Université Mostefa Ben Boulaïd - Batna 2
Faculty of Mathematics and Computer Science
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Algebra 3 Tutorial series 2, 3

Exercice 1 *Let* $A = \begin{pmatrix} 2 & 4 & -2 \\ 0 & -4 & 3 \\ -3 & -7 & 4 \end{pmatrix}$

Find the characteristic polynomial of A.

Exercice 2 *Let* $M \in M_3(\mathbb{R})$ *be the matrix*

$$M = \begin{pmatrix} -2 & 2 & 2 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

Compute the eigenvalues and the eigenvectors of M*. Is* M *diagonalizable ?*

Exercice 3 *Find a basis of* $\mathbb{R}^3$ *made of eigenvectors for the linear map* $L : \mathbb{R}^3 \to \mathbb{R}^3$ *that is represented by the matrix*

$$[L] = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

concerning the standard basis

Exercice 4 *Let* $M = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 0 \end{pmatrix} \in M_3(\mathbb{R}) :$

(a) *Determine the eigenvalues and the eigenvectors of* M *over the field* $\mathbb{R}$.

(b) *Say, justifying the answer, if there exist a matrix* $V \in M_3(\mathbb{R})$ *and a diagonal matrix* $D \in M_3(\mathbb{R})$ *such that* $M = PDP^{-1}$.

(c) *Determine a possible choice of such* V *and* D.

Exercice 5 *Let* A *be the matrix*

$$A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) *Find the eigenvalues and the eigenvectors of* A *over the field* $\mathbb{R}$*. Is* A *diagonalizable ?*

(b) *Find the eigenvalues and the eigenvectors of* A *over the field* $\mathbb{C}$*. Is* A *diagonalizable ?*

(c) *Find an invertible matrix* $V \in M_3(\mathbb{C})$ *such that* $P^{-1}AP$ *is diagonal.*

Exercice 6 *Let* $a \in \mathbb{R}$ *and consider the matrix*

$$A = \begin{pmatrix} 1 & -1 \\ 1 & a \end{pmatrix}$$

(**a**) *For which* $a \in \mathbb{R}$ *is* A *diagonalizable over* $\mathbb{R}$ *?*
(**b**) *For which* $a \in \mathbb{R}$ *is* $\lambda = 3$ *an eigenvalue for* A *?*
(**c**) *For which* $a \in \mathbb{R}$ *is* $v = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$ *an eigenvector for* A *?*

Exercice 7 *Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the endomorphism associated to the matrix*

$$\begin{pmatrix} 9 & 6 & 9 \\ 3 & 12 & 9 \\ 3 & 6 & 15 \end{pmatrix}$$

to the standard basis. Say if T is diagonalizable and, if so, compute a basis of $\mathbb{R}^3$ made of eigenvectors for T .

Exercice 8 *Let A be the matrix*

$$A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1. *Find the eigenvalues and the eigenvectors of A over the field $\mathbb{R}$ Is A diagonalizable?*
2. *Find the eigenvalues and the eigenvectors of A over the field $\mathbb{C}$. Is A diagonalizable?*
3. *Find an invertible matrix $V \in M_3(\mathbb{C})$ such that $P^{-1}AP$ is diagonal.*

Exercice 9 *Let $a \in \mathbb{R}$ and consider the matrix*

$$A = \begin{pmatrix} 1 & -1 \\ 1 & a \end{pmatrix}$$

1. *For which $a \in \mathbb{R}$ is A diagonalizable over $\mathbb{R}$?*
2. *For which $a \in \mathbb{R}$ is $\lambda = 3$ an eigenvalue for A?*
3. *For which $a \in \mathbb{R}$ is $v = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$ an eigenvector for A?*

Exercice 10 *Let $a \in \mathbb{R}$ be a real parameter ad let L_a be the linear endomorphism of $\mathbb{R}^3$ defined as*

$$L_a\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2ax + y + z \\ x + ay + z \\ -x + y + az \end{bmatrix}$$

Study the diagonalizability of L_a, depending on the parameter a.

Exercice 11 *Verify Cayley-Hamilton theorem for $A = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$*

Exercice 12 *Is $A = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & 1 & 3 \end{pmatrix}$ invertible ? If so, find A^{-1} (use Cayley-Hamilton theorem).*

Exercice 13 *Find the minimal polynomial of $A = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}$*

Exercice 14 *Let $p, q \in \mathbb{R}[x]$ be given by $p = x^2 - 2x - 3$, $q = x^3 - 2x^2 + 2x - 5$.*
Let $A \in M_2(\mathbb{R})$ and $B \in M_3(\mathbb{R})$ be given by

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

1. *Compute* $p(A)$, $p(B)$, $q(A)$, $q(B)$.
2. *Compute the characteristic polynomials of* A *and* B *, from question 1. What do you notice ?*
3. *Let* $\mathbb{F} = \mathbb{Z}_2$ *, the field of two elements and let* $p = x^2 + x \in \mathbb{F}[x]$.

Show that $p(t) = 0$*, for all* $t \in \mathbb{F}$*.*

Exercice 15 *Find a* 3×3 *matrix whose minimal polynomial is* x^2

Exercice 16 *Let* $A = \begin{pmatrix} 2 & 4 & -2 \\ 0 & -4 & 3 \\ -3 & -7 & 4 \end{pmatrix}$

(**a**) *Find the characteristic polynomial of* A*.*

(**b**) *Find the minimal polynomial of* A

(**c**) *Find the characteristic vectors and a basis* $\mathcal{B}$ *such that* $[A]_{\mathcal{B}}$ *is diagonal.*

Exercice 17 *What can the minimal polynomial of* $T : \mathbb{R}^4 \to \mathbb{R}^4$ *be if the characteristic polynomial of* $[T]_{\mathcal{B}}$ *is*

- $(t-1)(t^3+1)$
- $(t^2+1)^2$?

Here, $\mathcal{B}$ *is the standard basis of* $\mathbb{R}^4$