University of Batna 2 1^{st} **year LMD**
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Tutorial -1- Solutions
Algebra 2

Background

- **Vector space** : Recall that $(V, +, .)$ is a $\mathbb{R}$-vector space which satisfy the following conditions (called axioms)

  Axioms for vector addition

  1. Commutative : For all $u, v \in V$, $u + v = v + u$.
  2. Associative : For all $u, v, w \in V$, $(u + v) + w = u + (v + w)$.
  3. Additive identity : There exists an element $0_V \in V$, such that $0_V + u = u$ for all $u \in V$.
  4. Additive inverse : For every $u \in V$, there exists an element $-u \in V$ such that $u + (-u) = 0_V$.

  Axioms for scalar multiplication

  5. Associative : For all $\lambda, \mu \in \mathbb{R}$ and all $u \in V$, $\lambda.(\mu \cdot u) = (\lambda\mu) \cdot u$.
  6. Distributive 1 : For all $\lambda \in \mathbb{R}$ and all $u, v \in V$, $\lambda \cdot (u + v) = \lambda \cdot u + \lambda \cdot v$.
  7. Distributive 2 : For all $\lambda, \mu \in \mathbb{R}$ and all $u \in V$, $(\lambda + \mu) \cdot u = \lambda \cdot u + \mu \cdot u$.
  8. Unitarity : For all $u \in V$, $1 \cdot u = u$.

- **Vector subspace** : A subset F of V (vector space) is called a vector subspace on $\mathbb{K}$ of V if it satisfies the following three conditions :

  1) $0_V \in F$.

  2) $\forall x, y \in F$, $x + y \in F$ (F is closed under addition).

  3) $\forall \alpha \in \mathbb{K}$, $\forall x \in F$, $\alpha.x \in F$ (F is closed under scalar multiplication).

- **Linearly independent famillies** : A familly $\{v_1, v_2, ..., v_n\}$ of vectors of V (vector space) is linearly independent if :

$$\forall \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K}, \qquad \lambda_1 v_1 + ... + \lambda_n v_n = 0_V \Longrightarrow \lambda_1 = \lambda_2 = ... = \lambda_n = 0.$$

- **Generating families** : A familly $\{v_1, v_2, ..., v_n\}$ of vectors of V (vector space) is a generating familly of V if every vectors of V is a linear combiniation of of the vectors $v_1, ..., v_n$. In other words,

$$\forall v \in V, \exists \lambda_1, ..., \lambda_n \in \mathbb{K} : \ v = \lambda_1 v_1 + ... + \lambda_n v_n.$$

  We also say that the familly $\{v_1, v_2, ..., v_n\}$ generates V and we write $V = span\{v_1, v_2, ..., v_n\}$.

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- **Basis of a vector space** : A familly $\{v_1, v_2, ..., v_n\}$ of vectors in a vector space V is called a basis of V if it satisfies the following tow conditions :

  1) $\{v_1, v_2, ..., v_n\}$ is a linearly independent familly of V.

  2) $V = span\{v_1, v_2, ..., v_n\}$.

Solution 1

1. **Commutativity** : Let $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, then

$$\begin{aligned}(x_1, x_2) + (y_1, y_2) &= (x_1 + y_1, x_2 + y_2)\\ &= (y_1 + x_1, y_2 + x_2)\\ &= (y_1, y_2) + (x_1, x_2).\end{aligned}$$

   Therefore, axiom (1) holds.

2. **Associativity** : Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathbb{R}^2$, we have

$$\begin{aligned}((x_1, x_2) + (y_1, y_2)) + (z_1, z_2) &= (x_1 + y_1, x_2 + y_2) + (z_1, z_2)\\ &= (x_1 + y_1 + z_1, x_2 + y_2 + z_2)\\ &= (x_1, x_2) + (y_1 + z_1, y_2 + z_2)\\ &= (x_1, y_1) + ((y_1, y_2) + (z_1, z_2)).\end{aligned}$$

   Then, axiom (2) holds.

3. **Additive identity** : The vector $0_{\mathbb{R}^2} = (0, 0)$ satisfies the condition (3) since, for each $(x_1, x_2) \in \mathbb{R}^2$, we have

$$\begin{aligned}(0, 0) + (x_1, x_2) &= (0 + x_1, 0 + x_2)\\ &= (x_1, x_2).\end{aligned}$$

4. **Additive inverse** : Let $(x_1, x_2) \in \mathbb{R}^2$, then the vector $(-x_1, -x_2) \in \mathbb{R}^2$, satisfies the condition (4), since :

$$\begin{aligned}(x_1, x_2) + (-x_1, -x_2) &= (x_1 + (-x_1), x_2 + (-x_2))\\ &= (0, 0).\end{aligned}$$

5. **Associativity of multiplication** : Let $(x_1, x_2) \in \mathbb{R}^2$ and $\lambda, \mu \in \mathbb{R}$, we have

$$\begin{aligned}\lambda.(\mu \cdot (x_1, x_2)) &= \lambda.(\mu x_1, \mu x_2)\\ &= (\lambda\mu x_1, \lambda\mu x_2)\\ &= (\lambda\mu) \cdot (x_1, x_2).\end{aligned}$$

   Then, axiom (5) holds.

