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Rank one operators and norm of elementary operators

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Dedicated to Roger Horn on the occasion of his 65th birthday

Abstract

Let $\mathscr{A}$ be a standard operator algebra acting on a (real or complex) normed space E. For two n-tuples $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ of elements in $\mathscr{A}$, we define the elementary operator $R_{A,B}$ on $\mathscr{A}$ by the relation $R_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i$ for all X in $\mathscr{A}$. For a single operator $A \in \mathscr{A}$, we define the two particular elementary operators L_A and R_A on $\mathscr{A}$ by $L_A(X) = AX$ and $R_A(X) = XA$, for every X in $\mathscr{A}$. We denote by $d(R_{A,B})$ the supremum of the norm of $R_{A,B}(X)$ over all unit rank one operators on E. In this note, we shall characterize: (i) the supremun $d(R_{A,B})$, (ii) the relation $d(R_{A,B}) = \sum_{i=1}^{n} \|A_i\|\|B_i\|$, (iii) the relation $d(L_A - R_B) = \|A\| + \|B\|$, (iv) the relation $d(L_A R_B + L_B R_A) = 2\|A\|\|B\|$. Moreover, we shall show the lower estimate $d(L_A - R_B) \geqslant \max\{\sup_{\lambda \in V(B)} \|A - \lambda I\|, \sup_{\lambda \in V(A)} \|B - \lambda I\|\}$ (where $V(X)$ is the algebraic numerical range of X in $\mathscr{A}$).

1. Introduction

Let E be a normed space (not necessarily a Banach space) over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), let $B(E)$ be the normed algebra of all bounded linear operators acting on E and let $\mathscr{A}$ denote a standard operator algebra of $B(E)$ (it is a subalgebra of $B(E)$ that contains all finite rank operators on E).

For two n-tuples $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ of elements in $\mathscr{A}$, we define the elementary operator $R_{A,B}$ on $\mathscr{A}$ by $R_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i$ (if one of the operators A_i, B_i is equal to the identity I, $\mathscr{A}$ contains I).

For a single operator $A \in \mathscr{A}$, we define the following two particular elementary operators L_A and R_A on $\mathscr{A}$ by $L_A(X) = AX$ and $R_A(X) = XA$ for every X in $\mathscr{A}$ (are called the left

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multiplication and right multiplication by A, respectively); and also we define the other particular elementary operators: (i) the generalized derivation $\delta_{A,B} = L_A - R_B$, (ii) the inner derivation $\delta_A = \delta_{A,A}$, (iii) the multiplication operator $M_{A,B} = L_A R_B$, (iv) the operator $\varDelta_A = L_A + R_A$, (v) the operator $U_{A,B} = M_{A,B} + M_{B,A}$, (vi) the operator $V_{A,B} = M_{A,B} - M_{B,A}$.

Let F be any (real or complex) normed space. In this note we adopt the following notations and definitions:

(i) We denote by F' the topological dual space of F and by $(F)_1$ the unit sphere of F.
(ii) If F is an inner product space and $x, y \in F$, the relation $x \perp y$ holds if and only if $\inf_{\lambda \in \mathbb{K}} \|y + \lambda x\| = \|y\|$ (or equivalentely to $\|y + \lambda x\| \geqslant \|y\|$, for every λ in $\mathbb{K}$). This last condition makes sense in any normed space and therefore may be taken as a definition of the relation of the orthogonality in this general situation. In this general case, it is clear by using Hahn–Banach Theorem, that the relation $x \perp y$ holds if and only if there exists a unit element f in F' such that $f(x) = 0$ and $f(y) = \|y\|$ (this relation is not symmetric in a general situation of a normed space).
(iii) If F is an inner product space and $x, y \in F$, then the relation $x \parallel y$ (that means x, y are linearly dependent) holds if and only if $\|x + \lambda y\| = \|x\| + \|y\|$ for some unit scalar λ. The two conditions make sense on a normed space and the first condition implies the second but the converse is false in general. So we may adopt as definition of the parallelism relation in normed space as follows $x \parallel y$ if and only if $\|x + \lambda y\| = \|x\| + \|y\|$ for some unit scalar λ.

