J. Math. Anal. Appl. 351 (2009) 277–284

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the injective norm and characterization of some subclasses of normal operators by inequalities or equalities

Ameur Seddik

Department of Mathematics, Faculty of Science, King Abdulaziz University, PO Box 80203, Jeddah 21589, Saudi Arabia

ARTICLE INFO

Article history:
Received 17 March 2008
Available online 9 October 2008
Submitted by G. Corach

Dedicated to my wife

Keywords:
Tensor product space
Injective norm
Self-adjoint operator
Unitary operator
Normal operator

ABSTRACT

Let $\mathfrak{B}(H)$ be the C^*-algebra of all bounded linear operators acting on a complex Hilbert space H. In this note, we shall show that if S is an invertible normal operator in $\mathfrak{B}(H)$ the following estimation holds

$$\left\|S\otimes S^{-1}+S^{-1}\otimes S\right\|_{\lambda}\leqslant\|S\|\left\|S^{-1}\right\|+\frac{1}{\|S\|\|S^{-1}\|}$$

where $\|.\|_{\lambda}$ is the injective norm on the tensor product $\mathfrak{B}(H)\otimes\mathfrak{B}(H)$. This last inequality becomes an equality when S is invertible self-adjoint. On the other hand, we shall characterize the set of all invertible normal operators S in $\mathfrak{B}(H)$ satisfying the relation

$$\left\|S\otimes S^{-1}+S^{-1}\otimes S\right\|_{\lambda}=\|S\|\left\|S^{-1}\right\|+\frac{1}{\|S\|\|S^{-1}\|}$$

and also we shall give some characterizations of some subclasses of normal operators in $\mathfrak{B}(H)$ by inequalities or equalities.

1. Introduction

Let $\mathcal{A}$ be a standard operator algebra acting on a (real or complex) normed space (it is a subalgebra of bounded linear operators acting on a normed space that contains all finite rank operators). A linear operator $\mathcal{R}:\mathcal{A}\to\mathcal{A}$ defined by $\mathcal{R}(X)=\sum_{i=1}^{n}A_iXB_i$, where $A_i,B_i\in\mathcal{A}$ $(1\leqslant i\leqslant n)$ is called an elementary operator on $\mathcal{A}$; and it is denoted by $\mathcal{R}=\mathcal{R}_{A,B}$, where $A=(A_1,\dots,A_n)$ and $B=(B_1,\dots,B_n)$.

This concrete class includes many important operators on $\mathcal{A}$, such as the two-sided multiplication $\mathcal{M}_{A,B}:X\to AXB$, the inner derivation $\delta_A:X\to AX-XA$, the generalized derivation $\delta_{A,B}:X\to AX-XB$, the symmetrized two-sided multiplication $\mathcal{U}_{A,B}:X\to AXB+BXA$, for given $A,B\in\mathcal{A}$.

We denote by $\mathcal{A}\otimes\mathcal{A}$ the vector space given by

$$\mathcal{A}\otimes\mathcal{A}=\left\{\sum_{i=1}^{n}A_i\otimes B_i\colon\ n\geqslant1,\ \ A_i,B_i\in\mathcal{A},\ \ i=1,\dots,n\right\}$$

(called tensor product), and by $\mathcal{E}(\mathcal{A})$ the vector space of all elementary operators acting on $\mathcal{A}$. We may algebraically identify $\mathcal{A}\otimes\mathcal{A}$ with $\mathcal{E}(\mathcal{A})$ by the natural map $\Theta:\mathcal{A}\otimes\mathcal{A}\to\mathcal{E}(\mathcal{A})$, $\sum_{i=1}^{n}A_i\otimes B_i\to\Theta(\sum_{i=1}^{n}A_i\otimes B_i)=\mathcal{R}_{A,B}$ (where $A=(A_1,\dots,A_n)$ and $B=(B_1,\dots,B_n)$). We may endowed each of the two last vector spaces with norms that make the map Θ as an isometry. Indeed, the injective norm defined on $\mathcal{A}\otimes\mathcal{A}$ by $\|\sum_{i=1}^{n}A_i\otimes B_i\|_{\lambda}=\sup_{f,g\in(\mathcal{A}')_1}|\sum_{i=1}^{n}f(A_i)g(B_i)|$ (where $(\mathcal{A}')_1$

E-mail address: seddikameur@hotmail.com.

0022-247X/$ – see front matter © 2008 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2008.10.008

denotes the unit sphere of the dual of $\mathcal{A}$) and the norm $d(\mathcal{R}) = \sup\{\|\mathcal{R}(X)\|\colon \|X\| = 1 = \operatorname{rank} X\}$ defined on $\mathcal{E}(\mathcal{A})$ satisfy the relation $d(\Theta(\omega)) = \|\omega\|_\lambda$, for every $\omega \in \mathcal{A} \otimes \mathcal{A}$.

This last result has given recently in [4], and before was established by Magajna and Turnsek [2] in the particular case where $\mathcal{A}$ is the C^*-algebra of all bounded linear operators acting on a complex Hilbert space.

Also in [3,4], we have interested to characterize when the injective norm $d(\mathcal{R}_{A,B}) = \|\sum_{i=1}^n A_i \otimes B_i\|_\lambda$ gets its maximal value $D(\mathcal{R}_{A,B}) = \sum_{i=1}^n \|A_i\|\|B_i\|$, for arbitrary elementary operators and for some particular elementary operators.