6. **Distributivity 1** : Let $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, and $\lambda \in \mathbb{R}$, then

$$\begin{aligned}\lambda \cdot ((x_1, x_2) + (y_1, y_2)) &= \lambda \cdot (x_1 + y_1, x_2 + y_2)\\ &= (\lambda x_1 + \lambda y_1, \lambda x_2 + \lambda y_2)\\ &= (\lambda x_1, \lambda x_2) + (\lambda y_1, \lambda y_2)\\ &= \lambda.(x_1, x_2) + \lambda.(y_1, y_2).\end{aligned}$$

   Therefore, axiom (6) holds.

7. **Distributivity 2** : Let $\lambda, \mu \in \mathbb{R}$ and $(x_1, x_2) \in \mathbb{R}^2$, we have

$$\begin{aligned}(\lambda+\mu)\cdot(x_1,x_2) &= ((\lambda+\mu)x_1,(\lambda+\mu)x_2)\\ &= (\lambda x_1+\mu x_1, \lambda x_2+\mu x_2)\\ &= (\lambda x_1, \lambda x_2)+(\mu x_1, \mu x_2)\\ &= \lambda.(x_1,x_2)+\mu(x_1,x_2).\end{aligned}$$

   Then, axiom (7) holds.
8. **Unitarity** : Let $(x_1, x_2) \in \mathbb{R}^2$, $1 \cdot (x_1, x_2) = (x_1, x_2)$.

We conclude that $(\mathbb{R}^2, +, .)$ is a vectorial space on $\mathbb{R}$.

Solution 2

1. F_1 is a voctor subspace of $\mathbb{R}^3$. Indeed :

   (1) $0_V = 0_{\mathbb{R}^3} = (0,0,0) \in F_1$, because $0 - 0 + 0 = 0$.
   (2) Let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ be an element of F_1. Then

$$x_1 - y_1 + z_1 = 0, \quad (1)$$
$$x_2 - y_2 + z_2 = 0. \quad (2)$$

   Summing up (1) and (2), yields

$$x_1 + x_2 - (y_1 + y_2) + z_1 + z_2 = 0.$$

   Which implies that $u + v = (x_1 + y_1, x_2 + y_2, z_1 + z_2)$ inside in F_1, thus F_1 is closed under addition.

   (3) Let α be any scalar and $u = (x, y, z)$ be an element of F_1. Then $x - y + z = 0$, which leads $\alpha x - \alpha y + \alpha z = 0$, thus $\alpha.u = (\alpha x, \alpha y, \alpha z) \in F_1$. Therefore F_1 is closed under scalar multiplication.
   All three conditions of a subspace vector are satisfied for F_1, we conclude that F_1 is a subspace vector of $\mathbb{R}^3$.

2. Since $0_{\mathbb{R}^3} = (0,0,0) \notin F_2$, because $0 - 0 + 0 \neq 1$. Then, F_2 is not a subspace vector of $\mathbb{R}^3$.

3. The set F_3 contains $0_{\mathbb{R}^3}$ and the sum of tow vectors of F_3 still into F_3, but F_3 is not closed under scalar multiplication. For example, if we take $u = (-1, 0, 3) \in F_3$ and $\alpha = -2 \in \mathbb{R}$, then $-2.u = (2, 0, 6) \notin F_3$, because the first compenent of αu isn't negative. Thus, F_3 is not a subspace vector of $\mathbb{R}^3$.

Solution 3

We will show that the vectors $u = (-5, 6, 4), v = (1, 0, -2)$ and $w = (0, 3, 5)$ are linearly independent.

Indeed : let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, we have

$$\lambda_1 u + \lambda_2 v + \lambda_3 w = 0_{\mathbb{R}^3}.$$

Then

$$\lambda_1(-5, 6, 4) + \lambda_2(1, 0, -2) + \lambda_3(0, 3, 5) = (0, 0, 0).$$

Therefore

$$\begin{cases} -5\lambda_1 + \lambda_2 = 0, \\ 6\lambda_1 + 3\lambda_3 = 0, \\ 4\lambda_1 - 2\lambda_2 + 5\lambda_3 = 0. \end{cases}$$

Which leads

$$\lambda_1 = \lambda_2 = \lambda_3 = 0.$$

Then, the vectors u, v and w are linearly independent.

