorem and since $0 \in V(A) \cap V(B)$ we have

$$N(\delta_{A,B}) \geqslant \max\left\{ \sup_{\lambda \in V(B)} \|A-\lambda I\|,\ \sup_{\lambda \in V(A)} \|B-\lambda I\| \right\} \geqslant \max\left\{\|A\|,\ \|B\|\right\}.$$

From above and since $\max\left\{\|A\|, \|B\|\right\} = \left\|\delta_{A,B,\mathscr{B}(E)}\right\| \geqslant N(\delta_{A,B})$, it follows that

$$N(\delta_{A,B}) = \max\left\{ \sup_{\lambda \in V(B)} \|A-\lambda I\|,\ \sup_{\lambda \in V(A)} \|B-\lambda I\| \right\} = \max\left\{\|A\|,\ \|B\|\right\}.$$

□

REMARK 3. The estimate given in Theorem 3 is sharp in any unital standard operator algebra of a complex Hilbert space..

THEOREM 5. *Let $A = (A_1, ..., A_n)$ be a n-tuples of elements in $\mathscr{B}(H)$.Then we have*

$$N(I + R_{A^*,A}) \geqslant 1 + w^2(A).$$

Proof. It follows from Theorem 1 that $d(I + R_{A^*,A}) \geqslant \left|1 + \sum_{i=1}^{n} f(A_i^*) f(A_i)\right| = 1 + \sum_{i=1}^{n} |f(A_i)|^2$, for every state f on $\mathscr{A}$. So the result follows from the definition of $w(A)$. □

COROLLARY 4. *Let $A \in \mathscr{B}(H)$. Then $N(I + M_{A^*,A}) \geqslant 1 + w^2(A)$.*

THEOREM 6. *Let $A,\ B \in \mathscr{A}$. Then, the following lower estimate holds*

$$N(I + M_{A,B}) \geqslant \max\left\{ \sup_{\lambda \in V(B)} \|I + \lambda A\|,\ \sup_{\lambda \in V(A)} \|I + \lambda B\| \right\}.$$

Proof. The proof follows immediately from Theorem 1. □

REMARK 4. The above estimate is sharp. Indeed, it is easy to see that the above inequality becomes an equality for $B = I$, and the common value is $\|I + A\|$.

THEOREM 7. *Let* $A \in \mathscr{A}$. *Then* $d(U_{A,B}) \geqslant 2(\sqrt{2}-1)\,\|A\|\,\|B\|$.

Proof. The proof was given in [11] for the norm but it remains true for $d(U_{A,B})$. □

REMARK 5. This last estimation was given also in [7] by Magajna-Turnsek in the particular case of complex Hilbert space and where they proved that it is sharp (if $\dim H \geqslant 2$).

3. Characterization of normaloid operators

THEOREM 8. *Let* $A=(A_1,...,A_n)$ *be a n-tuple of elements in* $\mathscr{B}(H)$. *Then the following properties are equivalent:*

(i) A *is n-normaloid in* $(\mathscr{B}(H))^n$,

(ii) *there exist a state* f *on* $\mathscr{B}(H)$ *and* n *unit scalars* $\lambda_1,...,\lambda_n$ *such that* $f(A_i)=\lambda_i\,\|A_i\|$, $i=1,...,n$,

(iii) $d(I+R_{A^*,A}) = 1+\sum_{i=1}^{n}\|A_i\|^2$.

Proof. $(i)\Longrightarrow(ii)$. Since $V(A)$ is compact, then there exists a state f on $\mathscr{B}(H)$ such that $w^2(A)=\sum_{i=1}^{n}|f(A_i)|^2$. So from (i) it follows that $\sum_{i=1}^{n}|f(A_i)|^2=\sum_{i=1}^{n}\|A_i\|^2$. Since $|f(A_i)|\leqslant\|A_i\|$, for $i=1,...,n$, thus $|f(A_i)|=\|A_i\|$, for $i=1,...,n$. Therefore the result follows immediately.

$(ii)\Longrightarrow(iii)$. Since f is a state on $\mathscr{B}(H)$, it follows that $f(I)=1$, $f(A_i)=\lambda_i\,\|A_i\|$, $f(A_i^*)=\overline{\lambda_i}\,\|A_i^*\|$, $i=1,...,n$. Therefore the result follows immediately from Theorem 2.