In this note, we shall interest in the case where $\mathcal{A} = \mathfrak{B}(H)$ is the C^*-algebra of all bounded linear operators acting on a complex Hilbert space H.

In Section 2, we shall give some lower estimates for the injective norm $d(\mathcal{R}_{A,B}) = \|\sum_{i=1}^n A_i \otimes B_i\|_\lambda$, where A and B are two n-tuples of commuting operators in $\mathfrak{B}(H)$ and we shall characterize this norm for two n-tuples of commuting normal operators.

In Section 3, we apply the results of Section 2 to the injective norm of $S \otimes S^{-1} + S^{-1} \otimes S$ (where S is an invertible operator in $\mathfrak{B}(H)$). We shall show that the upper estimate $\|S \otimes S^{-1} + S^{-1} \otimes S\|_\lambda \leqslant \|S\|\|S^{-1}\| + \frac{1}{\|S\|\|S^{-1}\|}$ holds for every invertible normal operator S in $\mathfrak{B}(H)$. This last inequality becomes an equality when S is invertible self-adjoint. On the other hand, we shall characterize the set of all invertible normal operators S in $\mathfrak{B}(H)$ satisfying the relation

$$\left\|S \otimes S^{-1} + S^{-1} \otimes S\right\|_\lambda = \|S\|\left\|S^{-1}\right\| + \frac{1}{\|S\|\|S^{-1}\|}$$

and also we shall give some characterizations of some subclasses of normal operators in $\mathfrak{B}(H)$ by inequalities or equalities.

Some special notation used in this note (where $A \in \mathfrak{B}(H)$):

(i) $\sigma(A)$ the spectrum of A,
(ii) $r(A) = \sup_{\lambda \in \sigma(A)} |\lambda|$ the spectral radius of A,
(iii) $\sigma_1(A) = \{\alpha \in \sigma(A)\colon |\alpha| = \min_{\lambda \in \sigma(A)} |\lambda|\}$,
(iv) $\sigma_2(A) = \{\alpha \in \sigma(A)\colon |\alpha| = r(A)\}$,
(v) $\mathbf{S}_1$ the set of all unit bounded functionals acting on $\mathfrak{B}(H)$,
(vi) $|\mathbb{K}| = \sup_{\lambda \in \mathbb{K}} |\lambda|$, where $\mathbb{K}$ is a bounded subset of $\mathbb{C}$,
(vii) $L \circ M = \{\sum_{i=1}^n \alpha_i \beta_i\colon (\alpha_1, \ldots, \alpha_n) \in L,\ (\beta_1, \ldots, \beta_n) \in M\}$, where $L \subset \mathbb{C}^n$ and $K \subset \mathbb{C}^n$,
(viii) $|A| = (A^*A)^{1/2}$ the positive square root of A,
(ix) $\mathfrak{I}(H)$, the set of all invertible operators in $\mathfrak{B}(H)$,
(x) $\mathfrak{S}(H)$, the set of all invertible self-adjoint operators in $\mathfrak{B}(H)$,
(xi) $\mathfrak{U}(H)$, the set of all unitary operators in $\mathfrak{B}(H)$,
(xii) $\mathfrak{U}_s(H) = \mathfrak{S}(H) \cap \mathfrak{U}(H)$, the set of all unitary reflection operators in $\mathfrak{B}(H)$,
(xiii) $\mathfrak{N}(H)$, the set of all invertible normal operators in $\mathfrak{B}(H)$,
(xiv) $\mathfrak{L}_1(H) = \{X \in \mathfrak{L}(H)\colon \|X\| = 1\}$, where $\mathfrak{L}(H) \subset \mathfrak{B}(H)$,
(xv) $\mathcal{M}_S = \mathcal{M}_{S,S^{-1}}$ and $\Phi_S = \mathcal{U}_{S,S^{-1}}$, where $S \in \mathfrak{I}(H)$,
(xvi) D_θ the straight line passing through the origin with slope $\tan\theta$, for $\theta \in [0, \pi[$.

For a n-tuple $A = (A_1, \ldots, A_n)$ of commuting operators in $\mathfrak{B}(H)$, we denote by:

(xvii) Γ_A the set of all multiplicative functionals acting on the maximal commutative Banach algebra that contains the operators $A_1, \ldots, A_n$,
(xviii) $\sigma(A) = \{(\varphi(A_1), \ldots, \varphi(A_n))\colon \varphi \in \Gamma_A\}$ the joint spectrum of A.

For a n-tuple $A = (A_1, \ldots, A_n)$ of operators in $\mathfrak{B}(H)$, we denote by:

(xix) $V(A) = \{(f(A_1), \ldots, f(A_n))\colon f \in \mathbf{S}_1,\ f(I) = 1\}$ the joint algebraic numerical range of A.
(xx) For $x, y \in H$, we denote by $x \otimes y$ the operator defined on H by $(x \otimes y)z = \langle z, y\rangle x$ for every $z \in H$.

For the sake of completeness, we refer the reader to the following definition and propositions given in [3,4] which represents the basic source of all results obtained in this note:

Definition 1.

(i) Let $A \in \mathfrak{B}(H)$. A is called normaloid if $\|A\| = r(A)$.
(ii) Let $A, B \in \mathfrak{B}(H)$. We say that A is norm-parallel to B ($A \parallel B$) if $\|A + \lambda B\| = \|A\| + \|B\|$, for some unit scalar λ.

