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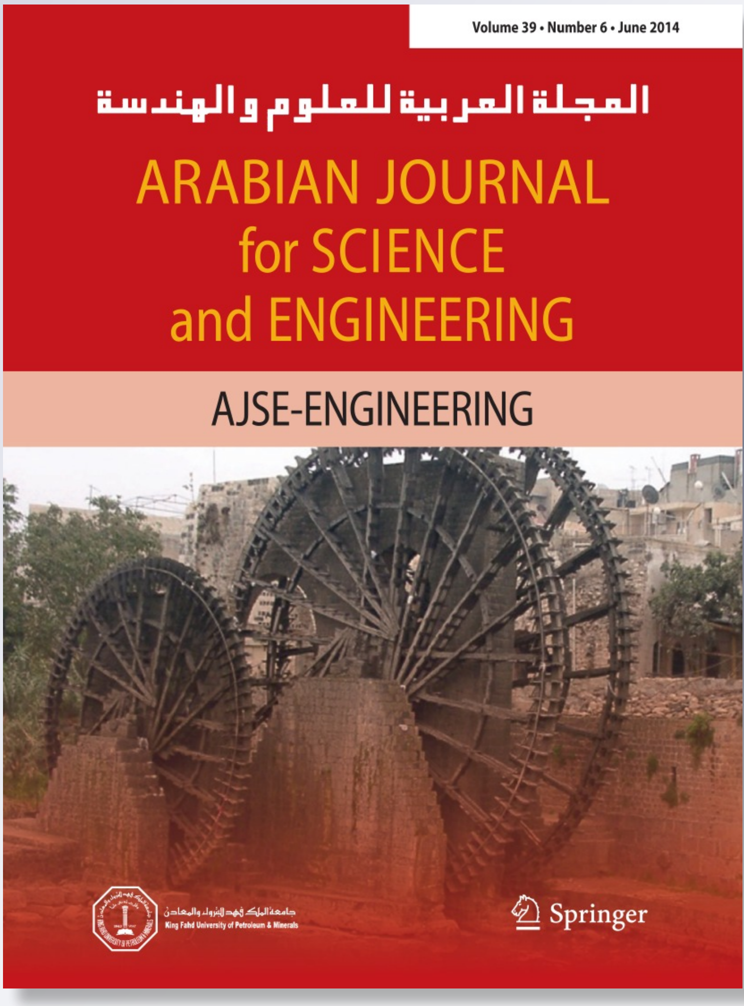
Arabian Journal for Science and Engineering

ISSN 1319-8025
Volume 39
Number 6

Arab J Sci Eng (2014) 39:5073-5084
DOI 10.1007/s13369-014-1036-4



Springer

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RESEARCH ARTICLE - MECHANICAL ENGINEERING

Simplified Method Using Homogenization Approach for Nonlinear Dynamic Analysis of Dissipative Energy in Multilayered Beams

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Received: 17 March 2012 / Accepted: 30 April 2013 / Published online: 22 April 2014

Abstract Based on known theoretical developments in linear dynamics of homogeneous beams, two homogenization approaches of composite beams are developed further to an anterior work using two equivalent properties: the physical and the mechanic–geometrical properties. Further to the assumption of Euler–Bernoulli beams, dynamic parameters are needed. Equations of a given beam structure subjected to free un-damped and/or damped vibration are established. The natural frequency responses of the first five modes are obtained from both approaches, and then compared with those obtained from a finite element model approach, taking into account different slenderness and boundary conditions. The result shows good agreement. An extension to the equivalent physical parameter homogenization method using the behaviour law in nonlinear state is presented here. Thus, the homogenization is extended to an elastically equivalent model elaboration for a system having an elastic–plastic and bilinear behaviour. The aim is to use analytical expressions from an elastically equivalent model for a nonlinear system of multilayer beam type to obtain the new corrected dynamic parameter. Based on the ductility factor method combined with the secant method that uses a substitute structure and secant stiffness to account for nonlinear behaviour, we developed a formula using global effective flexural modulus E_S^*, the equivalent mass density ρ^*, as well as the two types of viscous damping parameter (these types include a viscous resistance to transverse displacement C^* of the beam and a viscous resistance to straining of the beam material C_S^*) which can be incorporated into the formula without difficulty. The nonlinear analysis shows that the degradation of rigidity decreases the frequencies response curve, which tends to increase the vibration periods giving an additional storage space for the structure capacity to accumulate displacements of a higher degree compared to those of the elastic case. The importance of the nonlinearity assumption is shown, which fits better with the real mechanical performances of the structure.

Keywords Composite · Ductility factor · Damping factors of Rayleigh · Euler Bernoulli beam · Global equivalent mechanical–geometrical properties · Global equivalent physical properties · Homogenization · Law of elastic–plastic and bilinear behaviour · Vibration

الخلاصة

بناءا على التطورات النظرية المعروفة في السلوك الديناميكي الخطي للقضبان المتجانسة ، تم تطوير خصائص التجانس للعارضة المركبة بالرجوع إلى أعمال سابقة وباستعمال خاصيتي المرادفة: الأولى الخاصية الفيزيائية والثانية الخاصية الميكانيكية الهندسية. وبالإضافة إلى افتراض اولر- برنولي (Bernoulli-Euler) للقضبان ، هناك حاجة إلى خصائص ديناميكية. معادلات هيكل دو قضبان معين يتعرض لاهتزازات حرة غير مخمدة و / أو مخمدة قد أنشأت. وقد تم الحصول على استجابة الترددات الطبيعية للأوضاع الخمسة الأولى من كلا الطريقتين ، ومن ثم مقارنتها مع تلك التي تم الحصول عليها من نموذج العناصر المنتهية ، مع الأخذ بعين الاعتبار مختلف النحول وشروط الحدية. والنتيجة تظهر اتفاق جيد.

امتدادا لطريقة التجانس الفيزيائي باستخدام قانون السلوك غير خطي سنقوم بتفصيلها هنا. وبالتالي ، يتم توسيع التجانس بوضع نموذج مطاطي مرادف لنظام سلوك مرن-بلاستيك وشبه خطي. والهدف هو استخدام

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تعبيرات تحليلية من نموذج مطاطي مرادف لنظام غير خطي من نوع عارضة متعددة الطبقات للحصول على المعلمة (المتغيرة) الديناميكية الجديدة المصححة. وعلى أساس أسلوب عامل الليونة بالإضافة إلى طريقة القاطع التي تستعمل هيكل بديل والتصلب القاطع للأخذ بعين الاعتبار السلوك الغير خطي ، قمنا بتطوير صيغة باستخدام معامل الانحناء الفعلي Es^*، الكثافة الكتلية الفعلية ρ^* ، فضلا عن النوعين من خصائص التخميد اللزجي (التي تشمل مقاومة لزجية للحركة العرضية C^* للعارضة ومقاومة لزجية لاحتكاك العارضة CS^*) والتي يمكن إدراجها في الصيغة دون أية صعوبة.

ويبين التحليل غير الخطي أن تدهور الصلابة يقلل من منحنى استجابة الترددات ، والتي تميل إلى زيادة فترات اهتزاز تعطي مجال احتياطي إضافي لقدرة الهيكل لتحمل الانتقال (التطاول) بدرجة أعلى مقارنة بتلك في حالة المرونة. ان أهمية افتراض السلوك غير الخطي تظهر بشكل أفضل الأداء الميكانيكي الحقيقي للهيكل.