Solution 4

Let F be the subset of $\mathbb{R}^3$ defined as :

$$F = \left\{ (x_1, x_2, 0) \in \mathbb{R}^3 / x_1, x_2 \in \mathbb{R} \right\}.$$

1. F is a voctor subspace of $\mathbb{R}^3$. Indeed :

   (1) $0_V = 0_{\mathbb{R}^3} = (0,0,0) \in V$.
   (2) Let $u = (x_1, x_2, 0)$ and $v = (y_1, y_2, 0)$ be an element of F. Then $u+v = (x_1+y_1, x_2+y_2, 0)$, because $x_1 + y_1 \in \mathbb{R}$ and $x_2 + y_2 \in \mathbb{R}$ then $u + v$ inside in F. Thus F is closed under addition.
   (3) Let α be any scalar and $u = (x_1, x_2, 0)$ be an element of F. Then $\alpha.u = (\alpha x_1, \alpha x_2, 0) \in F$. Therefore F is closed under scalar multiplication.
   All three conditions of a subspace vector are satisfied for F, we conclude that F is a subspace vector of $\mathbb{R}^3$.

2. We are looking for a basis for F, we start by searching a set generating of F
   - For all $(x, y, 0) \in F$ we have : $(x, y, 0) = x(1,0,0)+y(1,0,0)$. Then $spanF = \{v_1 = (1,0,0), v_2 = (0,1,0)\}$.
   - Now, we show that the family $\{v_1, v_2\}$ is linearly independent.

     Let $\lambda_1, \lambda_2 \in \mathbb{R}$, such that $\lambda_1 v_1 + \lambda_2 v_2 = 0_{\mathbb{R}^3}$, then

     $$\lambda_1(1,0,0) + \lambda_2(0,1,0) = (0,0,0).$$

     Which in turn implies that

     $$\begin{cases} \lambda_1 = 0, \\ \lambda_2 = 0. \end{cases}$$

     Consequently, the vectors v_1 and v_2 are linearly independent, which proves that $\{v_1, v_2\}$ is a basis of F.

3. $\dim F = card\,\{v_1, v_2\} = 2$

Solution 5

1. We will show that the set $\{v_1 = (0,1,1), v_2 = (1,0,1), v_3 = (1,1,0)\}$ is a basis of $\mathbb{R}^3$.

   First, we show that the family is linearly independent. Indeed, let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, such that $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0_{\mathbb{R}^3}$, then

   $$\lambda_1(0,1,1) + \lambda_2(1,0,1) + \lambda_3(1,1,0) = (0,0,0).$$

   Which in turn implies that

   $$\begin{cases} \lambda_2 + \lambda_3 = 0, \\ \lambda_1 + \lambda_3 = 0, \\ \lambda_1 + \lambda_2 = 0. \end{cases}$$

   Which leads to

   $$\lambda_1 = \lambda_2 = \lambda_3 = 0.$$

Consequently, the vectors v_1, v_2 and v_3 are linearly independent.

Now, we are going to prove that the family $\{v_1, v_2, v_3\}$ is generating of $\mathbb{R}^3$.
Let $(x, y, z) \in \mathbb{R}^3$, we are looking for $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, such that :

$$(x, y, z) = \lambda_1(0, 1, 1) + \lambda_2(1, 0, 1) + \lambda_3(1, 1, 0).$$

Therefore

$$x = \lambda_2 + \lambda_3, \tag{3}$$
$$y = \lambda_1 + \lambda_3, \tag{4}$$
$$z = \lambda_1 + \lambda_2. \tag{5}$$

Substracting the equation (4) from the equation (3), yields

$$x - y = \lambda_2 - \lambda_1.$$

Summing up the last result with (5), we get

$$\lambda_2 = \frac{1}{2}(x - y + z). \tag{6}$$

Inserting (6) into (3), we obtain

$$\lambda_3 = \frac{1}{2}(x + y - z).$$

In the similar way, we insert (6) into (5), we have

$$\lambda_1 = \frac{1}{2}(-x + y + z).$$

So *span* $\{v_1, v_2, v_3\} = \mathbb{R}^3$, we conclude that $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$.

2. The components of the vector $w = (1, 1, 1)$ in the basis $\{v_1, v_2, v_3\}$ are : $\lambda_1 = \frac{1}{2}(-1 + 1 + 1) = \frac{1}{2}$, $\lambda_2 = \frac{1}{2}(1 - 1 + 1) = \frac{1}{2}$, $\lambda_3 = \frac{1}{2}(1 + 1 - 1) = \frac{1}{2}$.