Proposition 1. *(See [4].) Let $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ be two n-tuples of elements in $\mathfrak{B}(H)$. The following equalities hold*:

$$d(\mathcal{R}_{A,B}) = \sup_{f,g\in\mathbf{S}_1}\left|\sum_{i=1}^n f(A_i)g(B_i)\right| = \sup_{f\in\mathbf{S}_1}\left\|\sum_{i=1}^n f(B_i)A_i\right\| = \sup_{f\in\mathbf{S}_1}\left\|\sum_{i=1}^n f(A_i)B_i\right\|.$$

Proposition 2. *(See [4].) Let $A, B \in \mathfrak{B}(H)$. Then $d(\mathcal{U}_{A,B}) = 2\|A\|\|B\|$ if and only if $A \parallel B$.*

Proposition 3. *(See [3].) Let $A \in \mathfrak{B}(H)$. Then $d(\mathcal{U}_{A,A^*}) = 2\|A\|^2$ if and only if A is normaloid.*

2. On the injective norm of $\sum_{i=1}^n A_i \otimes B_i$

Lemma 1. *For every commuting normal operators $A_1, \ldots, A_n$ in $\mathfrak{B}(H)$, and for every scalars $\lambda_1, \ldots, \lambda_n$ the operator $\sum_{i=1}^n \lambda_i A_i$ is normal.*

Proof. The proof follows immediately from Putnam–Fuglede theorem. □

Theorem 1. *Let $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ be two n-tuples of commuting operators in $\mathfrak{B}(H)$. Then $\|\sum_{i=1}^n A_i \otimes B_i\|_\lambda \geqslant |\sigma(A)\circ\sigma(B)|$, and $\|\sum_{i=1}^n A_i \otimes B_i\|_\lambda = |\sigma(A)\circ\sigma(B)|$ if all A_i and B_i are normal operators.*

Proof. Let (φ, ψ) be an arbitrary pair in $\Gamma_A \times \Gamma_B$. Using Hahn–Banach theorem, we may extend φ and ψ to unit functionals f and g on $\mathfrak{B}(H)$, respectively. So it follows from Proposition 1 that $d(\mathcal{R}_{A,B}) \geqslant |\sum_{i=1}^n f(A_i)g(B_i)| = |\sum_{i=1}^n \varphi(A_i)\psi(B_i)|$. Therefore $\|\sum_{i=1}^n A_i \otimes B_i\|_\lambda \geqslant |\sigma(A)\circ\sigma(B)|$.

Now suppose that all A_i and B_i are normal operators. It suffice to prove that $|\sigma(A)\circ\sigma(B)| \geqslant d(\mathcal{R}_{A,B})$. Since $|\sigma(A)\circ\sigma(B)| \geqslant |\psi(\sum_{i=1}^n \varphi(A_i)B_i)|$ and $\sum_{i=1}^n \varphi(A_i)B_i$ is normal, for every $(\varphi,\psi) \in \Gamma_A \times \Gamma_B$, then $|\sigma(A)\circ\sigma(B)| \geqslant \sup_{\psi\in\Gamma_B} |\psi(\sum_{i=1}^n \varphi(A_i)B_i)| = \|\sum_{i=1}^n \varphi(A_i)B_i\|$, for every $\varphi \in \Gamma_A$. Thus $|\sigma(A)\circ\sigma(B)| \geqslant |\sum_{i=1}^n \varphi(A_i)f(B_i)| = |\varphi(\sum_{i=1}^n f(B_i)A_i)|$, for every $\varphi \in \Gamma_A$ and $f \in \mathbf{S}_1$. Hence, $|\sigma(A)\circ\sigma(B)| \geqslant \|\sum_{i=1}^n f(B_i)A_i\|$ for every $f \in \mathbf{S}_1$. So it follows from Proposition 1 that $|\sigma(A)\circ\sigma(B)| \geqslant d(\mathcal{R}_{A,B})$. □

Theorem 2. *Let $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ be two n-tuples of operators in $\mathfrak{B}(H)$. Then $\|\sum_{i=1}^n A_i \otimes B_i\|_\lambda \geqslant |V(A)\circ V(B)|$, and $\|\sum_{i=1}^n A_i \otimes B_i\|_\lambda = |V(A)\circ V(B)|$, if A and B are n-tuples of normal commuting operators.*

Proof. The inequality follows immediately from the definition of the joint numerical range and from Proposition 1.

The equality follows using the same argument as in the above proof. □

3. On the injective norm of $S \otimes S^{-1} + S^{-1} \otimes S$

For every $S \in \mathfrak{I}(H)$ it is known that:

(i) there exits $V \in \mathfrak{U}(H)$ such that $S = V|S|$ (polar decomposition of S),
(ii) S is normal if and only if $V|S| = |S|V$,
(iii) S is self-adjoint if and only if $V \in \mathfrak{U}_s(H)$ and $V|S| = |S|V$.