Abbreviations

n	Recurrence index
E^*	Equivalent longitudinal modulus of elasticity
ρ^*	Equivalent mass volume
C_s^*	Equivalent inner damping coefficient
C^*	Equivalent outer damping coefficient
$(\mathrm{EI})_{\mathrm{eq}}$	Bending stiffness factor
$(C_s\mathrm{I})_{\mathrm{eq}}$	Equivalent damping factor
$(\rho\mathrm{A})_{\mathrm{eq}}$	Transversal inertia factor
C_{eq}	Equivalent outer damping coefficient
ρ, E and G	Mass volume, Young modulus and shear modulus, respectively
A_i, h_i, ρ_i	Fold i relative characteristics relatives
N	Number of folds
M_n	Generalised mass
ξ_n, ξ_{eq}	Linear damping factor of nth mode, nonlinear equivalent damping factor
a_0 and a_1	Arbitrary coefficients
Φ_i	Vibration mode i
Y_i	Modal amplitude (normal or principal coordinates)
$v_i(x,t)$	Geometric displacement coordinates for mode i
ω_n	Natural frequency of vibration for mode N

1 Introduction

The lightness and robustness considerations have directed very early aeronautical and naval constructors, civilians and industrialists to composite material solutions [1]. The dynamic analysis of composite beams allows a better monitoring of their mechanical behaviour and facilitates thereafter their exploitation from the point of view of their design and their use. Although only a few studies were initiated within the theory described here, it will be partly validated by other theories based on the analysis of Pagano [2] as well as applications stated in reference [3].

The treatment of laminated beams developed in this work is based on a simple theory. The method of analysis is known as the Bernoulli–Euler theory of elementary mechanics of materials. Although the application of this theory is quite restricted, it yields considerable insight into the analysis of laminated structures and provides a natural introduction to the more general theory of geometrically-exact multilayer beams [4–6]. Two homogenization methods are presented, the first being based on the equivalent physical characteristics whilst the second is based on the equivalent mechanical–geometrical characteristics.

The general expression of equivalent transition factors developed through the two homogenization approaches is obtained while taking into account the effect of energy dissipation (internal and external damping). These factors are later modified to take account of the rigidity-weakening effect in the nonlinear stage. The dynamic responses are obtained based on natural frequencies and Rayleigh's damping factors. The analytical expressions of these responses are obtained in linear stage according to two approaches, physical approach (I), and mechanical–geometrical approach (II), respectively, and developed in the same manner as in the post-elastic phase.

The natural frequencies are determined and compared with those obtained from a finite element model. Afterwards, the effect of support conditions on the damping contribution is considered. The comparative study has shown a very good accuracy. The post-elastic behaviour doesn't affect the good concordance of the predictable responses obtained by the two approaches (I, II).

The aim is to produce a homogeneous formula that is obtaining relationships of similar behaviours to those of isotropic homogeneous beams, allowing the attainment of a quick and simple solution to the problem of frequency calculation of multilayered beams, taking into account the two cases of linear and nonlinear behaviours.

2 Characteristics of Straight Section

Figure 1 represents a straight beam of section (S) composed of a symmetrical pile of N isotropic materials of different thicknesses. If A, is the total surface, h the total height of the straight section (S) and A_i and h_i are, respectively, the surface and height of the ith fold, then the following equations are obtained:

$$A = \sum_{i=1}^{N} A_i \text{ and } h = \sum_{i=1}^{N} h_i$$

Moreover, the mechanical characteristics (ρ_i, E_i, CS_i and C_i) of the multilayered beam for each fold i, are mainly its mass ρ_i, modulus of elasticity E_i and the internal and external damping factors, CS_i, and C_i.

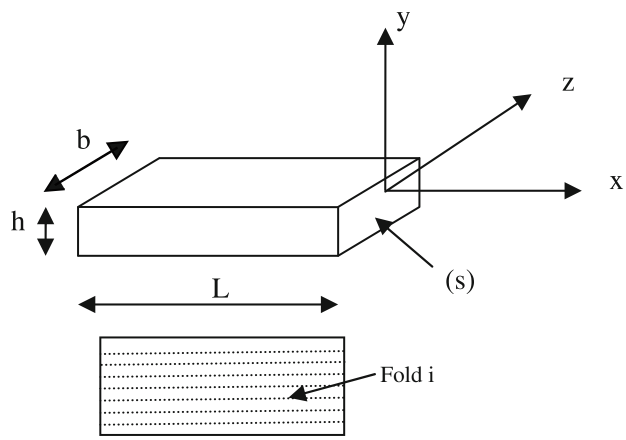

Fig. 1 Multilayered beam

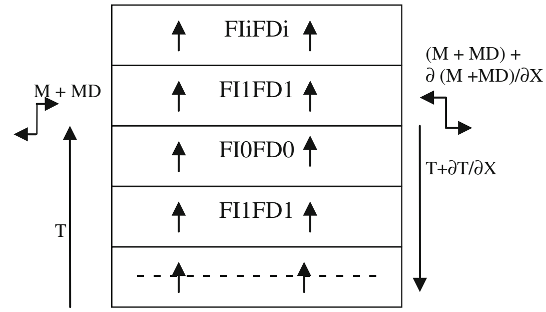

Fig. 2 Portion (dx) of a beam of N folds with forces acting on each fold

3 Homogenization Approaches

3.1 Case of Slender Beams

The movement equations are determined from the dynamic equilibrium studies as shown in equations [3,7,8]. The equilibrium equations, expressed throughout the study of a portion of a beam, are valid each time and are strictly the same as those of homogeneous isotropic beams because their demonstration doesn't involve the material properties. However, although the calculation of the beam's natural frequencies is a function of the beam's mechanical properties, it doesn't depend on the exciter forces involved. This enables the study to be carried out in the range of free vibrations.

Assumptions and reminder:

- The case of small harmonic movements of equilibrium status is respected around the equilibrium position.
- The plies are perfectly bound together, so that no slip occurs at ply interfaces.
- The beam has both geometric and material property symmetry about the neutral surface (i.e., the plies are symmetrically arranged about the xz plane.
- Plane sections, before deformation, remain plane after deformation.
- Each ply is linearly elastic with no shear coupling (i.e. ply orientations are either 0° or 90°).

In the case of transversal harmonic free vibrations of a non-damped beam and if no exciter force is involved, the movement equation is given by the following formula:

$$\frac{\partial^2}{\partial x^2}\left[\mathrm{EI}\frac{\partial^2}{\partial x^2}+C_s I\frac{\partial^3 v}{\partial x^2\partial t}\right]+\rho A\frac{\partial^2 v}{\partial t^2}+C\frac{\partial v}{\partial t}=0 \tag{1}$$

The solution is determined by the "variables separation", and the natural frequency expression in the case of beams of homogeneous rectangular section, is given by:

$$\omega = C\sqrt{\frac{\mathrm{EI}}{\rho A l^4}} \tag{2}$$

where
$C=(al)^2$ Factor depending on boundary conditions.

