TD n°2 : Calcul Intégral

Exo 1 : - Intégration des fcts usuelles :

+1 $\displaystyle\int \frac{e^{\cot an x}}{\sin^2 x}\,dx = -\int (\cot an x)' e^{\cot an x}\,dx$ (f')

$= -e^{\cot an x} + c$

sous forme $\int f'(x)\,dx = f(x)$

+2 $\displaystyle\int 3^x\,dx = \int \left[e^{x\ln 3}\right] dx = \frac{3^x}{\ln 3} + c$

+3 $\displaystyle\int_1^2 \frac{1}{t}(\ln t)^5\,dt = \left[\frac{1}{6}(\ln t)^6\right]_1^2 = \frac{\ln^6 2}{6}$

« sous forme $\int (f^n)' = \int [n f' f^{n-1}] = f^n$ »

+4 $\displaystyle\int \frac{\sin x}{\cos^2 x}\,dx = -\int -\frac{\sin x}{\cos^2 x}\,dx = \frac{1}{\cos x} + c$

« sous forme $\int \frac{-f'}{f^2} = \frac{1}{f}$ »

+5 $\displaystyle\int \frac{\sqrt{x} - x^3 e^{2x} + x^2}{x^3}\,dx = \int \left[x^{-\frac{5}{2}} - e^{2x} + \frac{1}{x}\right] dx$ « linéarité »

$\displaystyle = \frac{1}{-\frac{5}{2}+1} x^{(-5/2)+1} - \frac{1}{2}e^{2x} + \ln|x| + c$

$\displaystyle = \frac{-2}{3x\sqrt{x}} - \frac{1}{2}e^{2x} + \ln|x| + c$

Exo 2 : "Intégration par partie"

1/ $\displaystyle I_1 = \int \underbrace{(x^3+1)}_{u}\underbrace{e^{2x+1}}_{v'}\,dx$

$= uv - \int u'v$

$u(x) = x^3+1 \Rightarrow u'(x) = 3x^2$

$v'(x) = e^{2x+1} \Rightarrow$

$v(x) = \frac{1}{2}e^{2x+1}$

$\displaystyle I_1 \overset{P.P_1}{=} \frac{1}{2}(x^3+1)e^{2x+1} - \frac{1}{2}\int 3\underbrace{x^2}_{u}\underbrace{e^{2x+1}}_{v'}\,dx$

$\displaystyle \overset{P.P_2}{=} \frac{1}{2}(x^3+1)e^{2x+1} - \frac{3}{2}\left[\frac{1}{2}x^2 e^{2x+1} - \frac{2}{2}\int x e^{2x+1}\,dx\right]$

$\displaystyle = \left[\frac{1}{2}(x^3+1) - \frac{3}{4}x^2\right]e^{2x+1} + \frac{3}{2}\int \underbrace{x}_{u}\underbrace{e}_{v'}\,dx$

$\displaystyle \overset{P.P_3}{=} \left[\frac{1}{2}(x^3+1) - \frac{3}{4}x^2\right]e^{2x+1} + \frac{3}{2}\left[\frac{1}{2}x e^{2x+1} - \frac{1}{2}\int e^{2x+1}\,dx\right]$

$\displaystyle = \left[\frac{1}{2}(x^3+1) - \frac{3}{4}x^2 + \frac{3}{4}x - \frac{3}{8}\right]e^{2x+1} + c$

$\displaystyle = \left[\frac{1}{2}x^3 - \frac{3}{4}x^2 + \frac{3}{4}x + \frac{1}{8}\right]e^{2x+1} + c$

2/ $I_2 = \int \arcsin x\,dx$

$u(x) = \arcsin x \Rightarrow u'(x) = \frac{1}{\sqrt{1-x^2}}$

$v'(x) = 1 \Rightarrow v(x) = x$

$\displaystyle I_2 \overset{P.P}{=} x.\arcsin x + \int \frac{-2x}{2\sqrt{1-x^2}}\,dx$

$I_2 = x.\arcsin x + \sqrt{1-x^2} + c$

3/ $\displaystyle I_3 = \int \cancel{\frac{x^5}{\sqrt{x^3-1}}\,dx}$

$I_3 = \int x \arctan x\,dx$

$u(x) = \arctan x$
$\Rightarrow u'(x) = \frac{1}{1+x^2}$

$v'(x) = x \Rightarrow v(x) = \frac{1}{2}x^2$

$\displaystyle I_3 \overset{P.P}{=} \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{[x^2+1]-1}{x^2+1}\,dx$

$\displaystyle = \frac{x^2}{2}\arctan x - \frac{1}{2}\int \left(1 - \frac{1}{x^2+1}\right) dx$