Theorem 3. *Let $S \in \mathfrak{I}(H)$. Then we have*

$$\text{(i)}\quad \left\|S\otimes S^{-1} + S^{-1}\otimes S\right\|_\lambda \geqslant \sup_{\lambda,\mu\in\sigma(S)}\left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right|.$$

If S is normal, the above inequality becomes equality, and the following equality holds

$$\text{(ii)}\quad \left\|S^*\otimes S^{-1} + S^{-1}\otimes S^*\right\|_\lambda = \sup_{\lambda,\mu\in\sigma(S)}\left(\left|\frac{\lambda}{\mu}\right| + \left|\frac{\mu}{\lambda}\right|\right).$$

Proof. The proof follows immediately from Theorem 1. □

Remark 1. Using the fact that $\sigma(\mathcal{M}_S) = \sigma(S)\sigma(S^{-1})$ and $\sigma(\Phi_S) = \{\varphi(\mathcal{M}_S) + \frac{1}{\varphi(\mathcal{M}_S)}\colon \varphi \in \Gamma\}$ (where Γ is the set of all multiplicative functionals on the maximal commutative Banach algebra that contains $\mathcal{M}_S$), it is easy to see that

$$\text{(i)}\quad \sup_{\lambda,\mu\in\sigma(S)}\left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right| = \sup_{z\in\sigma(\mathcal{M}_S)}\left|z + \frac{1}{z}\right| = r(\Phi_S),$$

$$\text{(ii)}\quad \sup_{\lambda,\mu\in\sigma(S)}\left(\left|\frac{\lambda}{\mu}\right| + \left|\frac{\mu}{\lambda}\right|\right) = \sup_{z\in\sigma(\mathcal{M}_S)}\left(|z| + \left|\frac{1}{z}\right|\right).$$

Corollary 1. *Let P be an invertible positive operator in $\mathfrak{B}(H)$. Then we have*

$$\left\|P\otimes P^{-1}+P^{-1}\otimes P\right\|_\lambda=\|P\|\left\|P^{-1}\right\|+\frac{1}{\|P\|\|P^{-1}\|}.$$

Proof. From the above theorem and remark, it follows that $d(\Phi_P)=\sup_{t\in\sigma(\mathcal{M}_P)}(t+\frac{1}{t})$. It is clear that $\min\sigma(P)=\frac{1}{\|P^{-1}\|}$ and $\max\sigma(P)=\|P\|$, and since $\sigma(\mathcal{M}_P)=\sigma(P)\sigma(P^{-1})$, then $\min\sigma(\mathcal{M}_P)=\frac{1}{\|P\|\|P^{-1}\|}=p$ and $\max\sigma(\mathcal{M}_P)=\|P\|\|P^{-1}\|=\frac{1}{p}$. It is easy to see that $\max\{t+\frac{1}{t}:\ p\leqslant t\leqslant\frac{1}{p}\}=p+\frac{1}{p}$, this maximum is attainable in p and $\frac{1}{p}$. Thus, the result follows immediately from the fact that $p\in\sigma(\mathcal{M}_P)$. □

Remark 2. Let $S\in\mathfrak{I}(H)$.

(i) It follows immediately from the above theorem that $\|S\otimes S^{-1}+S^{-1}\otimes S\|_\lambda\geqslant 2$ and $\|S^*\otimes S^{-1}+S^{-1}\otimes S^*\|_\lambda\geqslant 2$.
(ii) It is easy to see that the two last inequalities become equalities when S is unitary.
(iii) If S is normal, then $\|S\otimes S^{-1}+S^{-1}\otimes S\|_\lambda=2$ if and only if $|\frac{\lambda}{\mu}+\frac{\mu}{\lambda}|\leqslant 2$ for every $\lambda,\mu\in\sigma(S)$.
(iv) If $\|S\otimes S^{-1}+S^{-1}\otimes S\|_\lambda=2$, then the interior of the spectrum of S is empty. Indeed, since $|\frac{\lambda}{\mu}+\frac{\mu}{\lambda}|\leqslant 2$ for every $\lambda,\mu\in\sigma(S)$, then every straight line D_θ $(0\leqslant\theta<\pi)$ intercept $\sigma(S)$ in at most two points.

Theorem 4. *The following properties hold*:

$$\text{(i)}\quad \forall S\in\mathfrak{I}(H),\quad \left\|S^*\otimes S^{-1}+S^{-1}\otimes S^*\right\|_\lambda=\|S\|\left\|S^{-1}\right\|+\frac{1}{\|S\|\|S^{-1}\|},$$

$$\text{(ii)}\quad \forall S\in\mathfrak{S}(H),\quad \left\|S\otimes S^{-1}+S^{-1}\otimes S\right\|_\lambda=\|S\|\left\|S^{-1}\right\|+\frac{1}{\|S\|\|S^{-1}\|},$$

$$\text{(iii)}\quad \forall S\in\mathfrak{N}(H),\quad \left\|S\otimes S^{-1}+S^{-1}\otimes S\right\|_\lambda\leqslant\|S\|\left\|S^{-1}\right\|+\frac{1}{\|S\|\|S^{-1}\|}.$$