3.1.1 Approach I : Homogenization Using Physical Properties: E^*, ρ^*, CS^*, C^*

This approach allows the equivalent physical properties of a multilayered beam to be found Fig. 2. The equilibrium of forces and moments is given by the following equations:

$$T+\frac{\partial T}{\partial x}.\mathrm{d}x - T - \left(\sum_{i=i}^{N}(t_{Ii}+t_{Di})\right)\mathrm{d}x = 0 \tag{3}$$

$$(M+M_D)+\frac{\partial(M+M_D)}{\partial x}\mathrm{d}x-(M+M_D)+T.\mathrm{d}x=0 \tag{4}$$

where

- T refers to the shear force,
- Index, 0, refers to the central fold,
- f_i, f_{Di} represent respectively the distributed transversal forces of inertia and damping of material i.

$$f_h=\rho_i A_i\frac{\partial^2 v(x,t)}{\partial^2 t}\, f_{di}=C_i(x)\frac{\partial v(x,t)}{\partial t}$$

From another side and according to the static study: The bending moment is given by:

$$M=\frac{\partial\alpha}{\partial x}\sum_{i=0}^{N}\int\int_A E_i y^2\,\mathrm{d}A \tag{5}$$

And the curvature is given by:

$$M = \frac{\partial \alpha}{\partial x} = \frac{\partial^2 v}{\partial x^2} \quad (6)$$

where

v is the vertical displacement.

The damping moment caused by the partial viscous resistance related to the component material strain of each fold is given by:

$$M_{Di} = C_{\mathrm{si}} I_i \frac{\partial^3 v(x,t)}{\partial x^3 \partial t}$$

By developing the integral of forces acting in the dynamic equilibrium of the differential element dx of the multilayered beam and after substituting the bending moment M and the shear force T with their respective values (3) and (5), the following governing differential equation of the free damped dynamic movement written in its general homogenized form (case of multilayer of N folds, distributed in "mirror symmetry" manner) is obtained, based on the recurrence index:

$$n = \frac{N_P - 1}{2} \quad (7)$$

The partial differential equation of free motion is given in a condensed form by:

$$\frac{\partial^2}{\partial x^2}\left(\sum_{i=1}^{Np} E_i I_i \frac{\partial^2 v(x,t)}{\partial x^2} + \sum_{i=1}^{Np} C_{Si} I_i(x) \frac{\partial^3 v(x,t)}{\partial x^2 \partial t}\right) + \sum_{i=1}^{Np} \rho_i A_i \frac{\partial^2 v(x,t)}{\partial t^2} + \sum_{i=1}^{Np} C_i(x) \frac{\partial v(x,t)}{\partial t} = 0 \quad (8.a)$$

Or in its explicit form by:

$$\left[E_0 + 2\left[\sum_{i=1}^{n} E_i \left[\left(\frac{A_i}{A_0}\right)^3 + 3\frac{A_i}{A_0}\left(\frac{A_i}{A_0} + 2\sum_{i=1}^{n} \frac{A_{(i-1)}}{A_0} - 1\right)^2\right]\right]\right] \times \left(\frac{1}{N_P^3}\right) I \frac{\partial^4 v}{\partial x^4}$$
$$+ \left[C_{s0} + 2\left[\sum_{i=1}^{n} C_{Si} \left[\left(\frac{A_i}{A_0}\right)^3 + 3\frac{A_i}{A_0}\left(\frac{A_i}{A_0} + 2\sum_{i=1}^{n} \frac{A_{(i-1)}}{A_0} - 1\right)\right]\right]\right] \times \left(\frac{1}{N_P^3}\right) I \frac{\partial^3 v}{\partial x^2 \partial t}$$
$$+ \left[\rho_0 + 2\left(\sum_{i=1}^{n} \rho_i \frac{A_i}{A_0}\right)\right]\left[1 + 2\sum_{i=1}^{n} \frac{A_i}{A_0}\right]^2 \left(\frac{1}{N_P}\right) A \frac{\partial^2 v}{\partial t^2}$$
$$+ \left[1 + 2\sum_{I+1}^{n} \frac{A_i}{A_0}\right]^3 \left[C_0 + \sum_{i=1}^{n} C_i\right]\left(\frac{1}{N_P^3}\right) \frac{\partial v}{\partial t} = 0 \quad (8.b)$$

Once homogenized, this equation has the same form as that obtained by Euler–Bernoulli for homogeneous beams, knowing that:

$$\frac{\partial^2}{\partial x^2}\left[E^* I \frac{\partial^2 v}{\partial x^2} + C_s I \frac{\partial^3 v}{\partial x^2 \partial t}\right] + \rho^* A \frac{\partial^2 v}{\partial t^2} + C^* \frac{\partial v}{\partial t} = 0 \quad (9)$$

By analogical identification of Eqs. (8.a) and (9), the equivalent modulus of elasticity is found and by dividing the stiffness (rigidity) by the inertia moment I of the whole straight section, the following is obtained:

$$E^* = \frac{\sum_{i=1}^{Np} E_i I_i}{I}\left[E_0 + 2\left[\sum_{i=1}^{n} E_i \left[\left(\frac{A_i}{A_0}\right)^3 + 3\frac{A_i}{A_0}\left(\frac{A_i}{A_0} + 2\sum_{i=1}^{n} \frac{A_{(i-1)}}{A_0} - 1\right)^2\right]\right]\frac{1}{N_P^3}\right] \quad (10)$$

The effective mass per unit volume is also given by:

$$\rho^* = \frac{\sum_{i=1}^{N_p} \rho_i A_i}{A} = \left[\rho_0 \frac{A_0}{A} + 2\left(\sum_{i=1}^{N_p} \rho_i \frac{A_i}{A}\right)\right] \quad (11.a)$$

$$\rho^* \left(\rho_0 + 2\sum_{i=1}^{n} \rho_i\right)\frac{1}{N_p} \quad (11.b)$$

The damping factors (coefficients) which represent, respectively, the resistance effect at global deformation speed (inner damping) is given by:

$$C_S^* = \frac{\sum_i^{N_P} C_{Si} I_i}{I} = \left[C_0 + 2\left[\sum_{i=1}^{n} C_i \left[\left(\frac{A_i}{A_0}\right)^3 + 3\frac{A_i}{A_0}\left(\frac{A_i}{A_0} + 2\sum_{i=1}^{n} \frac{A_{(i=1)}}{A_0} - 1\right)^2\right]\right]\right]\left(\frac{1}{N_P^3}\right) \quad (12)$$

and the viscous resistance effect at global transversal strain (distributed outer damping) by:

$$C^* = \frac{\sum_i^{N_P} C_i}{I} = \left[C_0 + 2\left(\sum_{i=1}^{n} C_i\right)\right] \times \left[1 + 2\sum_{i=1}^{n} \frac{A_i}{A_0}\right]^3 \left(\frac{1}{N_P^3}\right) \quad (13)$$

The general solution of Eqs. 8.a, 8.b is obtained similarly to the homogeneous case (1), by substitution of E by E^*, ρ by ρ^*, C^s by C^s*, and C by $C*$ into the homogeneous equation. Depending on the different cases of boundary conditions, the frequencies' equation is obtained according to boundary conditions (Table 1).