Let Ω be any (real or complex) normed algebra with unit I and let $A \in \Omega$. We define the algebraic numerical range of A by $V(A) = \{f(A) : f \in \mathscr{P}(\Omega)\}$, where $\mathscr{P}(\Omega) = \{f \in \Omega' : f(I) = \|f\| = 1\}$ (the elements of $\mathscr{P}(\Omega)$ are called states), and the numerical radius of A by $w(A) = \sup\{|\lambda| : \lambda \in V(A)\}$. It is known that $V(A)$ is non-empty, closed and convex (for more details see [3]). We put $V_{\mathrm{N}}(A) = V\left(\frac{A}{\|A\|}\right)$ for a non-zero element A in Ω (the normalized algebraic numerical range of A). A is called normaloid if $w(A) = \|A\|$. If $\Omega = B(E)$ and E is a complex Hilbert space, then A is normaloid if and only if $r(A) = \|A\|$ (where $r(A)$ is the spectral radius of A, see [7]).

For $(x, f) \in E \times E'$, we define the operator $x \otimes f$ on E by $(x \otimes f)y = f(y)x$. We denote by $\mathscr{F}_1$ the set of all unit rank one operators acting on E (it is clear that $\mathscr{F}_1 = \{x \otimes f : \|x\| = \|f\| = 1\}$), and by $d(R) = \sup_{X \in \mathscr{F}_1} \|R(X)\|$, for every $R \in B(\mathscr{A})$.

The norm problem for elementary operators consists in finding a formula which describes the norm of an elementary operator in terms of its coefficients. It is easy to see that the upper estimate $\|R_{A,B}\| \leqslant \sum_{i=1}^{n} \|A_i\| \|B_i\|$ is valid for any elementary operator, so the norm of any elementary operator is between 0 and $D(R_{A,B})$ (minimal and maximal value), where $D(R_{A,B})$ denotes the second member of this last estimation. It is clear that $d(R_{A,B}) \leqslant \|R_{A,B}\| \leqslant D(R_{A,B})$. The lower estimate for the particular elementary operator $U_{A,B}$ is studied by several authors in several algebras (see [1,4,5,8–10]). Recently, the best lower estimate of this operator acting on a Hilbert space is given in the two papers [2,14], that is $\|U_{A,B}\| \geqslant \|A\| \|B\|$. On the other hand in Hilbert space case Stampfli [13] has characterized the norm of $\delta_{A,B}$ by the relation $\|\delta_{A,B}\| = \inf\{\|A - \lambda I\| + \|B - \lambda I\| : \lambda \in \mathbb{C}\}$, and he has proved that $\|\delta_{A,B}\| = D(\delta_{A,B})$ if and only if $W_{\mathrm{N}}(A) \cap W_{\mathrm{N}}(-B) \neq \emptyset$ (where W_{N} denotes the normalized maximal numerical range and A and B are non-zero).

In this note, in Section 2, we shall characterize the supremum $d(R_{A,B})$ when it gets the maximal value $D(R_{A,B})$. It is clear that the condition $d(R_{A,B}) = D(R_{A,B})$ implies $\|R_{A,B}\| = D(R_{A,B})$ (we shall show that the converse is not true in general). We shall deduce for every non-zero elements A and B in $\mathscr{A}$ that $d(\delta_{A,B}) = D(\delta_{A,B})$ if and only if $V_{\mathrm{N}}(A) \cap V_{\mathrm{N}}(-B) \cap (\mathbb{K})_1 \neq \emptyset$,

$d(U_{A,B}) = D(U_{A,B})$ if and only if $A \parallel B$, $d(\Delta_A) = D(\Delta_A)$ if and only if $A \parallel I$, and if and only if A is normaloid (this gives a characterization of normaloid operators in $\mathscr{A}$).

In Section 3, we shall characterize when the norm of $R_{A,B}$ gets its minimal value 0.

In Section 4, we are interested to give some lower estimate for $d(\delta_{A,B})$ and for $d(U_{A,B})$. It is clear that every lower estimate for $d(R)$ is also a lower estimate for $\|R\|$, for every R in $B(\mathscr{A})$. We shall show the lower estimate $d(\delta_{A,B}) \geqslant \max\{\sup_{\lambda\in V(B)} \|A - \lambda I\|, \sup_{\lambda\in V(A)} \|B - \lambda I\|\}$ for every A, B in $\mathscr{A}$, and some other consequences.