Proof. (i) Let $S\in\mathfrak{I}(H)$. Then there exists $V\in\mathfrak{U}(H)$ such that $S=VP$ (where $P=|S|$). From Corollary 1 and from the fact that $\{X\in\mathcal{B}_1(H):\ \operatorname{rank}X=1\}=\{V^*X:\ X\in\mathcal{B}_1(H),\ \operatorname{rank}X=1\}$, and $\|S\|=\|P\|$, $\|S^{-1}\|=\|P^{-1}\|$ it follows that

$$\begin{aligned}\left\|S^*\otimes S^{-1}+S^{-1}\otimes S^*\right\|_\lambda&=\sup_{\|X\|=1=\operatorname{rank}X}\left\|S^*XS^{-1}+S^{-1}XS^*\right\|\\&=\sup_{\|X\|=1=\operatorname{rank}X}\left\|PV^*XP^{-1}V^*+P^{-1}V^*XPV^*\right\|\\&=\sup_{\|X\|=1=\operatorname{rank}X}\left\|P(V^*X)P^{-1}+P^{-1}(V^*X)P\right\|\\&=\sup_{\|X\|=1=\operatorname{rank}X}\left\|PXP^{-1}+P^{-1}XP\right\|\\&=\left\|P\otimes P^{-1}+P^{-1}\otimes P\right\|_\lambda\\&=\|P\|\left\|P^{-1}\right\|+\frac{1}{\|P\|\|P^{-1}\|}\\&=\|S\|\left\|S^{-1}\right\|+\frac{1}{\|S\|\|S^{-1}\|}.\end{aligned}$$

(ii) follows immediately from (i).

(iii) Let $S\in\mathfrak{N}(H)$. So from Theorem 3(i), it follows that $\|S\otimes S^{-1}+S^{-1}\otimes S\|_\lambda=\sup_{\lambda,\mu\in\sigma(S)}|\frac{\lambda}{\mu}+\frac{\mu}{\lambda}|$. Hence $\|S\otimes S^{-1}+S^{-1}\otimes S\|_\lambda\leqslant\sup_{\lambda,\mu\in\sigma(S)}(|\frac{\lambda}{\mu}|+|\frac{\mu}{\lambda}|)$. Thus from Theorem 3(ii) and the above property (i), we obtain $\|S\otimes S^{-1}+S^{-1}\otimes S\|_\lambda\leqslant\|S\|\|S^{-1}\|+\frac{1}{\|S\|\|S^{-1}\|}$. □

Remark 3. (i) The inequality in the above theorem may be strict. Indeed, for the invertible normal operator $S=\left[\begin{smallmatrix}1&0\\0&\frac{1+i}{2}\end{smallmatrix}\right]$ in $\mathcal{B}(\mathbb{C}^2)$, by a simple computation we find that

$$2=\left\|S\otimes S^{-1}+S^{-1}\otimes S\right\|_\lambda<\|S\|\left\|S^{-1}\right\|+\frac{1}{\|S\|\|S^{-1}\|}=\frac{3\sqrt{2}}{2}.$$

(ii) Denote by

$$\mathcal{R}(H)=\left\{S\in\mathfrak{N}(H):\ \left\|S\otimes S^{-1}+S^{-1}\otimes S\right\|_\lambda=\|S\|\left\|S^{-1}\right\|+\frac{1}{\|S\|\|S^{-1}\|}\right\}.$$

It is easy to see that $\mathbb{C}^*\mathfrak{U}(H) \cup \mathbb{C}^*\mathfrak{S}(H) \subset \mathcal{R}(H)$, and then if $\dim H \geqslant 2$, the inclusions $\mathbb{C}^*\mathfrak{U}(H) \subset \mathcal{R}(H)$, $\mathbb{C}^*\mathfrak{S}(H) \subset \mathcal{R}(H)$ are strict. In the following theorem, we give a complete characterization of the set $\mathcal{R}(H)$ using the spectral properties of normal operators.

Theorem 5. *Let $S \in \mathfrak{N}(H)$. Then the following properties are equivalent*:

(i) $\|S \otimes S^{-1} + S^{-1} \otimes S\|_\lambda = \|S\|\|S^{-1}\| + \frac{1}{\|S\|\|S^{-1}\|}$,
(ii) $\exists \theta \in [0, \pi[$, $D_\theta \cap \sigma_1(S) \neq \emptyset$, $D_\theta \cap \sigma_2(S) \neq \emptyset$.

Proof. (i) $\Rightarrow$ (ii). From Theorem 3(i) and from the compactness of $\sigma(S)$, we may choose λ, μ in $\sigma(S)$ such that $d(\Phi_S) = |\frac{\lambda}{\mu} + \frac{\mu}{\lambda}| = \|S\|\|S^{-1}\| + \frac{1}{\|S\|\|S^{-1}\|}$. So from Theorem 3(ii) and Theorem 4(i), we obtain that $\|S\|\|S^{-1}\| + \frac{1}{\|S\|\|S^{-1}\|} \leqslant |\frac{\lambda}{\mu}| + |\frac{\mu}{\lambda}| \leqslant \|S\|\|S^{-1}\| + \frac{1}{\|S\|\|S^{-1}\|}$. Thus $|\frac{\lambda}{\mu}| + |\frac{\mu}{\lambda}| = \|S\|\|S^{-1}\| + \frac{1}{\|S\|\|S^{-1}\|}$. Since S is normal, we may choose λ, μ in $\sigma(S)$ such that $|\lambda| = \|S\|$ and $|\mu| = \frac{1}{\|S^{-1}\|}$. Then, put $\lambda = \|S\|e^{i\theta}$ and $\mu = \frac{1}{\|S^{-1}\|}e^{i\varphi}$ for some reals θ, φ. Hence,

$$d(\Phi_S) = \left|\|S\|\left\|S^{-1}\right\|e^{i(\theta-\varphi)} + \frac{1}{\|S\|\|S^{-1}\|}e^{-i(\theta-\varphi)}\right| = \|S\|\left\|S^{-1}\right\| + \frac{1}{\|S\|\|S^{-1}\|}.$$

So it follows immediately that $\cos 2(\theta - \varphi) = 1$. Then $\theta \equiv \varphi[\pi]$. Therefore, the first implication follows immediately.