Table 1 Frequencies' equations for different boundary conditions

Schema	Boundary conditions	Frequency equations
	$x = 0,\, v(0,t) = M(0,t) = 0$	$\mathrm{Sin}(a_n 1) = 0$
	$x = 1,\, v(1,t) = M(1,t) = 0$	
	Or	
	$x = 0,\, \phi(0) = \phi''(0) = 0$	
	$x = 1,\, \phi(1) = \phi''(1) = 0$	
	$x = 0,\, v(0,t) = v'(0,t) = 0$	$\mathrm{Cos}(a_n 1).\mathrm{Cosh}(a_n 1) - 1 = 0$
	$x = 1,\, v(1,t) = v'(1,t) = 0$	
	Or	
	$x = 0,\, \phi(0) = \phi'(0) = 0$	
	$x = 1,\, \phi(1) = \phi'(1) = 0$	
	$X = 0, v(0,t) = v'(0,t) = 0$	$\mathrm{Cos}(a_n 1).\mathrm{Cosh}(a_n 1) + 1 = 0$
	$X = 1,\, T(1,t) = M(1,t) = 0$	
	Or	
	$x = 0,\, \phi(0) = \phi'(0) = 0$	
	$x = 1,\, \phi''(1) = \phi'''(1) = 0$	
	$x = 0,\, v(0,t) = v'(0,t) = 0$	$\mathrm{Tan}(a_n 1).\mathrm{Tanh}(a_n 1) - 1 = 0$
	$x = 1,\, v(1,t) = M(1,t) = 0$	
	Or	
	$x = 0,\, \phi(0) = \phi'(0) = 0$	
	$x = 1,\, \phi(1) = \phi''(1) = 0$	
	$X = 0,\, v(0,t) = v'(0,t) = 0$	$\mathrm{Tan}(a_n 1).\mathrm{Tanh}(a_n 1) - 1 = 0$
	$X = 1,\, v(1,t) = M(1,t) = 0$	
	Or	
	$x = 0,\, \phi(0) = \phi'(0) = 0$	
	$x = 1,\, \phi(1) = \phi''(1) = 0$	

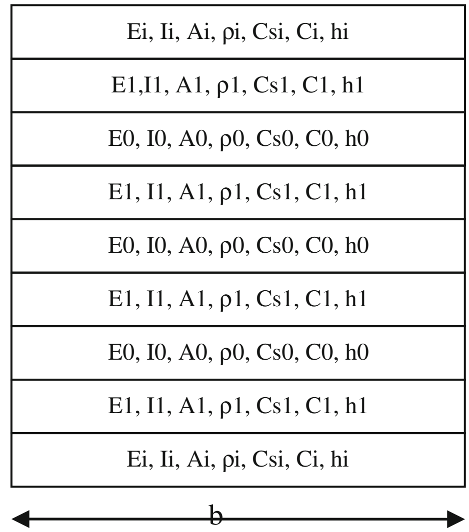

Fig. 3 Transversal section of a symmetric mirror multilayer

$$E^* = E_0 + \left[\sum_{i=1}^{n} 2E_i(1 + 12n^2)\right] \cdot \frac{1}{N_p^3}$$
$$\rho^* = \left(\rho_0 + 2\sum_{i=1}^{n} \rho_i\right)\frac{1}{N_p}$$
$$C_S^* = C_{S_0} + \left[\sum_{i=1}^{n} 2C_{S_i}(1 + 12n^2)\right] \cdot \frac{1}{N_p^3}$$
$$C^* = \left(C_0 + 2\sum_{i=1}^{n} C_i\right)\left(\frac{1}{N_P^3}\right)$$
$$n = \left(\frac{N-1}{2}\right). \tag{14}$$

3.1.2 *Theoretical Application: Computation of Homogenized Physical Properties of a Symmetrical Multilayer Formed by a Reversed Orthotropic Fold*

Even if the given example as presented in Fig. 3 looks very particular, it still remains needed within a very important theoretical application, as a common method used in the design of industrial applications or the development of symmetrical test models for experimental use. The homogenized physical properties are summarized by simple recurrence, under their general form, and are ready for use.

For a given value, N
If n is even then $i = 0$ for extreme layers
If n is odd then $i = 1$ for extreme layers

Homogenized physical parameters for a number N_P of plies having the same thickness

3.2 Approach II: Homogenization Approach Based on Equivalent Mechanical–Geometrical Properties: (EI)eq, (ρA)eq, (C_sI)eq and (Ceq)

By analogy, the dynamic equation of damped free motion of a homogenized multilayer composite beam can be expressed as a function of physical–mechanical properties in the following general shape:

$$\frac{\partial^2}{\partial x^2}\left[\begin{matrix}(\mathrm{EI})_{\mathrm{eq}}\frac{\partial^2 v}{\partial x^2} + \\ (C_s I)_{\mathrm{eq}}\frac{\partial^3 v}{\partial^2 x^2 \partial t}\end{matrix}\right] + (\rho A)_{\mathrm{eq}}\frac{\partial^2 v}{\partial t^2} + C_{\mathrm{eq}}\frac{\partial v}{\partial t} = 0 \tag{15}$$

The natural frequency is expressed as:

$$\omega = a^2\sqrt{\frac{(\mathrm{EI})_{\mathrm{eq}}}{\sum_{i=1}^{n}(\rho A)_i}} = C\sqrt{\frac{(\mathrm{EI})_{\mathrm{eq}}}{l^4\sum_{i=1}^{n}(\rho A)_i}} \tag{16}$$

with: $i = 1, 2, \ldots, n$ number of layers.

3.2.1 Bending Stiffness (Rigidity) Factor

Once the expression of the reduction efforts of both homogeneous and non-homogeneous assumptions is written, the equivalent parameters are then identified.

The equivalent parameter $(\mathrm{EI})_{\mathrm{eq}}$ is determined assuming a perfect adhesion at different interface levels by equalizing elastic curvatures relative to the two types of beams, which gives:

$$M_F(x) = \frac{1}{3}\sum_{i=1}^{N} bE\left(y_i^3 - y_{i-1}^3\right)\frac{\partial^2 v}{\partial x^2} \tag{17}$$

By identification with the elastic equation of a homogeneous beam:

$$M_F(x) = \mathrm{EI}\frac{\partial^2 v}{\partial x^2} \tag{18}$$

We have:

$$(\mathrm{EI})_{\mathrm{eq}} = \frac{1}{3}\sum_{i=1}^{N} bE(y_i^3 - y_{i-1}^3) \tag{19a}$$

For a width b, equal to a unit, the equivalent bending rigidity (stiffness) will be:

$$(\mathrm{EI})_{\mathrm{eq}} = \frac{1}{3}\sum E_i(y_i^3 - y_{i-1}^3) \tag{19b}$$

where
(y_i), $(y_i - 1)$—upper and lower position coordinates of fold (i).

Similarly, the set of forces in dynamic equilibrium of the differential element dx, is obtained by the following analogical development analysis:

3.2.2 Equivalent Damping Factor

$$M_{Di}(x) = \int_{y_{i-1}}^{y_i} \sigma_{Di}\, y\mathrm{d}A = \int_{y_{i-1}}^{y_i} C_{Si}\frac{\partial \varepsilon}{\partial t} yb\mathrm{d}y = \int_{y_{i-1}}^{y_i} C_{si}by^2\frac{\partial^3 v(x,t)}{\partial x^2 \partial t}\mathrm{d}y \tag{20a}$$

$$M_D(x) = \sum_{i=1}^{N} M_{Di} = \sum_{i=1}^{N} C_{Si}\frac{y_i^3 - y_i^3}{3}\frac{\partial^2 v}{\partial x^2 \partial t} b\mathrm{d}y \tag{20b}$$

Elsewhere, it is easy to show according to Navier–Bernoulli (deformation varies linearly according to the transversal section); the expression of the global damping moment is written as:

$$M_D = \int_{y_{i-1}}^{y_i} (C_S I)_{\mathrm{eq}}\left[\frac{\partial^3 v}{\partial x^2 \partial t}\right] b\mathrm{d}y \tag{20c}$$