$\displaystyle = \frac{1}{2}\left[x^2 \arctan x - (x - \arctan x)\right] + c$

4/ $\displaystyle \int \underbrace{x^7}_{v'}\underbrace{\ln x}_{u}\,dx = I_4$

$u(x) = \ln x \Rightarrow u'(x) = \frac{1}{x}$

$v'(x) = x^7 \Rightarrow v(x) = \frac{1}{2}x^8$

$\displaystyle I_4 = \frac{1}{8}x^8 \ln x - \frac{1}{8}\int x^7\,dx$

$\displaystyle = \frac{1}{8}x^3 \ln x - \frac{1}{64}x^8 + c$

5/ $\displaystyle I_5 = \int x^2 \cos^2 x\,dx = \int x^2\left[\frac{\cos 2x - 1}{2}\right] dx$

$\displaystyle I_5 = \frac{1}{2}\left[\int \underbrace{x^2}_{u}\underbrace{\cos 2x}_{v'}\,dx - \int x^2\,dx\right]$

$\displaystyle \overset{P.P_1}{=} \frac{1}{2}\left[\frac{1}{2}x^2 \sin 2x - \int (2x)\left(\frac{1}{2}\sin 2x\right)dx\right] - \frac{x^3}{6}$

$\displaystyle \overset{P.P_2}{=} \frac{1}{4}x^2 \sin 2x - \frac{1}{2}\left[-\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\,dx\right] - \frac{x^3}{6}$

$\displaystyle = \frac{1}{4}x^2 \sin 2x + \frac{1}{4}x\cos 2x - \frac{1}{8}\sin 2x - \frac{x^3}{6} + c$

Exo3 : - Intégration par Changement de Variable

1 $I_1 = \int_4^9 \frac{1+e^{\sqrt{x}}}{\sqrt{x}}\,dx$

$t = e^{\sqrt{x}} \Rightarrow dt = \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx$

$x=4 \longrightarrow t = e^2$

$x=9 \Rightarrow t = e^3$

$I_1 \overset{ch.v.}{=} \int_4^9 x^{-1/2}\,dx + \int_4^9 \frac{2e^{\sqrt{x}}}{2\sqrt{x}}$

$= \left[2\sqrt{x}\right]_4^9 + 2\int_{e^2}^{e^3} 1.\,dt$

$= 2\left(\sqrt{9}-\sqrt{4}\right) + 2\left[t\right]_{e^2}^{e^3} = 2\left(1+e^3-e^2\right)$

ou bien:

on pose : $t = \sqrt{x} \Rightarrow t^2 = x$ et $\begin{cases} x=4 \Rightarrow t=2 \\ x=9 \Rightarrow t=3 \end{cases}$

$\Rightarrow dx = (2t\,dt)$

$I_1 = \int_2^3 \frac{1+e^t}{\cancel{t}}\,(2\cancel{t}\,dt) = 2\int_2^3 (1+e^t)\,dt$

$= 2\left[t+e^t\right]_2^3 = 2\left[1+e^3-e^2\right].$

2: $I_2 = \int e^x \ln(\underbrace{1+e^x}_{t})\,dx$

ch.v

$t = (1+e^x)$

$\Rightarrow dt = e^x\,dx$

$I_2 \overset{ch.v}{=} \int \ln(t).\,dt$

p.p: $u(t) = \ln t \Rightarrow u'(t) = \frac{1}{t}$

$v'(t) = 1 \Rightarrow v(t) = t$

$\overset{p.p.}{=} t\ln(t) - \int \frac{t}{t}\,dt = t.\ln(t) - t + c$

$I_2 = (1+e^x)\ln(1+e^x) - (1+e^x) + C$

3: $I_3 = \int \frac{x^5}{\sqrt{x^3-1}}\,dx$

$t = \sqrt{x^3+1}$

$t^2+1 = x^3$

$2t\,dt = 3x^2\,dx$

$\frac{2}{3}t\,dt = x^2\,dx$

$I_3 = \int \frac{\overset{(t^2+1)}{x^3}}{\underbrace{\sqrt{x^3-1}}_{t}}\,\underbrace{x^2\,dx}_{\frac{2}{3}t\,dt}$

$\overset{ch.v.}{=} \int \frac{t^2+1}{\cancel{t}}\left(\frac{2}{3}\cancel{t}\,dt\right) = \frac{2}{3}\left[\frac{t^3}{3}+t\right] + C$

$= \frac{2}{9}\left(x^3+1\right)^{3/2} + \frac{2}{3}\left(\sqrt{x^3+1}\right) + C$