(ii) $\Rightarrow$ (i). Let $\alpha \in D_\theta \cap \sigma_1(S)$ and $\beta \in D_\theta \cap \sigma_2(S)$. Since S is normal, it follows that $\alpha = \frac{e^{i\theta}}{\|S^{-1}\|}$, $\beta = \|S\|e^{i(\theta+k\pi)}$, where $k \in \{0, 1\}$. Then $\|S \otimes S^{-1} + S^{-1} \otimes S\|_\lambda \geqslant |\frac{\alpha}{\beta} + \frac{\beta}{\alpha}| = \|S\|\|S^{-1}\| + \frac{1}{\|S\|\|S^{-1}\|}$. On the other hand, since S is normal and from Theorem 3 and Theorem 4(i), it follows that $\|S \otimes S^{-1} + S^{-1} \otimes S\|_\lambda \leqslant \sup_{\lambda,\mu\in\sigma(S)}(|\frac{\lambda}{\mu}| + |\frac{\mu}{\lambda}|) = \|S\|\|S^{-1}\| + \frac{1}{\|S\|\|S^{-1}\|}$. Therefore (i) follows immediately. □

Corollary 2.

(i) *If $\dim H \leqslant 2$, then $\mathcal{R}(H) = \mathbb{C}^*\mathfrak{U}(H) \cup \mathbb{C}^*\mathfrak{S}(H)$.*
(ii) *If $\dim H \geqslant 3$, the inclusion $\mathbb{C}^*\mathfrak{U}(H) \cup \mathbb{C}^*\mathfrak{S}(H) \subset \mathcal{R}(H)$ is strict.*

4. Characterization on the unitary reflection and unitary operators

We denote by

$$\mathfrak{D}(H) = \left\{S \in \mathfrak{I}(H)\colon\ \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| \geqslant 2\|X\|\right\}.$$

In [1], Corach, Porta and Recht proved that $\mathfrak{S}(H) \subset \mathfrak{D}(H)$; and since $\Phi_{\lambda S} = \Phi_S$ for every $\lambda \in \mathbb{C}^*$ and $S \in \mathfrak{I}(H)$, so it is easy to see that $\mathbb{C}^*\mathfrak{S}(H) \subset \mathfrak{D}(H)$.

In [5], we showed that this last inclusion is exactly an equality. That means

$$\mathbb{C}^*\mathfrak{S}(H) = \left\{S \in \mathfrak{I}(H)\colon\ \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| \geqslant 2\|X\|\right\}.$$

So the characterization of the class of operators $\mathbb{C}^*\mathfrak{S}(H)$ by inequalities was established. Using the polar decomposition of an invertible operator and the last formula it follows immediately that

$$\forall S \in \mathfrak{I}(H),\ \forall X \in \mathfrak{B}(H),\quad \left\|S^*XS^{-1} + S^{-1}XS^*\right\| \geqslant 2\|X\|. \tag{$*$}$$

Then we may deduce easily the two other characterizations of $\mathbb{C}^*\mathfrak{S}(H)$ given in the following proposition.

Proposition 4. *Let $S \in \mathfrak{I}(H)$. Then the following properties are equivalent*:

(i) $\forall X \in \mathfrak{B}(H)$, $\|SXS^{-1} + S^{-1}XS\| = \|S^*XS^{-1} + S^{-1}XS^*\|$,
(ii) $\forall X \in \mathfrak{B}(H)$, $\|SXS^{-1} + S^{-1}XS\| \geqslant \|S^*XS^{-1} + S^{-1}XS^*\|$,
(iii) $S \in \mathbb{C}^*\mathfrak{S}(H)$.

Proof. (i) $\Rightarrow$ (ii). The implication is trivial.

(ii) $\Rightarrow$ (iii). Let $S = VP$ the polar decomposition of S. Then we have

$$\forall X \in \mathfrak{B}(H)\quad \left\|SXS^{-1} + S^{-1}XS\right\| \geqslant \left\|S^*XS^{-1} + S^{-1}XS^*\right\| = \left\|P(V^*X)P^{-1} + P^{-1}(V^*X)P\right\| \geqslant 2\|V^*X\| = 2\|X\|.$$

Hence $S \in \mathbb{C}^*\mathfrak{S}(H)$.

(iii) $\Rightarrow$ (i). The implication is trivial. □

Also, it is easy to see that the set $\mathbb{C}^*\mathfrak{S}(H)$ is exactly the set of all invertible normal operators $S \in \mathfrak{B}(H)$ satisfying the condition $\sigma(S) \subset D_\theta$, for some $\theta \in [0, \pi[$ (see [5, Lemma 4.3]). Without this last condition, the characterization of set $\mathfrak{N}(H)$ of all invertible normal operators in $\mathfrak{B}(H)$ is given in the following proposition:

Proposition 5. *Let $S \in \mathfrak{I}(H)$. Then the following properties are equivalent*:

(i) $S \in \mathfrak{N}(H)$,
(ii) $\forall X \in \mathfrak{B}(H)$, $\|SXS^{-1}\| + \|S^{-1}XS\| = \|S^*XS^{-1}\| + \|S^{-1}XS^*\|$,
(iii) $\forall X \in \mathfrak{B}(H)$, $\|SXS^{-1}\| + \|S^{-1}XS\| \geqslant 2\|X\|$.