And by identification:

$$(C_S I)_{\mathrm{eq}} = \frac{1}{3}\sum bC_{Si}(y_i^3 - y_{i-1}^3). \tag{21}$$

3.2.3 Transversal Inertia Factor

$$F_I = (\rho A)_{\mathrm{eq}}\frac{\partial^2 v}{\partial t^2} = \sum_{i=1}^{N} F_{Ii} = \sum_{i=1}^{N}\rho_i A_i\frac{\partial^2 v}{\partial t^2}$$
$$(\rho A)_{\mathrm{eq}} = \sum_{i=1}^{N}\rho_i A_i. \tag{22}$$

3.2.4 Outer Damping Factor C_{eq}

$$F_D = (C(x))_{\mathrm{eq}}\left[\frac{\partial v}{\partial v}\right] bA = \sum_{i=1}^{N} f_{Di} = \sum_{i=1}^{N} C_i(x)\left[\frac{\partial v}{\partial t}\right] bA$$
$$(C(x))_{\mathrm{eq}} = \sum_{i=1}^{N} C_i(x). \tag{23}$$

It is noticed that this approach is more simply formulated and is for quick use; it can therefore be a good alternative for a first estimation. However, the expression of Eq. (15) obtained is similar to that obtained by the multilayer classical theory [8].

3.3 Homogenization Analysis of Damping Factors in Linear Phase

For the purpose of completeness of the dynamical analysis, from a formal point of view and to facilitate its future use for the forced damped cases, the partial differential equation governing the dynamic movement of systems of distributed characteristics is reformulated in its decoupled form, under normal coordinates.

3.3.1 Formulation of the Decoupled Dynamic Equation in Damped Bending

The dynamic equation for a free damped case is obtained as:

$$\frac{\partial^2}{\partial^2}\left(\mathrm{EI}\frac{\partial^2 v(x,t)}{\partial x^2} + C_S I\frac{\partial^3 v(x,t)}{\partial x^2 \partial t}\right) + \rho A\frac{\partial^2 v(x,t)}{\partial t^2} + C\frac{\partial v(x,t)}{\partial t} = 0 \tag{24}$$

After transformation, this equation is decoupled then obtained in its shape under normal coordinates.

$$v(x,t) = \sum_{i=1}^{\infty}\varphi_i(x)Y_i(t) \tag{25}$$

where
Φ_i: vibration mode i
Y_i: modal amplitude (normal or principal coordinates)
v_i(x,t): Geometrical displacement coordinates for mode i.

The decoupled equation with damping is given by:

$$\sum_{i=1}^{\infty} \rho A\varphi_i^2(x)\left(\frac{\mathrm{d}^2 Y_i(t)}{\mathrm{d}t^2}\right)+\sum_{i=1}^{\infty} C(x)\varphi_i(x)\left(\frac{\mathrm{d}Y_i(t)}{\mathrm{d}t}\right)$$
$$+\sum_{i=1}^{\infty}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[C_S I\frac{\mathrm{d}^2\varphi_i}{\mathrm{d}x^2}\right]\frac{\mathrm{d}Y_i(t)}{\mathrm{d}t}$$
$$+\sum_{i=1}^{\infty}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[EI\frac{\mathrm{d}^2\varphi_i}{\mathrm{d}x^2}\right]Y_i(t)=0. \quad (26)$$

3.3.2 Conditions for the Orthogonally Damping

It is presumed that the transformation to normal coordinates enables the decoupling of damping forces in the same manner as for the rigidity mass (stiffness mass). Rayleigh [7] showed that this assumption is possible, if the damping effects are taken proportionally to the rigidity mass as follows:

$C(x) = a_0 m(x) = a_0\rho A$ and $C_S = a_1 E$

where a_0 and $a_1$are arbitrary coefficients of proportionality. After substitution into the equation, a decoupled equation in normal coordinates is obtained as:

$$M_n\frac{\mathrm{d}^2 Y(t)}{\mathrm{d}t^2}+(a_0+a_1\omega_n^2 M_n)\frac{\mathrm{d}Y(t)}{\mathrm{d}t}+\omega_n^2 M_n Y(t)=0 \quad (27)$$

By definition, M_n, is called generalized mass of the beam for the mode $\hat{O}_n$, and is given by the integral:

$$M_n\int_0^L \varphi_n^2(x)\rho A\mathrm{d}x \quad (28)$$

This equation can be rearranged, dividing by M_n and putting the damping factor of the nth mode:

$$\zeta_n=\frac{a_0}{2\omega_n}+\frac{a_1\omega_n}{2} \quad (29)$$

The equation of damping factor (29) shows that for a damping proportional to the mass $[C(x) = a_0 m(x)]$, the damping factor is inversely proportional to the frequency. In addition, for a damping proportional to the rigidity ($C_s = a_1 E$), the damping factor is directly proportional to the frequency. Depending on the type of structure and for the extreme frequencies field, the damping factor ξ_n has a priority link with one or the other of the two factors of damping.

Finally, a standard form of the decoupled equation of dynamic free damped motion of the homogeneous multilayered beam is obtained, as:

$$\frac{\mathrm{d}^2 Y(t)}{\mathrm{d}t^2}+2\zeta_n\omega_n\frac{\mathrm{d}Y_n(t)}{\mathrm{d}t}+\omega_n^2 Y_n(t)=0 \quad (30)$$

Depending on the type of approach used I, or II, the equivalent parameters are added to obtain corresponding modal frequencies, modal vibrations and modal damping factors, respectively.

4 Comparative Study

4.1 Study According to Natural Frequencies

A finite element model [9], using a solid element, has been carried out to appreciate the convergence of the proposed approaches in the case of the Euler–Bernoulli assumption. Several boundary limit cases have been studied, by investigating the variation effect of geometrical slenderness (l/r). The results are obtained from a mirror symmetry multilayer example of five folds, of constant thickness, $h = 2$ mm (Table 2) with a Bernoulli slender beam ($L/r = 346$, 242, 121). The results are shown in Fig. (4a, b).

4.2 Discussion: Slender Beam

To validate approaches I and II, we applied the method used by reference [3] and [2] using Table 2. The comparative study showed the same results. The effective modulus formula as developed here is an alternative and slightly simpler scheme for finding E^*. We can observe the same remark as from reference [3] "the flexural modulus of the laminated beam, unlike the young's modulus of the homogeneous isotropic beam, depends on the ply stacking sequence and the ply module. That is, if the properties do not change through the thickness of a beam, the flexural modulus is the same as the Young's modulus".