4: $I_4 = \int \frac{\overbrace{\cos x\,dx}^{dt}}{\underbrace{3+\sin x}_{t}}$

$t = 3+\sin x$

$\Rightarrow dt = \cos x\,dx$

$I_4 \overset{ch.v}{=} \int \frac{dt}{t} = \ln|t| + C$

$= \ln|(3+\sin x)| + C$

$= \ln(3+\sin x) + C$

5: $I_5 = \int \sqrt{e^x-1}\,dx$

$t = \sqrt{e^x-1}$

$t^2+1 = e^x$

$2t\,dt = e^x\,dx$

$\frac{2t}{t^2+1}\,dt = dx$

$I_5 \overset{ch.v.}{=} \int t.\,\frac{2t}{t^2+1}\,dt$

$= 2\int \frac{(t^2+1)-1}{t^2+1}\,dt$

$= 2\int \left[1-\frac{1}{t^2+1}\right]dt$

$= 2\left[t - \arctan t\right] + C$

$I_5 = 2\left[\sqrt{e^x-1} - \arctan\left(\sqrt{e^x-1}\right)\right] + C$

Exo4 : - Intégration des Fcts. Rationnelle -

a $I_1 = \int_2^3 \frac{x-2}{x(x-1)}\,dx$ // $\frac{x-2}{x(x-1)} = \frac{a}{x-1} + \frac{b}{x}$

$= \underbrace{\frac{-1}{x-1} + \frac{2}{x}}_{\text{Par Identification}}$

$I_1 = \int_2^3 \frac{-dx}{x-1} + \int \frac{2\,dx}{x}$

$= \left[-\ln(x-1)\right]_2^3 + \left[2\ln x\right]_2^3$

$= \left[\ln\left(\frac{x^2}{x-1}\right)\right]_2^3 = \ln\left(\frac{9}{2}\right) - \ln(4) = \ln\left(\frac{9}{8}\right)$

b: $I_2 = \int_0^1 \frac{x}{(x+1)(x^2+1)}\,dx$

on pose : $\frac{x}{(x+1)(x^2+1)} = \frac{a}{x+1} + \frac{bx+c}{x^2+1}$

Par identification; on trouve : $\begin{cases} a = -\frac{1}{2} \\ b = \frac{1}{2} \\ c = \frac{1}{2} \end{cases}$

donc;

$I_2 = \int_0^1 \frac{(-1/2)\,dx}{(x+1)} + \frac{1}{2}\int_0^1 \frac{(x+1)}{(x^2+1)}\,dx$

$= \left[-\frac{1}{2}\ln(x+1)\right]_0^1 + \frac{1}{2}\left[\int_0^1 \frac{2x\,dx}{2(x^2+1)} + \int_0^1 \frac{dx}{(x^2+1)}\right]$

$= \left[-\frac{1}{2}\ln(x+1)\right]_0^1 + \frac{1}{4}\left[\ln(x^2+1)\right]_0^1 + \frac{1}{2}\left[\arctan x\right]_0^1$

$= -\frac{1}{2}\ln(2) + \frac{1}{4}\ln(2) + \frac{1}{2}\left(\frac{\pi}{4}\right)$

$= \frac{1}{2}\left[\ln(2) + \frac{\pi}{4}\right]$

Exo 5 :

$f(t) = -\frac{12}{5}t^2 + \frac{12}{5}at \quad ; t \in [0, a]$

$f(t)$: Débit du sang ($m\ell/mn$)

1. Tableau de variation :

$f'(t) = -\frac{24}{5}t + \frac{12}{5}a = \frac{12}{5}(a - 2t)$

$f'(t) \geq 0 \Rightarrow a - 2t \geq 0$

$\Rightarrow t \leq \frac{a}{2}$

t	0		$a/2$		a
$f'(t)$		$+$	0	$-$	
$f(t)$	0		$\frac{3a^2}{5}$		0

* $f_{max} = f\left(\frac{a}{2}\right) = \frac{3}{5}a^2$

2. $V = \int_0^a f(t)\,dt$; Volume total de sang prélevé.

$V = \frac{12}{5}\int_0^a (at - t^2)\,dt$

$= \frac{12}{5}\left[\frac{a}{2}t^2 - \frac{t^3}{3}\right]_0^a = \frac{12}{5}\left[\frac{a^3}{2} - \frac{a^3}{2}\right]_0^a$

$= \frac{12}{5}\left[\frac{a^3}{6}\right] = \frac{2}{5}a^3 \quad (m\ell)$

3. $a = ?$ sachant que : $V = 400 (m\ell)$

$V = 400 \Rightarrow \frac{2}{5}a^3 = 400$

$\Rightarrow a^3 = 1000$

$\Rightarrow a = 10 \; (mn)$

* $f_{max} = \frac{3}{5}a^2\Big|_{a=10} = \frac{3}{5}(10)^2 = 60 \; (m\ell)$