Proof. It is clear that the condition (i) is equivalent to each of the two following properties:

(1) $\forall X \in \mathfrak{B}(H)$, $\|SX\| = \|S^*X\|$,
(2) $\forall X \in \mathfrak{B}(H)$, $\|XS\| = \|XS^*\|$.

(i) $\Rightarrow$ (ii). The implication follows immediately from (1) and (2).
(ii) $\Rightarrow$ (iii). Using the property $(*)$, it follows immediately that

$$\forall X \in \mathfrak{B}(H) \quad \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| = \left\|S^*XS^{-1}\right\| + \left\|S^{-1}XS^*\right\| \geqslant \left\|S^*XS^{-1} + S^{-1}XS^*\right\| \geqslant 2\|X\|.$$

Therefore (iii) holds.
(iii) $\Rightarrow$ (i). Let $S = VP$ and $S^* = UQ$ be the polar decomposition of S and S^* (where $P = |S|$ and $Q = |S^*|$). It is easy to see that the condition (iii) is equivalent to the following condition:

(iv) $\forall X \in \mathfrak{B}(H)$, $\|PXP^{-1}\| + \|Q^{-1}XQ\| \geqslant 2\|X\|$.

Since $\sigma(P^2) = \sigma(Q^2)$, then from the spectral theorem it follows that $\sigma(P) = \sigma(Q)$. Using this last equality and the condition (iv), and applying [5, Theorem 3.6] it follows that $P = Q$. Therefore $S^*S = SS^*$. □

Remark 4. It follows from the above proposition that

$$\begin{aligned}\mathfrak{N}(H) &= \left\{S \in \mathfrak{I}(H)\colon\ \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| = \left\|S^*XS^{-1}\right\| + \left\|S^{-1}XS^*\right\|\right\}\\ &= \left\{S \in \mathfrak{I}(H)\colon\ \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geqslant 2\|X\|\right\}.\end{aligned}$$

In the rest of this section, we shall characterize another important subclasses of $\mathfrak{N}(H)$. We start with the following lemma which will be used in all the following characterizations.

Lemma 2.

(i) *Let $S \in \mathfrak{S}(H)$. Then $\|S\|\|S^{-1}\| = 1$ if and only if $S = \|S\|V$, for some $V \in \mathfrak{U}_s(H)$.*
(ii) *Let $S \in \mathfrak{I}(H)$. Then $\|S\|\|S^{-1}\| = 1$ if and only if $S = \|S\|V$, for some $V \in \mathfrak{U}(H)$.*

Proof. The proof is trivial. □

Theorem 6. *Let $S \in \mathfrak{I}(H)$. Then the following properties are equivalent*:

(i) $\forall X \in \mathfrak{B}(H)$, $\|S^*XS^{-1} + S^{-1}XS^*\| = 2\|X\|$,
(ii) $\|S^* \otimes S^{-1} + S^{-1} \otimes S^*\|_\lambda = 2$,
(iii) $S \in \mathbb{R}^*\mathfrak{U}(H)$,
(iv) $\forall X \in \mathfrak{B}(H)$, $\|SXS^{-1}\| + \|S^{-1}XS\| = 2\|X\|$.

Proof. (i) $\Rightarrow$ (ii). The implication is trivial.
(ii) $\Rightarrow$ (iii). Using Theorem 4(i), it follows that $2 = \|S^* \otimes S^{-1} + S^{-1} \otimes S^*\|_\lambda = \|S\|\|S^{-1}\| + \frac{1}{\|S\|\|S^{-1}\|}$. Hence $\|S\|\|S^{-1}\| = 1$. Therefore (iii) follows immediately from Lemma 2(ii).
(iii) $\Rightarrow$ (iv). The implication is trivial.
(iv) $\Rightarrow$ (i). Using the above proposition it follows that S is normal. So from the same proposition and the property $(*)$, we may deduce that

$$\forall X \in \mathfrak{B}(H) \quad 2\|X\| = \|SXS^{-1}\| + \|S^{-1}XS\| = \|S^*XS^{-1}\| + \|S^{-1}XS^*\| \geqslant \|S^*XS^{-1} + S^{-1}XS^*\| \geqslant 2\|X\|.$$

Therefore (i) holds. □

Remark 5. It follows from the above theorem that

$$\begin{aligned}\mathfrak{U}(H) &= \{S \in \mathfrak{I}_1(H)\colon\ \forall X \in \mathfrak{B}(H),\ \|S^*XS^{-1} + S^{-1}XS^*\| = 2\|X\|\} \\ &= \{S \in \mathfrak{I}_1(H)\colon\ \forall X \in \mathfrak{B}(H),\ \|SXS^{-1}\| + \|S^{-1}XS\| = 2\|X\|\} \\ &= \{S \in \mathfrak{I}_1(H)\colon\ \|S^* \otimes S^{-1} + S^{-1} \otimes S^*\|_\lambda = 2\}.\end{aligned}$$

Theorem 7. *Let $S \in \mathcal{R}(H)$. Then the following properties are equivalent*

(i) $\forall X \in \mathfrak{B}(H)$, $\|SXS^{-1} + S^{-1}XS\| \leqslant 2\|X\|$,
(ii) $\|S \otimes S^{-1} + S^{-1} \otimes S\|_\lambda = 2$,
(iii) $S \parallel S^{-1}$,
(iv) $S \in \mathbb{R}^*\mathfrak{U}(H)$.