Table 2 Presentation of study model

Number	Material type	Volume mass (t/m^3)	Young modulus ($E10^7$) (t/m^2)	Poisson modulus (μ)
2	Aluminium	2.7	0.72	0.3
1	Cupper	2.93	1.2	0.3
0	Steel	7.8	2	0.3
1	Cupper	2.93	1.2	0.3
2	Aluminium	2.7	0.72	0.3

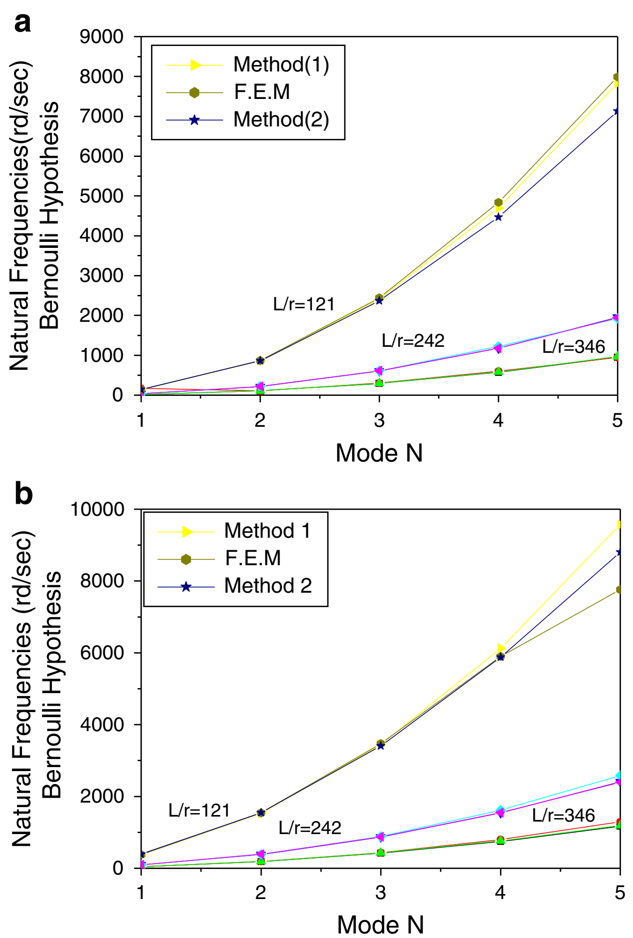

Fig. 4 **a** Natural frequencies of the first five modes of a multilayered beam of five folds ($N = 5$) for Fixed-Free ($F - Fr$) supports. **b** Natural frequencies of the first five modes of a multilayered beam of five folds ($N = 5$) for Simply-Supported ($S - S$) supports

Overall and from Fig. 4a, b, a very good convergence of results is noticed by the two proposed approaches, mainly for the first four modes. The curves show a very slight difference around 10 %) for the last mode. This divergence increases for low values of geometrical slenderness (l /r = 120). This could be due to the neglect of the shear force and rotational inertia effects.

4.3 Study According to Modal Damping Factor (Linear Case)

Four types of boundary support conditions, [SS (Supported-Supported), SC (Supported-clamped), CC (clamped-clamped), and CF (clamped-Free)] are considered. The comparative analysis between the four types of curves (Fig. 5) is also obtained in graphical form to appreciate the influence of boundary conditions on the modal dissipative potential contribution. For indication, the corresponding figures are obtained according to approach I, based on physical properties, for the case of a composite beam ($l/r = 346$), consisting of 5 isotropic layers distributed symmetrically to a middle plan (Fig. 3) and defined as shown in table N°II.

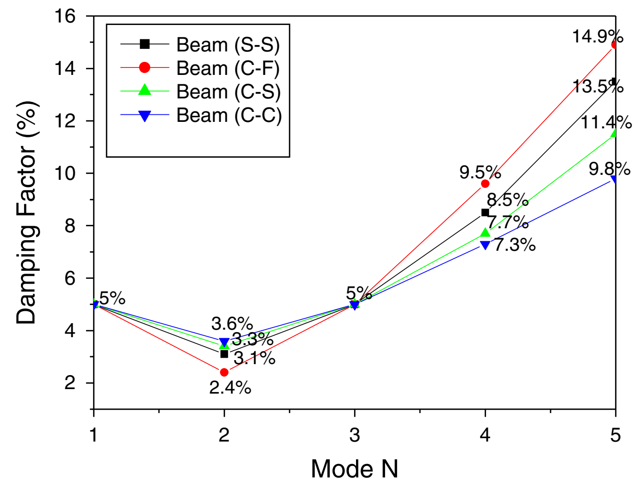

Fig. 5 Boundary conditions effect on the amount of modal damping for a slender beam ($L/r = 346$)

4.4 Discussion: Evaluation of Modal Damping Factors Linear Case of Isotropic Multilayer ($N = 5$ Layers)

We notice, according to Fig. 5, that within the lower modes range ($N < 3$), the amount of damping brought about by mass effect is dominant and that the comparative study of the damping curves for the different cases of boundary conditions shows that the damping rate is greater for hyperstatic beams (C–C) and (C–S). Inversely within the higher modes range, damping is essentially generated by wave's effect where Young's Modulus E plays a major role. At this modal level the comparative study of the damping curves is totally reversed compared to the lower modes range, where damping is found to be greater for the isostatic beams (C–F) and (S–S).

5 Equivalent Vibration Period t_{eq} and Equivalent Viscous Damping Factor ξ_{eq}-Bilinear Case

Some physical aspects of the nonlinear behaviour converted to an equivalent elastic linear system are better expressed by the equations of the variation of period T_{eq} and the damping factor ξ_{eq} with respect to the ductility factor μ [10,11]. For a bilinear System of a Single Degree Of Freedom (SSDOF) (Figs. 6 and 7), with a module in post-plastic state equal to α times the product of the initial module k_0 (elastic state), the relationship between the natural period of the equivalent system vibration (with a rigidity k_{sec}) and the original system

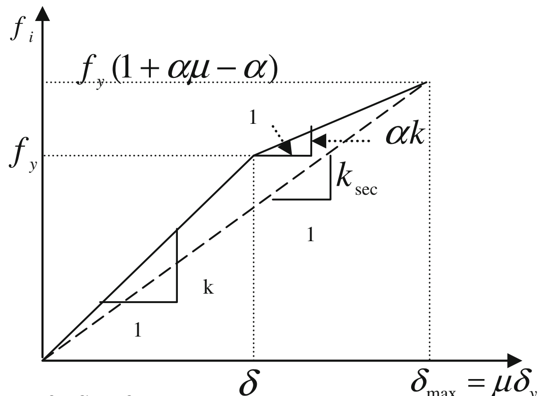

Fig. 6 Force–deformation relationship for a bilinear system of a single degree of freedom (SSDOF)

period ($T_0 = T_n$: natural period of system in vibration, in linearly elastic range, ($\delta\max \leq \delta y$) is given by:

5.1 Bilinear Case: Equivalent Period

$$\frac{T_{eq}}{T_n} = \sqrt{\frac{k_0}{k_{sec}}} = \sqrt{\frac{\mu}{1-\alpha+\alpha\mu}} \tag{31}$$

with:
$\mu = \delta_{max}/\delta_y$
μ : ductility factor.

5.2 Equivalent Damping Factor

The usual method [10,11], to define the equivalent damping of a system is to equalize the dissipative energy for one cycle of vibration of the non-elastic system with that of an equivalent linear system. Based on this concept, it is proven that the viscous damping factor for an equivalent linearly elastic system (Figs. 6 and 7) is given by:

$$\zeta_{eq} = \frac{1}{4\pi}\frac{E_D}{E_S} \tag{32}$$

with:

$$E_S = k_{sec}\frac{Y_m^2}{2} \tag{33}$$

After substitution of the two energetic quantities E_D and E_s in Eq. (31), the following is obtained

$$\zeta_{eq} = \xi_0 + \frac{2}{\pi}\left[\frac{(1-\alpha)(\mu-1)}{\mu-\alpha\mu+\alpha\mu^2}\right] \tag{34}$$

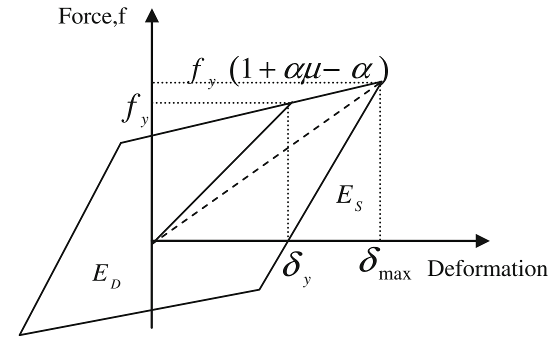

Fig. 7 Viscous damping due to the energy of hysteresis lens

The total viscous damping for the linearly equivalent system is:

$$\zeta_{eq} = \xi_0 + \xi_{eq} \tag{35}$$

where ξ_0 is the nominal linear viscous damping in the system and T is the small amplitude linear period (that is, the damping and period when $\mu = 1$).