$(iii)\Longrightarrow(i)$. From Theorem 2, there exist a unit functional f on $\mathscr{B}(H)$ and $n+1$ unit scalars λ_0, λ_1,..., λ_n such that $f(I)=\lambda_0$, $f(A_i)=\lambda_i\,\|A_i\|$, $i=1,...,n$. Put $g=\overline{\lambda_0}f$, thus g is a state on $\mathscr{B}(H)$. Hence $w^2(A)\geqslant\sum_{i=1}^{n}|g(A_i)|^2=\sum_{i=1}^{n}\|A_i\|^2\geqslant w^2(A)$. Therefore $w(A)=\|A\|$. □

THEOREM 9. *Let* $A,\ B\in\mathscr{B}(H)$ *such that* $V(B)=\{|z|\leqslant\|B\|\}$. *The following properties are equivalent:*

(i) A *is normaloid in* $\mathscr{B}(H)$,

(ii) $A\parallel I$,

(iii) $A^*\parallel A$,

(iv) (A^*,A) *is 2-normaloid in* $(\mathscr{B}(H))^2$,

(v) $d(I+M_{A^*,A})=1+\|A\|^2$,

(vi) $d(I+\lambda M_{A,A})=1+\|A\|^2$, *for some unit scalar* λ,

(vii) $d(I+M_{A,B})=1+\|A\|\,\|B\|$,

$(viii)$ $d(\delta_{A,B})=\|A\|+\|B\|$,

(ix) $d(I+U_{A^*,A})=1+2\,\|A\|^2$,

(x) $d(U_{A^*,A})=2\,\|A\|^2$,

(xi) $d(\Delta_A)=2\,\|A\|$.

Proof. $(i) \iff (ii)$ is trivial.

$(i) \implies (iii)$. There exist a state f on $\mathscr{B}(H)$ and a unit scalar λ such that $f(A) = \lambda \|A\|$. So we have $2\|A\| = f(\overline{\lambda}A + \lambda A^*) \leqslant \left\|\overline{\lambda}A + \lambda A^*\right\| \leqslant 2\|A\|$. Thus $\left\|\overline{\lambda}A + \lambda A^*\right\| = \left\|A + \lambda^2 A^*\right\| = \|A\| + \|A^*\|$, where $\left|\lambda^2\right| = 1$.

$(iii) \implies (i)$. There exists a unit scalar λ such that $\|A^* + \lambda A\| = 2\|A\|$. Since $A^* + \lambda A$ is normal in $\mathscr{B}(H)$, then there exists a state f on $\mathscr{B}(H)$ such that $\left|f\overline{(A)} + \lambda f(A)\right| = |f(A^* + \lambda A)| = 2\|A\| \leqslant 2|f(A)| \leqslant 2\|A\|$. Thus $|f(A)| = \|A\|$. Therefore A is normaloid in $\mathscr{B}(H)$.

$(i) \iff (iv)$ is trivial.

$(i) \iff (v)$ follows from the above Theorem for $n = 1$.

$(iv) \iff (ix)$ follows from the above Theorem for $n = 2$ and $A_1 = A^* = B_2$, $B_1 = A = A_2$.

$(iii) \iff (x)$ follows from $[10,\ \textit{Corollary}\ 2.5]$.

$(ii) \iff (xi)$ follows from $[10,\ \textit{Corollary}\ 2.6]$.

$(vi) \implies (i)$ follows from Corollary 1.

$(i) \implies (vi)$. There exist a unit functional f on $\mathscr{B}(H)$ and a unit scalar α such that $f(A) = \alpha\|A\|$. Put $\lambda = \overline{\alpha}^2$, so we obtain from Theorem 1 that $1 + \|A\|^2 \geqslant d(I + \lambda M_{A,A}) \geqslant \left|f(I)^2 + \lambda f(A)^2\right| = 1 + \|A\|^2$.

$(i) \iff (vii)$. This follows immediately from Corollary 3.

$(i) \iff (viii)$. This follows from Corollary 1 and $[10,\ \textit{Corollary}\ 2.3]$. $\square$

REMARK 6. Suppose that H is separable and let B denote the unilateral shift on H. Since $V(B) = \{|z| \leqslant \|B\| = 1\}$, then for every normaloid operator A in $\mathscr{B}(H)$, we obtain $N(I + M_{A,B}) = N(\delta_{A,B}) = 1 + \|A\|$.