Proof. (i) ⇒ (ii). This implication is trivial.

(ii) ⇒ (iii). It is clear that $2 = d(\Phi_S) = \|S\|\|S^{-1}\| + \frac{1}{\|S\|\|S^{-1}\|}$, and hence $\|S\|\|S^{-1}\| = 1$. So from Lemma 2(ii) we obtain that $S = \|S\|V$, where $V \in \mathfrak{U}(H)$. Since V is normal, then from Proposition 3, it follows that $d(\mathcal{U}_{S,S^{-1}}) = d(\Phi_S) = d(\Phi_V) = d(\mathcal{U}_{V^*,V}) = 2\|V\|^2 = 2\|S\|\|S^{-1}\|$. Therefore, from Proposition 2, it follows that $S \parallel S^{-1}$.

(iii) ⇒ (iv). Since $2\|S\|\|S^{-1}\| = d(\Phi_S) = \|S\|\|S^{-1}\| + \frac{1}{\|S\|\|S^{-1}\|}$, then $\|S\|\|S^{-1}\| = 1$. Therefore (iv) follows immediately from Lemma 2(ii).

(iv) ⇒ (i). This implication is trivial. □

Remark 6. It follows from the above that

$$\begin{aligned}\mathfrak{U}(H) &= \{S \in \mathcal{R}_1(H)\colon\ \forall X \in \mathfrak{B}(H),\ \|SXS^{-1} + S^{-1}XS\| \leqslant 2\|X\|\} \\ &= \{S \in \mathcal{R}_1(H)\colon\ \|S \otimes S^{-1} + S^{-1} \otimes S\|_\lambda = 2\} \\ &= \{S \in \mathcal{R}_1(H)\colon\ S \parallel S^{-1}\}.\end{aligned}$$

Problem 1. Is it true that $\mathfrak{U}(H)$ is characterized by

$$\mathfrak{U}(H) = \{S \in \mathfrak{I}_1(H)\colon\ \forall X \in \mathfrak{B}(H),\ \|SXS^{-1} + S^{-1}XS\| \leqslant 2\|X\|\}?$$

Theorem 8. *Let $S \in \mathfrak{S}(H)$. Then the following properties are equivalent*

(i) $\forall X \in \mathfrak{B}(H)$, $\|SXS^{-1} + S^{-1}XS\| = 2\|X\|$,
(ii) $\|S \otimes S^{-1} + S^{-1} \otimes S\|_\lambda = 2$,
(iii) $S \parallel S^{-1}$,
(iv) $S \in \mathbb{R}^*\mathfrak{U}_s(H)$.

Proof. The proof is similar than the proof of Theorem 7, only we use Lemma 2(i) instead of Lemma 2(ii). □

Remark 7. (i) It follows from the above theorem that the class of unitary reflection operators in $\mathcal{B}(H)$ is characterized by

$$\begin{aligned}\mathfrak{U}_s(H) &= \{S \in \mathfrak{S}_1(H)\colon\ \forall X \in \mathcal{B}(H),\ \|SXS^{-1} + S^{-1}XS\| = 2\|X\|\} \\ &= \{S \in \mathfrak{S}_1(H)\colon\ \|S \otimes S^{-1} + S^{-1} \otimes S\|_\lambda = 2\} \\ &= \{S \in \mathfrak{S}_1(H)\colon\ S \parallel S^{-1}\}.\end{aligned}$$

(ii) From the above theorem and from the characterization of the class $\mathbb{C}^*\mathfrak{S}(H)$, we find easily that the class $\mathbb{C}^*\mathfrak{U}_s(H)$ is characterized by

$$\mathbb{C}^*\mathfrak{U}_s(H) = \{S \in \mathfrak{I}(H)\colon\ \forall X \in \mathcal{B}(H),\ \|SXS^{-1} + S^{-1}XS\| = 2\|X\|\}.$$

(iii) From above, the relation "$S \parallel S^{-1}$, $S \in \mathfrak{S}_1(H)$" characterize the set $\mathfrak{U}_s(H)$; and the relation "$S \parallel S^{-1}$, $S \in \mathcal{R}_1(H)$" characterize the set $\mathfrak{U}(H)$. It has been showed in [3, Theorem 9] that the relation "$A \parallel A^*$, $A \in \mathfrak{B}(H)$" characterize the set of all normaloid operators in $\mathfrak{B}(H)$.

Acknowledgment

I would like to thank the referee for his/her careful reading.

References

[1] G. Corach, H. Porta, L. Recht, An operator inequality, Linear Algebra Appl. 142 (1990) 153–158.
[2] B. Magajna, A. Turnsek, On the norm of symmetrized two-sided multiplications, Bull. Austral. Math. Soc. 67 (2003) 27–38.
[3] A. Seddik, On the injective norm of $\sum_{i=1}^{n} A_i \otimes B_i$ and characterization of normaloid operators, Oper. Matrices 2 (2008) 67–77.
[4] A. Seddik, Rank one operators and norm of elementary operators, Linear Algebra Appl. 424 (2007) 177–183.
[5] A. Seddik, Some results related to Corach–Porta–Recht inequality, Proc. Amer. Math. Soc. 129 (2001) 3009–3015.