2. Characterization of the relation $d(R_{A,B}) = D(R_{A,B})$

Theorem 2.1. *Let $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ be two n-tuples of elements in $\mathscr{A}$. The following equalities hold*:

$$\begin{aligned} d(R_{A,B}) &= \sup_{f,g\in(\mathscr{A}')_1} \left| \sum_{i=1}^{n} f(A_i) g(B_i) \right|, \\ &= \sup_{f\in(\mathscr{A}')_1} \left\| \sum_{i=1}^{n} f(B_i) A_i \right\|, \\ &= \sup_{f\in(\mathscr{A}')_1} \left\| \sum_{i=1}^{n} f(A_i) B_i \right\|. \end{aligned}$$

Proof. We denote by k_1, k_2 and k_3 the above supremum cited in Theorem 2.1 in the same order. Let $x, y \in (E)_1$, $h \in (E')_1$ and $f, g \in (\mathscr{A}')_1$. Since $d(R_{A,B}) \geqslant \left\| \sum_{i=1}^n A_i(x\otimes h)B_i y \right\| = \left\| \left(\sum_{i=1}^n h(B_i y)A_i \right) x \right\|$, then $d(R_{A,B}) \geqslant \left\| \sum_{i=1}^n h(B_i y)A_i \right\|$. So that $d(R_{A,B}) \geqslant \left| \sum_{i=1}^n f(A_i) \times h(B_i y) \right| = \left| h\left(\sum_{i=1}^n f(A_i)B_i y \right) \right|$, then from Hahn–Banach Theorem it follows that $d(R_{A,B}) \geqslant \left\| \left(\sum_{i=1}^n f(A_i)B_i \right) y \right\|$. Thus $d(R_{A,B}) \geqslant \left\| \sum_{i=1}^n f(A_i)B_i \right\|$ and so $d(R_{A,B}) \geqslant \left| g\left(\sum_{i=1}^n f(A_i) \times B_i \right) \right| = \left| \sum_{i=1}^n f(A_i)g(B_i) \right|$. Therefore $d(R_{A,B}) \geqslant k_1$. It is clear that $k_1 \geqslant \left\| f\left(\sum_{i=1}^n g(B_i)A_i \right) \right\|$, then $k_1 \geqslant \left\| \sum_{i=1}^n g(B_i)A_i \right\|$, so that $k_1 \geqslant k_2$. Since $k_2 \geqslant \left\| f\left(\sum_{i=1}^n g(B_i)A_i \right) \right\| = \left\| g\left(\sum_{i=1}^n \times f(A_i)B_i \right) \right\|$, it follows that $k_2 \geqslant \left\| \sum_{i=1}^n f(A_i)B_i \right\|$. Therefore $k_2 \geqslant k_3$. Since $k_3 \geqslant \left| h\left(\sum_{i=1}^n \times f(A_i)B_i y \right) \right| = \left| \sum_{i=1}^n f(A_i)h(B_i y) \right| = \left| f\left(\sum_{i=1}^n h(B_i y)A_i \right) \right|$, then $k_3 \geqslant \left\| \sum_{i=1}^n h(B_i y)A_i \right\| \geqslant \left\| \sum_{i=1}^n h(B_i y)A_i x \right\| = \left\| \left(\sum_{i=1}^n A_i(x\otimes h)B_i \right) y \right\|$. Therefore $k_3 \geqslant d(R_{A,B})$, this completes the proof. □

Lemma 2.1. *Let F be a normed space and $x_1, \ldots, x_n \in F$. Then $\left\| \sum_{i=1}^n x_i \right\| = \sum_{i=1}^n \|x_i\|$ if and only if there exists $f \in (F')_1$ such that $f(x_i) = \|x_i\|$ for $i = 1, \ldots, n$.*

Proof. This follows immediately form Hahn–Banach Theorem. □

Theorem 2.2. *Let $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ be two n-tuples of non-zero elements in $\mathscr{A}$. The following properties are equivalent*:

(i) $d(R_{A,B}) = D(R_{A,B})$,

(ii) *there exist two unit elements f, g in $\mathscr{A}'$ and n unit scalars $\lambda_1, \ldots, \lambda_n$ such that $f(A_i) = \lambda_i\|A_i\|$ and $g(B_i) = \overline{\lambda_i}\|B_i\|$ for $i = 1, \ldots, n$,*

(iii) $\left\| \sum_{i=1}^n \lambda_i A_i \right\| = \sum_{i=1}^n \|A_i\|$ *and* $\left\| \sum_{i=1}^n \overline{\lambda_i} B_i \right\| = \sum_{i=1}^n \|B_i\|$ *for some unit scalars $\lambda_1, \ldots, \lambda_n$.*

Proof. The equivalence (ii) $\Longleftrightarrow$ (iii) follows immediately from Lemma 2.1.