5.3 Elastic–Plastic Case: Equivalent Period ($\alpha = 0$)

$$\frac{T_{eq}}{T} = \sqrt{\frac{k_0}{k_S}} = \sqrt{\mu} \tag{36}$$

5.4 Elastic–Plastic Case: Equivalent Damping Factor ($\alpha = 0$)

$$\zeta_{eq} = \xi_0 + \frac{2}{\pi}\left[1 - \frac{1}{\mu}\right]. \tag{37}$$

6 General Expressions of Homogenized Parameters Developed for Both Approaches: SI and SII

The two homogenization approaches, one, based on physical properties (SI), and the other, based on mechanical–geometric properties (SII), are reviewed to take account of the nonlinear behaviour effect (Figs. 8, 9).

6.1 Bilinear Case, Approach SI

$$\frac{f_n}{f_s} = \frac{\omega_n}{\omega_s} = \frac{\left(\sqrt{\frac{E^*I}{\rho^*AL^4}}\right)}{\left(\sqrt{\frac{E_s^*I}{\rho^*AL^4}}\right)} = \sqrt{\frac{E^*}{E_s^*}} = \sqrt{\frac{\mu}{1-\alpha+\alpha\mu}} \tag{38}$$

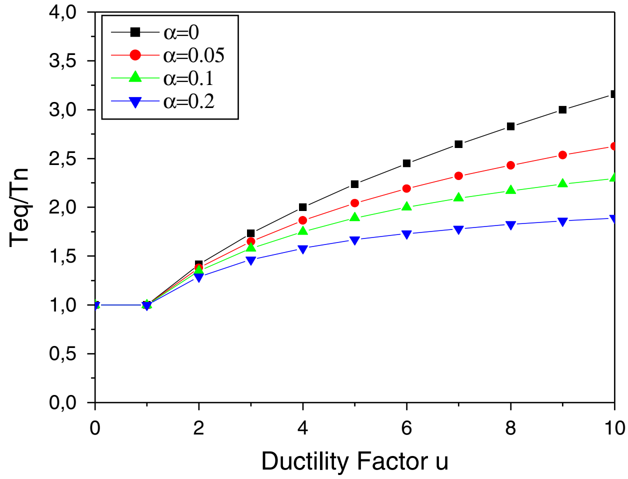

Fig. 8 Variation of the equivalent linear period of system with the ductility factor μ

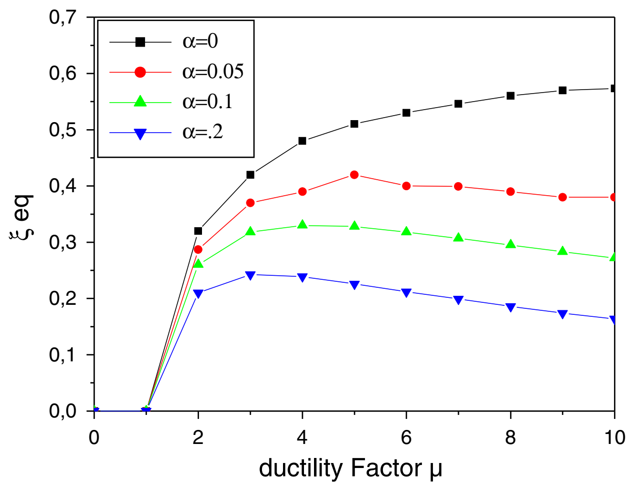

Fig. 9 Variation of equivalent damping factor $\boldsymbol{\xi}_{\text{eq}}$ with ductility Factor μ for $\boldsymbol{\alpha}$, values

6.2 Elastic–Plastic Case, Approach SI: $\alpha = 0$

$$\frac{f_n}{f_s} = \frac{\omega_n}{\omega_s} = \frac{\left(\sqrt{\frac{E^* I}{\rho^* A L^4}}\right)}{\left(\sqrt{\frac{E_s^* I}{\rho^* A L^4}}\right)} = \sqrt{\frac{E^*}{E_s^*}} = \sqrt{\frac{\mu}{1}} \tag{39}$$

6.3 Bilinear Case, Approach SII

$$\frac{f_n}{f_s} = \frac{\omega_n}{\omega_s} = \frac{\left(\sqrt{\frac{(\text{EI})_{\text{eq}}}{(\rho A)_{\text{eq}} L^4}}\right)}{\left(\sqrt{\frac{(\text{EI})_{\text{Seq}}}{(\rho A)_{\text{eq}} L^4}}\right)} = \sqrt{\frac{(\text{EI})_{\text{eq}}}{(\text{EI})_{\text{Seq}}}} = \sqrt{\frac{\mu}{1 - \alpha + \alpha\mu}} \tag{40}$$

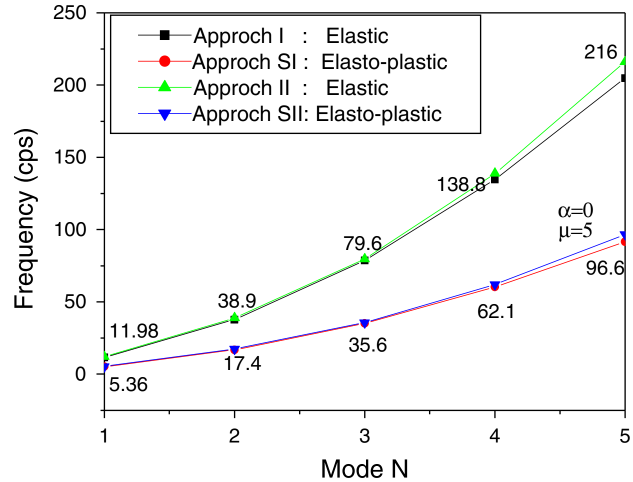

Fig. 10 Comparative curve of linear and nonlinear frequencies, for the case of the (F-Fr) multilayered beam ($N = 5$), of slenderness ($L/R = 346$)

6.4 Elastic–Plastic Case, Approach SII: $\alpha = 0$

$$\frac{f_n}{f_s} = \frac{\omega_n}{\omega_s} = \frac{\left(\sqrt{\frac{(\text{EI})_{\text{eq}}}{(\rho A)_{\text{eq}} L^4}}\right)}{\left(\sqrt{\frac{(\text{EI})_{\text{Seq}}}{(\rho A)_{\text{eq}} L^4}}\right)} = \sqrt{\frac{(\text{EI})_{\text{eq}}}{(\text{EI})_{\text{Seq}}}} = \sqrt{\frac{\mu}{1}} \tag{41}$$

6.4.1 Typical Application to the Case of a Multilayered Beam (F-Fr) with l/r $= 346$ and an Elastic–Plastic Behaviour Characterized by: $\alpha = 0$ and $\mu = 5$

The frequency curves (Fig. 10) for the case of a slender beam of $L/r = 346$ (Bernoulli assumption) are used as an example to compare them with the curves of frequencies obtained from this method, based on the secant rigidity (stiffness) modulus. The behaviour model is supposed to be elastic–plastic with $\alpha = 0$ and a ductility factor taken arbitrarily equal to 5 ($\mu = 5$).