4. Some sufficient conditions for the estimate $d(U_{A,B}) \geqslant \|A\|\,\|B\|$

THEOREM 10. *Let* $A, B, C, D \in \mathscr{A}$ *such that* $A \perp C$ *or* $B \perp D$ *then* $M_{A,B} \perp M_{C,D}$.

Proof. Assume $A \perp C$. So there exists a unit element f in $\mathscr{A}'$ such that $f(A) = 0$ and $f(C) = \|C\|$. Thus by using Theorem 1, it follows that $d(M_{C,D} + \lambda M_{A,B}) \geqslant \|f(C)D + \lambda f(A)B\| = \|C\|\,\|D\| = \|M_{C,D}\|$ for all scalar λ. Hence $\|M_{C,D} + \lambda M_{A,B}\| \geqslant \|M_{C,D}\|$ for all scalar λ. Therefore $M_{A,B} \perp M_{C,D}$.

The second implication follows also by the same argument. $\square$

COROLLARY 5. *Let* $A, B \in \mathscr{A}$ *such that* $A \perp B$ *or* $B \perp A$. *Then the two following properties hold:*

(i) $M_{A,B} \perp M_{B,A}$ *and* $M_{B,A} \perp M_{A,B}$,

(ii) $d(U_{A,B}) \geqslant \|A\|\,\|B\|$.

THEOREM 11. *Let* $A, B \in \mathscr{A}$. *Then the lower estimate* $d(U_{A,B}) \geqslant \|A\|\,\|B\|$ *holds if one of the following properties is satisfied:*

(i) $\inf_{\lambda \in \mathbb{K}} \|A + \lambda B\| \leqslant \dfrac{\|A\|}{2}$,

$(ii)\quad \inf_{\lambda\in\mathbb{K}} \|B+\lambda A\| \leqslant \dfrac{\|B\|}{2}.$

Proof. (i) Suppose that $\inf_{\lambda\in\mathbb{K}} \|A+\lambda B\| \leqslant \dfrac{\|A\|}{2}$ and put $V_{A,B} = M_{A,B} - M_{B,A}$. It is easy to see that $V_{A+\lambda B,B} = V_{A,B}$, for all scalar λ. Hence, $\|V_{A,B}\| \leqslant 2\|A+\lambda B\|\,\|B\|$, for all scalar λ. So it follows from the hypothesis that $d(V_{A,B}) \leqslant \|V_{A,B}\| \leqslant \|A\|\,\|B\|$. On the other hand, it is clear that $d(U_{A,B}) + d(V_{A,B}) \geqslant 2d(M_{A,B}) = 2\|A\|\,\|B\|$. Therefore, from the two last inequalities it follows that $d(U_{A,B}) \geqslant \|A\|\,\|B\|$.

By the same argument, the estimation follows with the condition (ii). □

REMARK 7. From above, the lower estimate $d(U_{A,B}) \geqslant \|A\|\,\|B\|$ holds if one of the following properties is satisfied:

(i) $\inf_{\lambda\in\mathbb{K}} \|A+\lambda B\| \leqslant \dfrac{\|A\|}{2}$,

(ii) $\inf_{\lambda\in\mathbb{K}} \|B+\lambda A\| \leqslant \dfrac{\|B\|}{2}$,

(iii) $\inf_{\lambda\in\mathbb{K}} \|A+\lambda B\| = \|A\|$,

(iv) $\inf_{\lambda\in\mathbb{K}} \|B+\lambda A\| = \|B\|$.

So it is clear that if $d(U_{A,B}) < \|A\|\,\|B\|$, then we must have:

(v) $\frac{\|A\|}{2} < \inf_{\lambda\in\mathbb{K}} \|A+\lambda B\| < \|A\|$ and $\frac{\|B\|}{2} < \inf_{\lambda\in\mathbb{K}} \|B+\lambda A\| < \|B\|$.

So we may ask the following Problem.

PROBLEM 1. *Does the lower estimate $d(U_{A,B}) \geqslant \|A\|\,\|B\|$ remain true for some pairs $(A,B) \in \mathscr{A}^2$ satisfying the condition (v) given in the above Remark?*

In the remaining results we specialize to the case when $\mathscr{A} \subset \mathscr{B}(H)$.