(ii) $\Longrightarrow$ (i) is also trivial since $\sum_{i=1}^n \|A_i\|\|B_i\| = \left|\sum_{i=1}^n f(A_i)g(B_i)\right| \leqslant d(R_{A,B}) \leqslant \sum_{i=1}^n \times \|A_i\|\|B_i\|$, for f and g given in the condition (ii).

(i) $\Longrightarrow$ (ii) The map $f \to \left\|\sum_{i=1}^n f(A_i)B_i\right\|$ is w^*-continuous on $\mathscr{A}'$ and $(\mathscr{A}')_1$ is w^*-compact, so it follows that $d(R_{A,B}) = \left\|\sum_{i=1}^n f(A_i)B_i\right\|$, for some element f in $(\mathscr{A}')_1$. The Hahn–Banach Theorem guaranties also the existence of an element g in $(\mathscr{A}')_1$ such that $\sum_{i=1}^n \|A_i\| \|B_i\| = d(R_{A,B}) = \sum_{i=1}^n f(A_i)g(B_i)$. Since A_i, B_i are non-zero and $|f(A_i)| \leqslant \|A_i\|, |g(B_i)| \leqslant \|B_i\|$, for $i = 1, \dots, n$ then $|f(A_i)| = \|A_i\|$, $|g(B_i)| = \|B_i\|$ and $f(A_i)g(B_i) = \|A_i\|\|B_i\|$ for $i = 1, \dots, n$. Thus $f(A_i) = \lambda_i\|A_i\|$ and $g(B_i) = \overline{\lambda_i}\|B_i\|$ for some unit scalars $\lambda_1, \dots, \lambda_n$. □

Corollary 2.1. *Let $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$ be two n-tuples of non-zero elements in $\mathscr{A}$ (with $n \geqslant 2$) such that $A_2 = B_1 = I$ and $d(R_{A,B}) = D(R_{A,B})$. Then all operators A_i and B_i are normaloid in $\mathscr{A}$.*

Proof. Using Theorem 2.2, there exist two unit elements f and g in $\mathscr{A}'$ and n unit scalars $\lambda_1, \dots, \lambda_n$ such that $f(A_1) = \lambda_1\|A_1\|$, $g(I) = \overline{\lambda_1}$, $f(I) = \lambda_2$, $g(B_2) = \overline{\lambda_2}\|B_2\|$ and if $n \geqslant 3$, then $f(A_i) = \lambda_i\|A_i\|$, $g(B_i) = \overline{\lambda_i}\|B_i\|$ for $i = 3, \dots, n$. So it follows immediately that $(\overline{\lambda_2}f)(A_1) = (\lambda_1\overline{\lambda_2})\|A_1\|$, $(\overline{\lambda_2}f)(I) = 1$, $(\lambda_1 g)(B_2) = (\lambda_1\overline{\lambda_2})\|B_2\|$, $(\lambda_1 g)(I) = 1$, and if $n \geqslant 3$, it follows also that $(\overline{\lambda_2}f)(A_i) = (\overline{\lambda_2}\lambda_i)\|A_i\|$, $(\lambda_1 g)(B_i) = (\lambda_1\overline{\lambda_i})\|B_i\|$ for $i = 3, \dots, n$. This completes the proof. □

Corollary 2.2. *Let A, B be two non-zero elements in $\mathscr{A}$. The following properties are equivalent*:

(i) $d(\delta_{A,B}) = \|A\| + \|B\|$,
(ii) $V_{\mathrm{N}}(A) \cap V_{\mathrm{N}}(-B) \cap (\mathbb{K})_1 \neq \emptyset$,
(iii) $\|I + \lambda A\| = 1 + \|A\|$ *and* $\|I - \lambda B\| = 1 + \|B\|$ *for some unit scalar* λ.

Proof. It is clear that $D(\delta_{A,B}) = \|A\| + \|B\|$.

(i) $\Longrightarrow$ (ii) This implication follows from the proof of the above Corollary for $n = 2$.

(ii) $\Longrightarrow$ (iii) This implication is trivial.