Discussion

In order to better grasp some physical aspects of the nonlinear behaviour brought back to an equivalent elastic linear system, the variation curves of the period T_{eq} and of the damping factor ξ_{eq} are obtained compared to the factor of ductility μ (Figs. 8 and 9). The frequency curves in the case of the beam of slenderness $L/r = 346$ are used as an example to be compared with those obtained according to the method of secant module of rigidity. The model behaviour is supposed to be elastic–plastic with, $\alpha = 0$ and a ductility factor taken arbitrarily as $\mu = 5$.

Referring to both the elastic linear and the elastic–plastic cases, the natural periods are shown on the graphs (Fig. 10).

The frequency curves of the elastic–plastic case decrease on average by up to 50 % compared to those of the linear elastic case. The convergence of both approaches SI and SII is maintained like that of the case of the elastic state (Approach I and II).

7 Conclusions

For various engineering applications, the most commonly used theory in structural mechanics, for modelling the behaviour of the dynamic bending of beams is the formulation of Euler–Bernoulli. Consequently, two approaches of homogenization have been proposed for the dynamic beam case of free damped motion. Comparative study with respect to the method from reference [3] showed excellent concordance.

The natural responses, investigated by the two approaches are satisfactory for the relatively slender multilayered beams. They are very useful because of their simple form for the computation of dimension design and comparisons. In addition the nonlinearity effect doesn't affect the accuracy of the frequency responses given by the two approaches, SI and SII, and based on the elastic–plastic and bilinear models of behaviour.

The considered example, which takes into account the ductility effect and the elastic–plastic behaviour law, shows the importance of the nonlinear assumption, which copes better with the real mechanical performances of the structure. The deterioration of the global rigidities reduces the frequencies, which tends to increase the vibration periods, thus giving supplementary storage space for the structure capacity to accumulate displacements of higher degree relatively to the considered structure in elastic state.

However, the nonlinear study has to be enlarged on, to take into account some phenomena such as rotational inertia and transversal shear so that they can be generalized to Timoshenko beams. In addition, results for the case with the extension of the present approach to the general setting of multilayer beams, with the ability of the model to accommodate shear distortion in each layer, a large deformation and a large overall motion, should be further discussed in a future publication".

The transverse dynamic loading study can give other useful indications which allow the completion of the evaluation of the homogenization parameters according to different linear and nonlinear assumptions. However extensive experimental work is still needed, to validate the present work.

Annex

Validation of effective modulus using comparison between the author's formula and Gibson's formula [3] through application from Table 2 and using a section of laminated beam of depth $h = 10$ mm and width $b = 10$ mm. The total number of plies is $N = 5$. The ply thicknesses are the same.

Let's give an answer from Author's formula (10):

$$E^* = \frac{\sum_{i=1}^{N_i} EI}{I} = \left[E_0 + 2\sum_{i=1}^{n} E_i \left[\left(\frac{A_i}{A_0}\right)^3\right.\right.$$
$$\left.\left. +3\frac{A_i}{A_0}\left(\frac{A_1}{A_0} + 2\sum_{i=1}^{n}\frac{A_{(i-1)}}{A_0} - 1\right)\right]\frac{1}{N_p^3}\right.$$

$$E^* = \frac{1}{N_P^3}\left[\begin{array}{l} E_{\mathrm{Fe}} + 2E_{\mathrm{Cu}}\left\{\left(\frac{A_{\mathrm{Cu}}}{A_{\mathrm{Fe}}}\right)^3 + 3\frac{A_{\mathrm{Cu}}}{A_{\mathrm{Fe}}}\left(\frac{A_{\mathrm{Cu}}}{A_{\mathrm{Fe}}} + 1\right)^2\right\} \\ +2E_{\mathrm{AL}}\left\{\left(\frac{A_{\mathrm{AL}}}{A_{\mathrm{Fe}}}\right)^3 + 3\frac{A_{\mathrm{AL}}}{A_{\mathrm{Cu}}}\left(\frac{A_{\mathrm{AL}}}{A_{\mathrm{Fe}}} + 2\frac{A_{\mathrm{Cu}}}{A_{\mathrm{Fe}}} + 1\right)^2\right\}\end{array}\right]$$

The ply thicknesses are the same therefore we have also the same cross section for each ply

$$A_{\mathrm{Fe}} = A_{\mathrm{Cu}} = A_{\mathrm{AL}} = Cte$$
$$E^* = \frac{1}{5^3}[E_{\mathrm{Fe}} + 26E_{\mathrm{Cu}} + 98E_{\mathrm{AL}}] = 0.8304(10)^7\frac{t}{m^2}$$

Alternatively, the simplified formula (14) could be used:

$$E^* = \left[E_0 + \sum_{i=1}^{n} 2E_i(1 + 12n^2)\right]\frac{1}{N_p^3}$$
$$n = \left(\frac{N-1}{2}\right) = \frac{5-1}{2} = 2$$

$$E^* = \frac{1}{N_P^3}E_{\mathrm{Fe}} + 2\left[\begin{array}{l} E_{\mathrm{Cu}}\{1 + 12(1)\} \\ + \\ E_{\mathrm{AL}}\{1 + 12(4)\}\end{array}\right]$$
$$= 0.8304(10)^7\frac{t}{m^2}$$

Let's give an answer from reference's [3] and [2] using known formula:

$$E_f = \frac{8}{h^3}\sum_{j=1}^{\frac{N}{2}}(E_x)(z_j^3 - z_{j-1}^3)$$

E_x: Young's modulus of jth ply along the x direction

- Z: distance from neutral surface defined by the xy-plane
- N: the total number of plies
- Z_j: the distance from the neutral surface to the outside of the jth ply

For an even number of plies of uniform thickness ($h =$ constant), we have

$$z_j = \frac{jh}{N}$$

where J: position of the jth ply

The explicit formula for an even number of plies is then given by:

$$E_f = \frac{8}{N^3}\sum_{j=1}^{\frac{N}{2}}(E_x)(3j^2 - 3j + 1)$$

To use this formula for our purpose for an odd number of plies, we simply need to divide each ply into identical plies having half the thickness of the original ply ($h = 1$ mm), so that the total number of plies is now even $N = 5 \times 2 = 10$.

Again we obtain the same result

$$E^* = \frac{1}{5^3}\left[\, E_{\mathrm{Fe}} + 26E_{\mathrm{Cu}} + 98E_{\mathrm{AL}} \,\right] = 0.8304(10)^7\frac{t}{m^2}$$

Finally we can obtain the global effective modulus related to elastic–plastic case (39)

$$E_s^* = \frac{E^*}{\mu} = \frac{0.8304(10^7)}{5} = 0.16608(10^7)\frac{t}{m^2}$$
$$\frac{f_n}{f_s} = \frac{\omega_n}{\omega_s} = \frac{\sqrt{5}}{1} = 2.236$$

For both linear (2) and nonlinear (39) cases the circular frequency for the first mode of vibration of a cantilever beam is easily obtained equal to

$$\omega_n = 3.516\sqrt{\frac{E^*I}{\mathrm{AL}^4}}$$
$$\omega_s = \frac{\omega_n}{\sqrt{5}} = 1.573\sqrt{\frac{E^*I}{\mathrm{AL}^4}}.$$

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