LEMMA 1. *Assume that $\mathscr{A} \subset \mathscr{B}(H)$ and let $A,\ B \in \mathscr{A}$. Then we have*

$$d(U_{A,B}) = \sup\{|\langle Ax,y\rangle\,\langle Bu,v\rangle + \langle Au,v\rangle\,\langle Bx,y\rangle| : x,y,u,v \in (H)_1\}.$$

Proof. The proof is similar to the proof of Theorem 1. □

THEOREM 12. *Assume that $\mathscr{A} \subset \mathscr{B}(H)$ and let $A,\ B \in \mathscr{A}$. Then the lower estimate $d(U_{A,B}) \geqslant \|A\|\,\|B\|$ holds if one of the following properties is satisfied:*

(i) *A and B are positive,*

(ii) *A^*B or AB^* is positive.*

Proof. (i) From the above Lemma, we obtain $d(U_{A,B}) \geqslant \langle Ax,x\rangle\,\langle By,y\rangle$ for every unit vectors x and y in H. Thus $d(U_{A,B}) \geqslant \sup_{\|x\|=1} \langle Ax,x\rangle \sup_{\|y\|=1} \langle By,y\rangle = \|A\|\,\|B\|$.

(ii)Suppose that $A^*B \geqslant 0$. Let x, y be two unit vectors in H such that $Ax \neq 0$ and $By \neq 0$. From Lemma 1, we obtain

$$\begin{aligned} d(U_{A,B}) &\geqslant \left| \left\langle Ax, \frac{1}{\|Ax\|}Ax \right\rangle \left\langle By, \frac{1}{\|By\|}By \right\rangle + \left\langle Ay, \frac{1}{\|By\|}By \right\rangle \left\langle Bx, \frac{1}{\|Ax\|}Ax \right\rangle \right| \\ &\geqslant \left| \|Ax\| \, \|By\| + \frac{1}{\|Ax\| \, \|By\|} \left\langle B^*Ay, y \right\rangle \left\langle A^*Bx, x \right\rangle \right| \\ &\geqslant \|Ax\| \, \|By\| \end{aligned}$$

Hence $d(U_{A,B}) \geqslant \|Ax\| \, \|By\|$, for every unit vectors x, y in H. Therefore the result follows immediately.

By the same argument and since $d(U_{A,B}) = d(U_{A^*,B^*})$, the estimate holds with the second condition. □

REMARK 8. The Theorem 12.(i) is false in general if we replace the condition "positive" by the condition "self-adjoint", this may be seen from the example given in [7, Remark 3.1].

COROLLARY 6. *Assume that* $\mathscr{A} \subset \mathscr{B}(H)$ *and let* $A, B \in \mathscr{A}$. *such that* $A = UP$ *and* $B = UQ$, *where* $P = |A|$, $Q = |B|$ *and* U *is unitary in* $\mathscr{B}(H)$. *Then the lower estimate* $d(U_{A,B}) \geqslant \|A\| \, \|B\|$ *holds.*

Proof. From Lemma 1 and Theorem 12.(i), we have

$$\begin{aligned} d(U_{A,B}) &= \sup\{\|\langle Bx, y\rangle A + \langle Ax, y\rangle B\| : \|x\| = \|y\| = 1\} \\ &= \sup\{\|\langle Qx, U^*y\rangle UP + \langle Px, U^*y\rangle UQ\| : \|x\| = \|y\| = 1\} \\ &= \sup\{\|\langle Qx, y\rangle P + \langle Px, y\rangle Q\| : \|x\| = \|y\| = 1\} \\ &= d(U_{P,Q}) \\ &\geqslant \|P\| \, \|Q\| \\ &= \|A\| \, \|B\| \end{aligned}$$

□

THEOREM 13. *Assume that* $\mathscr{A} \subset \mathscr{B}(H)$ *and let* $A, B \in \mathscr{A}$. *If there exist a sequence* $\{x_n\}$ *of unit vectors in* H *such that* $\|Ax_n\| \to \|A\|$ *and* $\|Bx_n\| \to \|B\|$, *then the lower estimate* $d(U_{A,B}) \geqslant \|A\| \, \|B\|$ *holds.*