(iii) $\Longrightarrow$ (i) This implication follows immediately from Theorem 2.2. □

Remark 2.1. In [13, Corollary 1], Stampfli has proved (in Hilbert space case) that $\|\delta_{A,B}\| = \|A\| + \|B\|$, if $\|A\| = 1$, $W_0(A) = \{\|z\| \leqslant 1\}$ and B is arbitrary in $B(E)$, so that in this situation $d(\delta_{A,B}) < \|\delta_{A,B}\|$ for any non-normaloid operator B in $B(E)$.

Corollary 2.3. *Let A, B be two normaloid in $\mathscr{A}$. Then the following properties hold*:

(i) $d(\delta_{A,\lambda B}) = \|A\| + \|B\|$, *for some unit scalar* λ,
(ii) *if* $V(A) = \{|z| \leqslant \|A\|\}$ *then* $d(\delta_{A,B}) = \|A\| + \|B\|$.

Proof. (i) The result (i) is trivial if $A = 0$ or $B = 0$. If A and B are non-zero then there exist two unit scalars α and β in $V_{\mathrm{N}}(A)$ and $V_{\mathrm{N}}(B)$, respectively. So the result (i) follows immediately from the above corollary if we take $\lambda = -\frac{\alpha}{\beta}$.

(ii) The result (ii) is trivial if $B = 0$. If $B \neq 0$ then there exists a unit scalar λ in $V_{\mathrm{N}}(B)$, since B is normaloid. Thus $-\lambda \in V_{\mathrm{N}}(A) \cap V_{\mathrm{N}}(-B) \cap (\mathbb{K})_1$, since $V_{\mathrm{N}}(A) = \{|z| \leqslant 1\}$. Therefore the result follows immediately from the above corollary. □

Corollary 2.4. *Let $A \in \mathscr{A}$. The following properties are equivalent*:

(i) $d(\delta_A) = 2\|A\|$,
(ii) $\lambda\|A\|$ *and* $-\lambda\|A\|$ *belong to* $V(A)$ *for some unit scalar* λ,
(iii) $\operatorname{diam} V(A) = 2\|A\|$ (*the diameter of* $V(A)$),
(iv) $\|I + \lambda A\| = \|I - \lambda A\| = 1 + \|A\|$ *for some unit scalar* λ.

Proof. The equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv) follows from Corollary 2.2 and the equivalence (ii) $\Leftrightarrow$ (iii) is trivial. □

Corollary 2.5. *Let $A, B \in \mathscr{A}$. The following properties are equivalent*:

(i) $d(U_{A,B}) = 2\|A\|\|B\|$,
(ii) $A \parallel B$,
(iii) *there exist a unit scalar* λ *and a unit element* f *in* $\mathscr{A}'$ *such that* $f(A) = \|A\|$ *and* $f(B) = \lambda\|B\|$.

Proof. This follows immediately from Theorem 2.2. □

Corollary 2.6. *Let $A \in \mathscr{A}$. The following properties are equivalent*:

(i) A *is normaloid*,
(ii) $A \parallel I$,
(iii) $d(\varDelta_A) = 2\|A\|$.

Proof. This follows from the fact that $\varDelta_A = U_{A,I}$ and by using the above Corollary. □

Remark 2.2. (i) Corollary 2.6 gives a characterization of the normaloid operators in $\mathscr{A}$. It is clear that $\|\varDelta_A\| = 2\|A\|$ for any A in $\mathscr{A}$, so that $d(\varDelta_A) < \|\varDelta_A\|$ for any non-normaloid operator A in $\mathscr{A}$.

(ii) Stacho–Zalar mentioned in their paper [12] that the condition $\|U_{A,B}\| = 2\|A\|\|B\|$ should correspond to "being parallel". That is not true because $\|U_{A,I}\| = 2\|A\|\|I\|$ for every $A \in \mathscr{A}$ (A is not necessarily parallel to I), but the condition $d(U_{A,B}) = 2\|A\|\|B\|$ correspond exactely to $A \parallel B$.

3. Characterization of the relation $R_{A,B} = 0$

Let $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ be two n-tuples of elements in $\mathscr{A}$. We may arrange the operators B_i so that $B_1, \ldots, B_m$ form a maximal linearly independent subset of $B_1, \ldots, B_n$. If $m < n$, we put $B_k = \sum_{i=1}^{m} c_{ik} B_i$, for $k = m+1, \ldots, n$ and some constants c_{ik} ($1 \leqslant i \leqslant m,\ m+1 \leqslant k \leqslant n$); in this case $R_{A,B} = R_{C,D}$, where $C = (C_1, \ldots, C_m)$, $D = (B_1, \ldots, B_m)$ and $C_i = A_i + \sum_{j=m+1}^{n} c_{ij} A_j$ for $i = 1, \ldots, m$.

Theorem 3.1. *The following properties are equivalent*:

(i) $d(R_{A,B}) = 0$,

(ii) $A_1 = \cdots = A_n = 0$ *when* $m = n$, *and* $A_i = -\sum_{j=m+1}^{n} c_{ij} A_j$, *for* $i = 1, \ldots, m$ *when* $m < n$,

(iii) $R_{A,B} = 0$.

Proof. (i) $\Longrightarrow$ (ii) Since $d(R_{A,B}) = 0$, by Theorem 2.1, we obtain $\sum_{i=1}^{n} f(A_i)B_i = 0$ for every f in $(\mathscr{A}')_1$ when $m = n$ and $\sum_{i=1}^{m} f(C_i)B_i = 0$ for every f in $(\mathscr{A}')_1$ when $m < n$. So from Hahn–Banach Theorem it follows that $A_1 = \cdots = A_n = 0$ when $m = n$ and $A_i = -\sum_{j=m+1}^{n} c_{ij} A_j$ for $i = 1, \ldots, m$ when $m < n$.

(ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (i) are trivial. $\square$

Remark 3.1. The above result is proved in [6] by another method but our proof follows immediately from our Theorem 2.1.

4. Lower estimate bound for $d(\delta_{A,B})$ and $d(U_{A,B})$

Theorem 4.1. *Let $A, B \in \mathscr{A}$. We have the following lower estimate*:

$$d(\delta_{A,B}) \geqslant \max\left\{ \sup_{\lambda \in V(B)} \|A - \lambda I\|, \sup_{\lambda \in V(A)} \|B - \lambda I\| \right\}.$$

Proof. Let $\lambda \in V(A)$ and $\mu \in V(B)$. Then there exist two states f, g on $\mathscr{A}$ such that $f(A) = \lambda$ and $g(B) = \mu$. So from Theorem 2.1, we obtain $d(\delta_{A,B}) \geqslant \|f(A)I - f(I)B\| = \|B - \lambda I\|$ and $d(\delta_{A,B}) \geqslant \|g(I)A - g(B)I\| = \|A - \mu I\|$. The result follows immediately. $\square$

Corollary 4.1. *Let $A \in \mathscr{A}$. Then $d(\delta_A) \geqslant \sup_{\lambda \in V(A)} \|A - \lambda I\|$.*

Corollary 4.2. *Let $A, B \in \mathscr{A}$. Then the following properties hold*:

(i) $d(\delta_{A,B}) \geqslant \|A\|$ *if* $0 \in V(B)$,

(ii) $d(\delta_{A,B}) \geqslant \|B\|$ *if* $0 \in V(A)$,

(iii) $d(\delta_A) \geqslant \|A\|$ *if* $0 \in V(A)$.

Theorem 4.2. *Let $A, B \in \mathscr{A}$. Then $d(U_{A,B}) \geqslant 2(\sqrt{2} - 1)\|A\|\|B\|$.*

Proof. The proof is given in [11] for $\|U_{A,B}\|$ but the proof is valid for $d(U_{A,B})$. $\square$

Theorem 4.3. *Let $A, B, C, D \in \mathscr{A}$ such that $C \perp A$ or $D \perp B$ then $M_{A,B} \perp M_{C,D}$.*

Proof. Assume $C \perp A$. So there exists a unit element f in $\mathscr{A}'$ such that $f(C) = 0$ and $f(A) = \|A\|$. Thus by using Theorem 2.1, it follows that $d(M_{A,B} + \lambda M_{C,D}) \geqslant \|f(A)B + \lambda f(C)D\| = \|A\|\|B\| = \|M_{A,B}\|$ for all complex λ. Therefore $\|M_{A,B} + \lambda M_{C,D}\| \geqslant \|M_{A,B}\|$ for all complex λ.

The second implication follows also by the same argument. $\square$

Corollary 4.3. *Let $A, B \in \mathscr{A}$ such that $A \perp B$ or $B \perp A$. Then $d(U_{A,B}) \geqslant \|A\|\|B\|$ and $d(V_{A,B}) \geqslant \|A\|\|B\|$.*

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