Proof. Let x be a unit vector in H. It follows from Lemma 1, that $d(U_{A,B}) \geqslant 2|\langle Ax_n, x\rangle \langle Bx_n, x\rangle| = 2|\langle (Ax_n \otimes Bx_n)x, x\rangle|$, for every integer $n \geqslant 1$. Hence $d(U_{A,B}) \geqslant 2 \sup_{\|x\|=1} |\langle (Ax_n \otimes Bx_n)x, x\rangle| = 2w(Ax_n \otimes Bx_n) \geqslant \|Ax_n \otimes Bx_n\| = \|Ax_n\| \, \|Bx_n\|$. Letting $n \to \infty$, the result follows immediately. □

COROLLARY 7. *Assume that* $\mathscr{A} \subset \mathscr{B}(H)$ *and let* $A, B \in \mathscr{A}$. *If* A *or* B *is an isometry (or a co-isometry) in* $\mathscr{B}(H)$, *then the lower estimate* $d(U_{A,B}) \geqslant \|A\| \, \|B\|$ *holds.*

THEOREM 14. *Assume that* $\mathscr{A} \subset \mathscr{B}(H)$ *and let* $A,\ B \in \mathscr{A}$. *The lower estimate* $d(U_{A,B}) \geqslant \|A\|\,\|B\|$ *holds if* $\|B\|^2\,(A^*A) \leqslant \|A\|^2\,(B^*B)$ *or* $\|A\|^2\,(B^*B) \leqslant \|B\|^2\,(A^*A)$.

Proof. We may assume without loss of the generality, that $\|A\| = \|B\| = 1$.

Suppose that $A^*A \leqslant B^*B$. For every unit vector x in H such that $Ax \neq 0,\ Bx \neq 0$, it follows from the proof of the Theorem 12.(ii) that $d(U_{A,B}) \geqslant \|Ax\|\,\|Bx\| + \frac{|\langle A^*Bx,x\rangle|^2}{\|Ax\|\|Bx\|}$. Hence $d(U_{A,B}) \geqslant \|Ax\|\,\|Bx\|$ for every unit vector x in H. From the hypothesis, we obtain $d(U_{A,B}) \geqslant \|Ax\|^2$. Therefore $d(U_{A,B}) \geqslant 1$. By the same argument, the estimate holds with the second condition. □

REMARK 9. The relation $\leqslant$ defined on the set of all self-adjoint operators in $\mathscr{B}(H)$ by "$A \leqslant B$ if and only if $B - A$ is positive " is an order relation (total if $\dim H = 1$ and partial if $\dim H \geqslant 2$). Two self-adjoint operators A and B are called comparable if $A \leqslant B$ or $B \leqslant A$. So we may reformulate the above Theorem as follows:

Let $A,\ B \in \mathscr{A}$. The lower estimate $d(U_{A,B}) \geqslant \|A\|\,\|B\|$ holds if the two positive operators $\|B\|^2\,(A^*A)$ and $\|A\|^2\,(B^*B)$ are comparable.

THEOREM 15. *Assume that* $\mathscr{A} = \mathscr{B}(H)$ *and let* S *be an invertible element in* $\mathscr{B}(H)$. *Then the lower estimate* $d(U_{A,B}) \geqslant \|A\|\,\|B\| + \frac{1}{\|A\|\|B\|}$ *holds, where* $A = S$ *and* $B = (S^*)^{-1}$.

Proof. It is clear that there exist two sequences $\{x_n\}$ and $\{y_n\}$ of unit vectors in H such that $\|Ax_n\| \to \|A\|$ and $\|By_n\| \to \|B\|$ and $Ax_n \neq 0,\ By_n \neq 0$ for $n \geqslant 1$. Since $A^*B = I$ and from the proof of Theorem 12.(ii), it follows that

$$d(U_{A,B}) \geqslant \|Ax_n\|\,\|By_n\| + \frac{1}{\|Ax_n\|\,\|By_n\|}$$

Letting $n \to \infty$, the result follows immediately. □

COROLLARY 8. *Assume that* $\mathscr{A} = \mathscr{B}(H)$ *and let* S *be an invertible self-adjoint operator in* $\mathscr{B}(H)$. *The lower estimate* $d(\varphi_S) \geqslant \|S\|\,\|S^{-1}\| + \frac{1}{\|S\|\|S^{-1}\|}$ *holds, where* $\varphi_S = U_{S,S^{-1}}$.

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